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Introduction

To give the reader the " flavor " of our main result, we first recall some basic results on similarity of complex square matrices. The set of invertible $n \times n$ complex matrices GL_n acts on the set of $n \times n$ complex matrices $\mathfrak{gl}_n$ by conjugation. The orbits are similarity classes of matrices. The theory of the Jordan form allows us to uniquely decompose a matrix A of $\mathfrak{gl}_n$ as a sum of two commuting matrices: a matrix A_s, called the semisimple part of A, and a matrix A_n, called the nilpotent part of A. Furthermore there is an element P of GL_n that conjugates A_s and A_n to a diagonal matrix and a canonical nilpotent matrix respectively. A canonical nilpotent matrix is a sum of blocks having 1's just above their diagonal and 0's elsewhere. Therefore it is enough to investigate two kinds of similarity classes: those represented by diagonal matrices and those represented by nilpotent matrices. It turns out that while the number of similarity classes of diagonal matrices is infinite, that of similarity classes of nilpotent matrices is finite. In fact the nilpotent classes of $\mathfrak{gl}_n$ are parametrized by partitions of n.

Let $\mathbf{d} = [d_1, \ldots d_k]$ be a partition of n. Then such a partition is associated to the

conjugacy class of the matrix:

$$X = \begin{pmatrix} N_1 & 0 & \dots & 0 \\ 0 & N_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & N_k \end{pmatrix},$$

where N_i is a $d_i \times d_i$ matrix of the form:

$$N_i = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

The matrix X is contained in a minimum Levi subalgebra l of $\mathfrak{gl}_n$such that

$$l \cong \mathfrak{gl}_{d_1} \oplus \mathfrak{gl}_{d_2} \cdots \oplus \mathfrak{gl}_{d_k}.$$

And the canonical Borel subalgebra inside of the semisimple algebra $[l, l]$ is

$$b \cong \mathfrak{n}_{d_1} \oplus \mathfrak{n}_{d_2} \cdots \oplus \mathfrak{n}_{d_k},$$

where $\mathfrak{n}_{d_i}$ is a $d_i \times d_i$ upper triangular matrix of trace zero. Let us consider the mapping Φ which associates each such X to the pair (l, b) derived from it. Clearly Φ is a one to one map. Furthermore this map induces a map from nilpotent orbits $GL_n \cdot X$ to GL_n-conjugacy classes of pairs (l, b). This is the basic idea behind the Bala-Carter classification of complex nilpotent orbits of semisimple complex Lie algebras. For example if we consider the partition $[4, 2, 1]$ of 7 then

$$X = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence,

$$l \cong \begin{pmatrix} * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * \end{pmatrix} \quad \text{and} \quad b \cong \begin{pmatrix} * & * & * & * & 0 & 0 & 0 \\ 0 & * & * & * & 0 & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We shall develop a similar theory for reductive real Lie algebras. First, we give a brief description of the Bala-Carter classification.

1. The Bala-Carter classification

Let $\mathfrak{g}_{\mathbb{C}}$ be a semisimple Lie algebra and $G_{\mathbb{C}}$ its adjoint group. We say that an element x of $\mathfrak{g}_{\mathbb{C}}$ is nilpotent if and only if, $ad_x : y \to [x, y]$ for all $y \in \mathfrak{g}_{\mathbb{C}}$, is a nilpotent endomorphism of $\mathfrak{g}_{\mathbb{C}}$. B. Kostant (see also Dynkin [**Dy**]) has shown, in his fundamental 1959 paper [**Ko**], that the number of nilpotent orbits of $G_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$ is finite and bounded by $3^{rank(\mathfrak{g}_{\mathbb{C}})}$ and that there is a one-to-one correspondence between nilpotent orbits of $\mathfrak{g}_{\mathbb{C}}$ and a certain collection of semisimple orbits which, in turn, is in one to one correspondence with certain labeled or weighted Dynkin diagrams. A description of the labels gives a parametrization of the nilpotent orbits.

This is the Dynkin-Kostant classification.

