allez

Chapter 5

Noticed Principal Orbits in Simple Real Lie Algebras

Introduction

We shall investigate the behaviour of a principal nilpotent element of $\mathfrak{p}_{\mathbb{C}}$. Such an element and its orbit under the action of $K_{\mathbb{C}}$ have been extensively studied by Kostant and Rallis in their seminal 1971 paper [**K-R**]. The principal orbit plays also an important role in Representation Theory. We shall prove that the semisimple part of a minimal (θ, σ)-stable Levi subalgebra containing a principal nilpotent of $\mathfrak{p}_{\mathbb{C}}$ is the complexification of a quasi-split reductive subalgebra of $\mathfrak{g}$.

5.1 Principal nilpotent element

Let e be in $\mathfrak{p}_{\mathbb{C}}$. Then e is principal if and only if $K_{\mathbb{C}} \cdot e$ is a maximal $K_{\mathbb{C}}$-orbit in $\mathfrak{p}_{\mathbb{C}}$

[**K-R**] that is if and only if

$$\dim K_{\mathbb{C}}\cdot e \geq \dim K_{\mathbb{C}}\cdot e' \qquad \text{for all } e' \in \mathfrak{p}_{\mathbb{C}}.$$

If the orbit $G\cdot\lambda_{\mathbb{R}}$ corresponds to the orbit $K_{\mathbb{C}}\cdot\lambda_{\mathbb{C}}$, then

$$\dim_{\mathbb{C}} K_{\mathbb{C}}\cdot\lambda_{\mathbb{C}} = \tfrac{1}{2}\dim_{\mathbb{R}} G\cdot\lambda_{\mathbb{R}} = \tfrac{1}{2}\dim_{\mathbb{C}} G_{\mathbb{C}}\cdot\lambda_{\mathbb{C}}$$

Kostant and Rallis give several characterizations of a principal nilpotent of $\mathfrak{p}_{\mathbb{C}}$. We will use the following criterion due to them. We will say that e is principal if and only if

$$\dim\ K_{\mathbb{C}}\cdot e = \dim\ \mathfrak{p}_{\mathbb{C}} \text{ - } \dim\ \mathfrak{a}_{\mathbb{C}},$$

where $\mathfrak{a}_{\mathbb{C}}$ is the complexification of a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. The dimension of $\mathfrak{a}$ is called the *real rank* of $\mathfrak{g}$ (see Helgason [**He**]). The following theorem gives a characterization of the *quasi-split* simple real Lie algebras $\mathfrak{g}$ in terms of the noticed principal elements.

Theorem 5.1.1. *Let $\mathfrak{g}$ be a simple real Lie algebra. Then $\mathfrak{g}$ is quasi-split if and only there exists a nilpotent element $\lambda_{\mathbb{C}}$ of $\mathfrak{p}_{\mathbb{C}}$ such that $\lambda_{\mathbb{C}}$ is noticed and principal.*

Proof. If $\mathfrak{g}$ is an exceptional simple real Lie algebra then an analysis of D. Djoković's tables [**D1**], [**D2**] shows that the principal nilpotent orbit is noticed if and only if $\mathfrak{g}$ is quasisplit. Now assume that g is a classical real Lie algebra. If $\mathfrak{g}$ is equal to $\mathfrak{sl}_n(\mathbb{R})$, $\mathfrak{su}(p,q)$, or $\mathfrak{sp}_n(\mathbb{R})$ then the theorems 4.2.1 and 4.2.3 tell us that the principal nilpotent is noticed because it is also regular in each case. Moreover, since $\mathfrak{sp}(p,q)$, $\mathfrak{su}^*_{2n}$ and $\mathfrak{so}^*_{2n}$ have no non-zero noticed elements, we only need to

give a proof for the case where $\mathfrak{g} = \mathfrak{so}(p,q)$. First we will give a general description of the principal orbit of $\mathfrak{g}$ which corresponds to a maximal $K_{\mathbb{C}}$-orbit in $\mathfrak{p}_{\mathbb{C}}$ under the Kostant-Sekiguchi correspondence, and then we shall prove that such an orbit is noticed if and only if $\mathfrak{g}$ is quasi-split. In our case the principal orbit is characterized by its dimension $pq - q$. The dimension of the complex nilpotent orbits of type B_n and D_n are given by

