Nilpotent Orbits and Θ-Stable Parabolic Subalgebras

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by

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ABSTRACT OF DISSERTATION

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ABSTRACT

In this work, we present a new classification of nilpotent orbits in a real reductive Lie algebra $\mathfrak{g}$ under the action of its adjoint group. The notion of a θ-stable parabolic subalgebra, which is also important in representation theory, (e.g Zuckerman and Vogan's work on Cohomological Induction) is at the basis of our classification which generalizes the Bala-Carter classification of the nilpotent orbits of complex semisimple Lie algebras. Our theory takes full advantage of the work of Kostant and Rallis on $\mathfrak{p}_{\mathbb{C}}$, the "complex symmetric space associated with $\mathfrak{g}$". The Kostant-Sekiguchi correspondence, a bijection between nilpotent orbits in $\mathfrak{g}$ and nilpotent orbits in $\mathfrak{p}_{\mathbb{C}}$, is also used. We identify a fundamental set of *noticed* nilpotents in $\mathfrak{p}_{\mathbb{C}}$ and show that they allow us to recover all other nilpotents. Finally, we study the behaviour of a principal orbit, that is an orbit of maximal dimension, under our classification. This is not done in the other classification schemes currently available in the literature. Some of the results from this work were announced in the proceedings of the DIMACS workshop for African-American Researchers in the Mathematical Sciences. [**No**]

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