Summations

What does $\sum_{j=m}^{n} a_j$ stand for ?

It represents the sum $a_m + a_{m+1} + a_{m+2} + \cdots + a_n$.

The variable j is called the index of summation, running from its **lower limit** m to its upper limit n. We could as well have used any other letter to denote this index.

Summations

How can we express the sum of the first 1000 terms of the sequence $\{a_n\}$ with $a_n=n^2$ for n = 1, 2, 3, … ?

We write as $\sum_{j=1}^{100} a_j$

What is the value of $\sum_{j=1}^{6} a_j$

It is 1 + 2 + 3 + 4 + 5 + 6 = 21.

What is the value of $\sum_{j=1}^{100} j$

It is 1 + 2 + 3 + 4 + … + 100 = much of work to calculate this…

Summations

It is said that Carl Friedrich Gauss came up with the following formula:

$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$$

When you have such a formula, the result of any summation can be calculated much more easily, for example:

$$\sum_{j=1}^{100} j = \frac{100(100+1)}{2} = \frac{10100}{2} = 5050$$

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Double Summations

Corresponding to nested loops in C or Java, there is also double (or triple etc.) summation:

Example: $$\sum_{i=1}^{5}\sum_{j=1}^{2} ij$$

$$= \sum_{i=1}^{5} (i + 2i)$$

$$= \sum_{i=1}^{5} 3i$$

$$= 3 + 6 + 9 + 12 + 15 = 45$$

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Matrices

A **matrix** is a rectangular array of numbers.

A matrix with m rows and n columns is called an **m×n matrix**

Example:

$$A = \begin{bmatrix} -1 & 1 \\ 2.5 & -0.3 \\ 8 & 0 \end{bmatrix}$$ is a 3×2 matrix.

A matrix with the same number of rows and columns is called **square**.

Two matrices are **equal** if they have the same number of rows and columns and the corresponding entries in every position are equal.

Matrices

A general description of an m×n matrix A = $[a_{ij}]$:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \qquad \begin{bmatrix} a_{1j} \\ a_{2j} \\ . \\ . \\ . \\ a_{mj} \end{bmatrix}$$ j-th column of A

$$[a_{i1}, \quad a_{i2}, \quad \dots, \quad a_{in}]$$

i-th row of A

Matrix Addition

Let A = [a_{ij}] and B = [b_{ij}] be m×n matrices.
The sum of A and B, denoted by A+B, is the m×n matrix that has a_{ij} + b_{ij} as its (i, j)th element.

In other words, A+B = [a_{ij} + b_{ij}].

Example:

$$\begin{bmatrix} -2 & 1 \\ 4 & 8 \\ -3 & 0 \end{bmatrix} + \begin{bmatrix} 5 & 9 \\ -3 & 6 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 10 \\ 1 & 14 \\ -7 & 1 \end{bmatrix}$$

Matrix Multiplication

Let A be an m×k matrix and B be a k×n matrix.

The product of A and B, denoted by AB, is the m×n
matrix with (i, j)th entry equal to the sum of the products of the corresponding elements from the i-th row of A and the j-th column of B.

In other words, if AB = [c_{ij}] then

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{ik}b_{kj} = \sum_{t=1}^{k} a_{it}\, b_{tj}$$

Matrix Multiplication

A more intuitive description of calculating C = AB:

$$A = \begin{bmatrix} \boxed{3 \quad 0 \quad 1} \\ -2 \quad -1 \quad 4 \\ 0 \quad 0 \quad 5 \\ -1 \quad 1 \quad 0 \end{bmatrix} \qquad B = \begin{bmatrix} \boxed{\begin{matrix} 2 \\ 0 \\ 3 \end{matrix}} & \begin{matrix} 1 \\ -1 \\ 4 \end{matrix} \end{bmatrix}$$

- Take the first column of B
- Turn it counterclockwise by 90^o and superimpose it on the first row of A
- Multiply corresponding entries in A and B and add the products: 3*2 + 0*0 + 1*3 = 9
- Enter the result in the upper-left corner of C

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Matrix Multiplication

Now superimpose the first column of B on the second, third, …, m-th row of A to obtain the entries in the first column of C (same order).

Then repeat this procedure with the second, third, …, n-th column of B, to obtain to obtain the remaining columns in C (same order).

After completing this algorithm, the new matrix C contains the product AB.

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Matrix Multiplication

Let us calculate the complete matrix C:

$$A = \begin{bmatrix} 3 & 0 & 1 \\ -2 & -1 & 4 \\ 0 & 0 & 5 \\ -1 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \\ 0 & -1 \\ 3 & 4 \end{bmatrix}$$

$$C = \begin{bmatrix} 9 & 7 \\ 8 & 15 \\ 15 & 20 \\ -2 & -2 \end{bmatrix}$$

Identity Matrices

The identity matrix of order n is the n×n matrix $I_n = [\delta_{ij}]$, where δ_{ij} = 1 if i = j and δ_{ij} = 0 if i $\neq$ j:

$$A = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

Multiplying an m×n matrix A by an identity matrix of appropriate size does not change this matrix:

$$AI_n = I_mA = A$$

Powers and Transposes of Matrices

The power function can be defined for square matrices. If A is an n×n matrix, we have:

$$A^0 = I_n,$$
$$A^r = AAA \dots A \ (r \ times \ the \ letter \ A)$$

The transpose of an m×n matrix $A = [a_{ij}]$, denoted by A^t, is the n×m matrix obtained by interchanging the rows and columns of A.