$$2n^2 + n - \frac{1}{2}\sum_i {s_i}^2 + \frac{1}{2}\sum_{i\ odd} r_i \qquad (5.1.2)$$

and

$$2n^2 - n - \frac{1}{2}\sum_i {s_i}^2 + \frac{1}{2}\sum_{i\ odd} r_i \qquad (5.1.3),$$

respectively, where r_i and s_i are defined as follows: if $[d_1, d_2, \ldots, d_l]$ is the partition associated with the nilpotent orbit. Put $r_i = |\{j | d_j = i\}|$ and $s_i = |\{j | d_j \geq i\}|$. See **[C-Mc]**

We consider two cases:

case 1: $p = q$

Consider the orbit $G{\cdot}\lambda_{\mathbb{R}}$ parametrized by the following Young diagram:

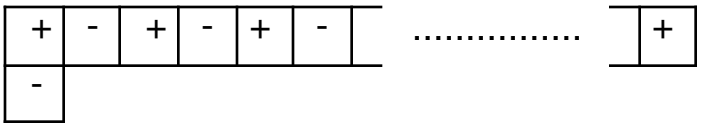

figure 2

If p is even then there are two such orbits and the Young diagram should be labeled with Roman numerals I or II see **[C-Mc]**.

The associated complex orbit $G_{\mathbb{C}}\cdot\lambda_{\mathbb{C}}$ is parametrized by $[2p-1,1]$. Then $r_1 = 1$, $r_{2p-1} = 1$, $s_1 = 2$, and $s_i = 1$ for $2 \leq i \leq 2p-1$. Using formula 5.1.3 with $n = p$ we find that

$$\dim\, G_{\mathbb{C}}\cdot\lambda_{\mathbb{C}} = 2p^2 - p - \tfrac{1}{2}(2p+2) + 1 = 2p^2 - 2p$$

Hence

$$\dim\, K_{\mathbb{C}}\cdot\lambda_{\mathbb{C}} = p^2 - p$$

Therefore the above orbit is principal. From theorem 4.2.2 it is also noticed. Finally $\mathfrak{so}(p,q)$ is quasi-split.

case 2: $p > q$

Consider the orbit $G\cdot\lambda_{\mathbb{R}}$ parametrized by the following Young diagram:

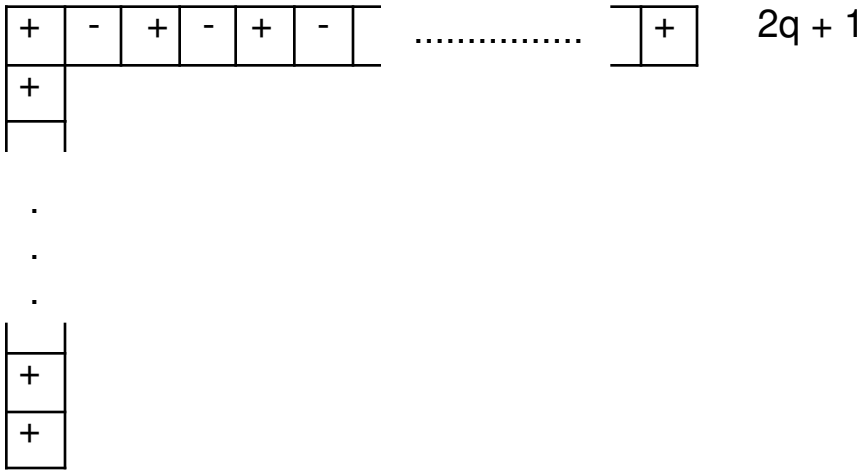

figure 3

where the first row has length $2q+1$ and the remaining $p-q-1$ rows have length 1.

If q is even then there are two such orbits and the Young diagram should be labeled with Roman numerals I or II see **[C-Mc]**.