Each node of a weighted Dynkin diagram is labeled with 0,1, or 2. For certain algebras the number of nilpotent orbits is very small in comparison to the actual $3^{rank(\mathfrak{g}_{\mathbb{C}})}$ possible weighted Dynkin diagrams. However the Dynkin-Kostant classification does not provide a simple algorithm for determining whether or not a given weighted diagram represents a nilpotent class. The Bala-Carter correspondence **[B-C1], [B-C2]** gives a conceptually simpler description of the nilpotent orbits. As in the case of the Dynkin-Kostant classification it uses the Jacobson-Morozov theorem relating nilpotent element of $\mathfrak{g}_{\mathbb{C}}$ to 3-dimensional subalgebras in $\mathfrak{g}_{\mathbb{C}}$ that are isomorphic to $\mathfrak{sl}_2(\mathbb{C})$ first determined by Dynkin **[Dy]**. It also uses a theorem of R. Richardson **[R]** to relate nilpotent classes of $\mathfrak{g}_{\mathbb{C}}$ to certain *distinguished* parabolic subalgebras of $\mathfrak{g}_{\mathbb{C}}$. For example Borel subalgebras are distinguished. More will be said about this later in chapters 2 and 3 **[Lemma 2.1.3]**.

The Bala-Carter classification can be expressed as follows:

There is a one-to-one correspondence between nilpotent orbits of $\mathfrak{g}_{\mathbb{C}}$ and conjugacy classes of pairs (m, p_m), where m is a Levi subalgebra of $\mathfrak{g}_{\mathbb{C}}$ and p_m is a distinguished parabolic subalgebra of the semisimple algebra $[m, m]$. In this correspondence, the nilpotent $\mathfrak{g}_{\mathbb{C}}$ orbit comes from the Richardson orbit of P_m, the connected subgroup of $G_{\mathbb{C}}$ with Lie algebra p_m, on the nilradical of p_m.

We also mention that in the case of the classical Lie algebras the classes of *distinguished* parabolic subalgebras can be parametrized by certain partitions.

Finally, we note that the Bala-Carter theory is also valid in characteristic $p > 3(h-1)$ where h is the Coxeter number of $G_{\mathbb{C}}$.

2. A concise outline of the work

This work shows how a theory similar to the Bala-Carter classification can be used to parametrize nilpotent orbits of a semisimple real Lie algebra under the action of its adjoint group. Complex semisimple Lie algebras regarded as real do not give rise to any new nilpotent orbits. Their compact real forms contain no non-zero nilpotent. Therefore we can limit our analysis to their non-compact real forms. Such simple real Lie algebras were classified by Cartan and can be found in Helgason [**He**]. Note that a complex conjugacy class of a real nilpotent can split under the action of the real adjoint group. For example the two matrices

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

are conjugate under the complex group $PSL_2(\mathbb{C})$ but not under the real group $PSL_2(\mathbb{R})$.

A real reductive Lie algebra $\mathfrak{g}$ has a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for a Cartan involution θ. Hence, by complexification we obtain $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}}$. Denote by σ the conjugation of $\mathfrak{g}_{\mathbb{C}}$ with regard to $\mathfrak{g}$. Let H, E , F be elements of $\mathfrak{g}$ such that $[H, E] = 2E$, $[H, F] = -2F$, $[E, F] = H$. We call (H, E, F) a standard triple in $\mathfrak{g}$. Let G be the adjoint group of $\mathfrak{g}$. Let $K_{\mathbb{C}}$ be the complexification of the connected subgroup K of G with Lie algebra $\mathfrak{k}$. Sekiguchi [**Se**] proves that there is a one to one correspondence between the G-orbits in $\mathfrak{g}$ and the $K_{\mathbb{C}}$-orbits in $\mathfrak{p}_{\mathbb{C}}$. We call this correspondence the Kostant-Sekiguchi correspondence since it was first conjectured by Kostant. This allows us to exploit the rich theory of symmetric spaces [**K-R**].