The associated complex orbit $G_{\mathbb{C}}\cdot\lambda_{\mathbb{C}}$ is parametrized by $[2q+1, 1^{p-q-1}]$. Thus $r_1 = p-q-1$, $r_{2q+1}=1$, $s_1 = p-q$, and $s_i = 1$ for $2 \leq i \leq 2q+1$. Using formula 5.1.3 with $n = \frac{p+q}{2}$ we find that

$$\dim G_{\mathbb{C}}\cdot\lambda_{\mathbb{C}} = \tfrac{(p+q)^2}{2} - \tfrac{(p+q)}{2} - \tfrac{1}{2}((p-q)^2+2q) + \tfrac{1}{2}(p-q) = 2pq - 2q.$$

Using formula 5.1.4 with $n = \frac{p+q-1}{2}$ we have:

$$\dim G_{\mathbb{C}}\cdot\lambda_{\mathbb{C}} = \tfrac{(p+q)^2-2(p+q)+1}{2} - \tfrac{(p+q-1)}{2} - \tfrac{1}{2}((p-q)^2+2q) + \tfrac{1}{2}(p-q) = 2pq - 2q.$$

In either case we obtain

$$\dim K_{\mathbb{C}}\cdot\lambda_{\mathbb{C}} = pq - q.$$

Again the orbit $K_{\mathbb{C}}\cdot\lambda_{\mathbb{C}}$ is principal and noticed if and only if $[2q+1, 1^{p-q-1}]$ dominates every other admissible partition that is if and only if $|p-q| \leq 2$ ([**C-Mc**]) that is if and only if $\mathfrak{g}$ is quasi-split. The result follows.

■

We shall need the following lemma.

Lemma 5.1.4. *Let l be a minimal (σ,θ)-stable Levi subalgebra containing a principal nilpotent e of $\mathfrak{p}_{\mathbb{C}}$. Then l is a minimal Levi subalgebra containing e.*

Proof. Let (x,e,f) be a KS-triple containing e. We have proved (Proposition 1.1.3) that $l = \mathfrak{g}_{\mathbb{C}}^{\mathfrak{t}}$ where $\mathfrak{t}$ is a maximal toral subalgebra of $\mathfrak{k}_{\mathbb{C}}^{e}$. In fact $\mathfrak{t} \subseteq \mathfrak{k}_{\mathbb{C}}^{(x,e,f)}$. From King [**Ki**] we know that if e is principal then

$$\mathfrak{g}_{\mathbb{C}}^{(x,e,f)} = \mathfrak{k}_{\mathbb{C}}^{(x,e,f)}.$$

Hence $\mathfrak{t}$ is a maximal toral subalgebra of $\mathfrak{g}_{\mathbb{C}}^{e}$ and l is a minimal Levi subalgebra containing e. [**C-Mc**]

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We wish to thank David Vogan from MIT and Roger Carter from the University of Warwick (England) for their suggestions about the proof of the following proposi-

tion. The result was also mentioned by Collingwood and McGovern [**C-Mc**]. But they did not provide a proof.

The Bala-Carter theory associates each $G_{\mathbb{C}}$-nilpotent orbits of $\mathfrak{g}_{\mathbb{C}}$ to pair (m, p_m) where m is a Levi subalgebra of $\mathfrak{g}_{\mathbb{C}}$ and p_m is a distinguished parabolic of the semi-simple algebra $[m, m]$.(See [**Ca**]). Maintaining the above notations we have:

Proposition 5.1.5. *Using the above notations (m, p_m) and (m, q_m) are conjugate under $G_{\mathbb{C}}$ if and only p_m and q_m are conjugate under M, the connected subgroup of $G_{\mathbb{C}}$ with Lie algebra m.*

Proof. If p_m and q_m are M-conjugate then they are necessarily $G_{\mathbb{C}}$-conjugate since $M \subseteq G_{\mathbb{C}}$. Hence (m, p_m) and (m, q_m) are $G_{\mathbb{C}}$-conjugate.

Suppose that (m, p_m) and (m, q_m) are conjugate under $G_{\mathbb{C}}$. Let $g \in G_{\mathbb{C}}$ be a conjugating element. Then g induces an automorphism on the semisimple part $[m, m]$ of m. Consequently to prove that p_m and q_m are conjugate under M it is enough to show that $\frac{N(m)}{M}$ acts trivially on conjugacy classes of distinguished parabolic subalgebras, where $N(m)$ denotes the normalizer of m in $G_{\mathbb{C}}$. Let $\frac{Aut(m)}{Int(m)}$ denotes the outer automorphisms of m. It is isomorphic to the group of automorphism of the Dynkin diagram of m (see [**Kn**]). $Int(m)$ should be understood as the set of inner automorphisms of m.

We have $M \subseteq Ker(\Pi \circ \varphi)$ where Π and φ are the projection and inclusion maps respectively, as defined in the following sequence:

$$N(m) \xrightarrow{\varphi} Aut(m) \xrightarrow{\Pi} \frac{Aut(m)}{Int(m)} \longrightarrow 1$$

giving

$$\frac{N(m)}{Ker(\Pi \circ \varphi)} \subseteq \frac{Aut(m)}{Int(m)}.$$

and

$$1 \longrightarrow \frac{Ker(\Pi \circ \varphi)}{M} \longrightarrow \frac{N(m)}{M} \longrightarrow \frac{N(m)}{Ker(\Pi \circ \varphi)} \longrightarrow 1 \qquad 5.1.6.$$

Hence, the only important elements of $\frac{N(m)}{M}$ are those outside of $\frac{Ker(\Pi \circ \varphi)}{M}$.

We shall prove that the outer automorphisms of m do not change the conjugacy classes of distinguished parabolic subalgebras. It is known that the group of automorphisms of the Dynkin diagram is a cyclic group of order 2 for types A_l ($l \geq 2$), D_l ($l > 4$), and E_6. It is $\mathfrak{S}_3$, the permutation group on 3 letters, for D_4. Otherwise it is trivial. see [**He**].

Bala and Carter, [**B-C2**], give a description of the classes of semisimple subalgebras of parabolic type. In every case, all but at most one simple component has type A_l. They, [**B-C1**], also give a description of the weigthed Dynkin diagram of each class of distinguished simple parabolic subalgebras. In type A_l, the only distinguished parabolic subalgebras are the Borel subalgebras. Thus, if all the simple parts are of type A_l ($l \geq 2$) then p_m and q_m are conjugate under M. Suppose there is a simple component not of type A_l. A careful analysis of the weighted Dynkin of classes of distinguished simple parabolics subalgebras shows that they are invariant under their non trivial diagram automorphisms. Thus they are invariant under the outer automorphisms of m. Hence they are invariant under $\frac{N(m)}{M}$. It follows that p_m and q_m must be M-conjugate.

■

Moreover we have:

Proposition 5.1.7. *Let e be a nilpotent element of $\mathfrak{g}_{\mathbb{C}}$ contained in a minimal Levi subalgebra l. Then $G_{\mathbb{C}}\cdot e \cap l = L\cdot e$, where L is the connected subgroup of $G_{\mathbb{C}}$ with Lie algebra l.*

Proof. Since $L \subseteq G_{\mathbb{C}}$ it is clear that

$$L\cdot e \subseteq G_{\mathbb{C}}\cdot e \cap l$$

Let e' be a nilpotent element of $G_{\mathbb{C}}\cdot e \cap l$. From the Bala-Carter theory we can associate a pair (l, p_l) to $G_{\mathbb{C}}\cdot e$ where p_l is a distinguished parabolic subalgebra of $[l, l]$. Similarly we can associate a pair $(l', p_l^{'})$ to $G_{\mathbb{C}}\cdot e'$. Since e and e' are $G_{\mathbb{C}}$-conjugate the two pairs (l, p_l) and $(l', p_l^{'})$ are also conjugate under $G_{\mathbb{C}}$. In particular l is $G_{\mathbb{C}}$-conjugate to l'. Therefore l is a minimum Levi subalgebra conaining e'. Then the Bala-Carter theory tells us that we can find a pair (l, q_l), where q_l is a distinguished parabolic subalgebra of $[l, l]$ containing e', which is $G_{\mathbb{C}}$-conjugate to (l, p_l). From proposition 5.1.5 we know that p_l and q_l are conjugate under L. From the Bala-Carter theory we know that e and e' are Richardson elements of p_l and q_l repectively. Since the Levi parts of p_l and q_l are L-conjugate they are "associated" in Jonston and Richardson's sense (see **[J-R, H2]**). Thus $e' \in L\cdot e$. The proposition follows.