In chapter 1, we explain the nature of the Kostant-Sekiguchi correspondence. Given a nilpotent element e in $\mathfrak{p}_{\mathbb{C}}$, we give a method for constructing a minimal θ-stable

Levi subalgebra containing e **Proposition [1.1.3]**. A KS-triple (x, e, f) in $\mathfrak{g}_{\mathbb{C}}$, that is a normal triple in Kostant-Rallis' sense with the additional property that $\sigma(e) = f$, is associated to a θ-stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}_{\mathbb{C}}$. Several important Richardson-type theorems are also proved. **[Theorems 1.2.1, 1.2.3, 1.2.6]**. For example , let Q be the connected subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{q}$. If $\mathfrak{q} = l \oplus u$ is the Levi decomposition of $\mathfrak{q}$ and if e is even we have $\overline{Q \cap K_{\mathbb{C}} \cdot e} = u \cap \mathfrak{p}_{\mathbb{C}}$. This explains the fact that our theory is so close to that of Bala and Carter for even orbits. Define L to be the connected subgroup of $G_{\mathbb{C}}$ with Lie algebra l.

In chapter 2 we introduce the notion of *noticed* nilpotent element. Such an element e of $\mathfrak{p}_{\mathbb{C}}$ is characterized by the fact that the reductive centralizer $\mathfrak{k}_{\mathbb{C}}^{(x,e,f)}$ is trivial **[Theorem 2.1.1]**. In fact a nilpotent is always noticed in the minimal (θ, σ)-stable Levi subalgebra that contains it. Furthermore e is even and noticed if and only if $\mathfrak{q} = l \oplus u$ and $\dim\, l \cap \mathfrak{k}_{\mathbb{C}} = \dim\, \frac{u \cap \mathfrak{p}_{\mathbb{C}}}{[u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]}$ **[Theorem 2.1.6]**. This dimension criteria is very similar to the one that Bala and Carter give for their distinguished parabolic subalgebras. In the last section we show that there are, in fact, non even noticed elements whose associated θ-stable parabolic subalgebra does not satisfy the above dimension condition , by giving an example (see **section 2.2**) in $\mathfrak{sl}_7(\mathbb{C})$ which uses the partition $[4, 2, 1]$ discussed above. This implies that our classification is different from the Bala-Carter classification, where all the distinguished nilpotent elements are even. A distinguished element in $\mathfrak{p}_{\mathbb{C}}$ is noticed but not vice versa.

The main classification theorem is proved is chapter 3. We prove that the orbits $K_{\mathbb{C}} \cdot e$ are in one-to-one correspondence with the triples of the form $(l, q_l, \mathfrak{w})$, where e is a non zero nilpotent in $\mathfrak{p}_{\mathbb{C}}$, l is a minimal (θ, σ)-stable Levi subalgebra of $\mathfrak{g}_{\mathbb{C}}$ containing e, q_l is a θ stable parabolic subalgebra of $[l, l]$ and $\mathfrak{w}$ is a certain $L \cap K_{\mathbb{C}}$ prehomogeneous subspace of $q_l \cap \mathfrak{p}_{\mathbb{C}}$ containing e **[Theorem 3.2.4]**. We note that

Kawanaka has obtained related results. [**Ka**]

In chapter 4 we give a description of the noticed orbits of the classical simple real Lie algebras in terms of signed Young diagrams. Among other results we show that the non-zero noticed nilpotent orbits of $\mathfrak{sl}_n(\mathbb{R})$ are parametrized by partitions of n with distinct parts. The noticed orbits of most of the real simple algebras are even. The classical algebras $\mathfrak{so}^*_{2n}$, $\mathfrak{su}^*_{2n}$, $\mathfrak{sp}(p,q)$ have no non-zero noticed nilpotent orbits [**Theorem 4.2.1**]. An exceptional simple real Lie algebra contains a non-zero noticed element if and only if it is quasi-split [**Proposition 4.1.0**].

In the last chapter we analyze the behaviour of a principal nilpotent element of $\mathfrak{p}_{\mathbb{C}}$, i.e. an element whose $K_{\mathbb{C}}$-orbit has maximal dimension among all $K_{\mathbb{C}}$-nilpotent orbits, under the classification. The main result is that a principal nilpotent element e is regular in the minimal (θ,σ)-stable Levi subalgebra l of $\mathfrak{g}_{\mathbb{C}}$ containing it and the real form l_0 of l is quasi-split [**Theorem 5.1.8**].