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Now, we are ready to prove the main result on principal nilpotents in $\mathfrak{p}_{\mathbb{C}}$.

Theorem 5.1.8. *Let e be a nilpotent principal element of $\mathfrak{p}_{\mathbb{C}}$. Then for any normal triple (x, e, f) corresponding to a triple $(l, \mathfrak{q}_l, \mathfrak{w}_l)$ as described in the classification Theorem 3.2.4:*

1. *e is regular in $l \cap \mathfrak{p}_{\mathbb{C}}$*
2. *the the real form l_0 of l is quasi-split*
3. *if $\mathfrak{q}_l = m \oplus v$ is a Levi decomposition of $\mathfrak{q}_l$ then $dim\, m \cap \mathfrak{k}_{\mathbb{C}} = dim\, \frac{v \cap \mathfrak{p}_{\mathbb{C}}}{[v \cap \mathfrak{k}_{\mathbb{C}}, v \cap \mathfrak{p}_{\mathbb{C}}]}$ and q_l is a Borel subalgebra of l*
4. *$\mathfrak{g}$ and l_0 have the same real rank.*

Proof. From Kostant and Rallis [**K-R**] e lies in the closure of a principal orbit of $L \cap K_{\mathbb{C}}$ on $l \cap \mathfrak{p}_{\mathbb{C}}$. Therefore we can find a nilpotent $e' \in l \cap \mathfrak{p}_{\mathbb{C}}$ such that:

$$e \in \overline{L \cap K_{\mathbb{C}} \cdot e'} \subseteq \overline{K_{\mathbb{C}} \cdot e'}.$$

Thus

$$K_{\mathbb{C}} \cdot e \subseteq K_{\mathbb{C}} \cdot e'.$$

Since e is principal

$$K_{\mathbb{C}} \cdot e = K_{\mathbb{C}} \cdot e'.$$

It follows that e' is $G_{\mathbb{C}}$-conjugate to e. From lemma 5.1.4 and proposition 5.1.7

$$G_{\mathbb{C}} \cdot e \cap l = L \cdot e.$$

Thus e and e' are conjugate under L. But

$$\dim L \cap K_{\mathbb{C}} \cdot e = \dim L \cap K_{\mathbb{C}} \cdot e' = \tfrac{1}{2} \dim L \cdot e.$$

This implies that $L \cap K_{\mathbb{C}} \cdot e$ is a principal orbit in $l \cap \mathfrak{p}_{\mathbb{C}}$.

Moreover, e is noticed in l by definition. Thus by theorem 5.1.1 the real form l_0 of l is quasi-split.

The principal element is always even and lies in a Borel subalgebra. Therefore q_l is a Borel subalgebra of l and

$$\dim m \cap \mathfrak{k}_{\mathbb{C}} = \dim \frac{v \cap \mathfrak{p}_{\mathbb{C}}}{[v \cap \mathfrak{k}_{\mathbb{C}}, v \cap \mathfrak{p}_{\mathbb{C}}]} \quad \text{from theorem 2.1.6.}$$

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ and let $\mathfrak{m}$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. By the Kostant-Sekiguchi correspondence there is a real KS-triple (H, E, F) in $\mathfrak{g}$ such that $l_o = \mathfrak{k}^{\mathfrak{t}}$ where $\mathfrak{t}$ is a maximal toral subalgebra of $\mathfrak{k}^{(H,E,F)}$. Thus we have

$$\mathfrak{k}^{(H,E,F)} \subseteq \mathfrak{k}^{H} = \mathfrak{m}.$$

Hence $\mathfrak{a} \subseteq l_0$. This implies that the real rank of l_0 is equal to the dimension of the subspace $\mathfrak{a}$ which is also the real rank of $\mathfrak{g}$ by definition. The desired result follows.

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