In other words, if $A^t = [b_{ij}]$, then $b_{ij} = a_{ji}$ for i = 1, 2, …, n and j = 1, 2, …, m.

Powers and Transposes of Matrices

Example:

$$A = \begin{bmatrix} 2 & 1 \\ 0 & -1 \\ 3 & 4 \end{bmatrix} \qquad A^t = \begin{bmatrix} 2 & 0 & 3 \\ 1 & -1 & 4 \end{bmatrix}$$

A square matrix A is called **symmetric** if A = A^t. Thus A = $[a_{ij}]$ is symmetric if $a_{ij} = a_{ji}$ for all i = 1, 2, …, n and j = 1, 2, …, n.

$$A = \begin{bmatrix} 5 & 1 & 3 \\ 1 & 2 & -9 \\ 3 & -9 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix}$$

A is symmetric, B is not.

Zero-One Matrices

A matrix with entries that are either 0 or 1 is called **a zero-one matrix**. Zero-one matrices are often used like a "table" to represent discrete structures.

We can define Boolean operations on the entries in zero-one matrices:

a	b	$a \wedge b$
0	0	0
0	1	0
1	0	0
1	1	1

a	b	$a \vee b$
0	0	0
0	1	1
1	0	1
1	1	1

Zero-One Matrices

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $m \times n$ zero-one matrices.

Then the **join** of A and B is the zero-one matrix with (i, j)th entry $a_{ij} \vee b_{ij}$. The join of A and B is denoted by $A \vee B$.

The **meet** of A and B is the zero-one matrix with (i, j)th entry $a_{ij} \wedge b_{ij}$. The meet of A and B is denoted by $A \wedge B$.

Zero-One Matrices

Example:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Join: $$A \vee B = \begin{bmatrix} 1 \vee 0 & 1 \vee 1 \\ 0 \vee 1 & 1 \vee 1 \\ 1 \vee 0 & 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Meet: $$A \wedge B = \begin{bmatrix} 1 \wedge 0 & 1 \wedge 1 \\ 0 \wedge 1 & 1 \wedge 1 \\ 1 \wedge 0 & 0 \wedge 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Zero-One Matrices

Let A = $[a_{ij}]$ be an m×k zero-one matrix and B = $[b_{ij}]$ be a k×n zero-one matrix.

Then the **Boolean product** of A and B, denoted by A·B, is the m×n matrix with (i, j)th entry $[c_{ij}]$, where

$$c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2i}) \vee \ldots \vee (a_{ik} \wedge b_{kj}).$$

Note that the actual Boolean product symbol has a dot in its center.

Basically, Boolean multiplication works like the multiplication of matrices, but with computing ∧ instead of the product and ∨ instead of the sum.

Zero-One Matrices

Example:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$A \circ B = \begin{bmatrix} (1 \wedge 0) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) \\ (1 \wedge 0) \vee (1 \wedge 0) & (1 \wedge 1) \vee (1 \wedge 1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Zero-One Matrices

Let A be a square zero-one matrix and r be a positive integer.

The **r-th Boolean power** of A is the Boolean product of r factors of A. The r-th Boolean power of A is denoted by $A^{[r]}$.

$A^{[0]} = I_n$,
$A^{[r]} = A \circ A \circ \cdots \circ A$ (r times the letter A)

In-Class Exercise

Find a matrix M such that

$$\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} M + \begin{bmatrix} -6 & 3 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 4 & 27 \\ 6 & 15 \end{bmatrix}$$

In-Class Exercise

Let $M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$. Then we get the following equations:

(1) $2m_{11} + 4m_{21} + (-6) = 4 \Rightarrow 2m_{11} + 4m_{21} = 10$
(2) $m_{11} + 3m_{21} + 0 = 6 \Rightarrow m_{11} + 3m_{21} = 6$
(3) $2m_{12} + 4m_{22} + 3 = 27 \Rightarrow 2m_{12} + 4m_{22} = 24$
(4) $m_{12} + 3m_{22} + (-2) = 15 \Rightarrow m_{12} + 3m_{22} = 17$

From (2) we get:

(5) $m_{11} = 6 - 3m_{21}$

If we use this to substitute m_{11} in (1) we get:

$12 - 6m_{21} + 4m_{21} = 10 \Rightarrow -2m_{21} = -2 \Rightarrow m_{21} = 1$

From (5) it follows:

$m_{11} = 6 - 3 \cdot 1 = 3$

In-Class Exercise

From (4) we get:

(6) $m_{12} = 17 - 3m_{22}$

If we use this to substitute m_{12} in (3) we get:

$34 - 6m_{22} + 4m_{22} = 24 \Rightarrow -2m_{22} = -10 \Rightarrow m_{22} = 5$

From (6) it follows:

$m_{12} = 17 - 3 \cdot 5 = 2$

Therefore, the solution is $M = \begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}$.

Relations

Chapter 9 in the textbook

Relations

If we want to describe a relationship between elements of two sets A and B, we can use **ordered pairs** with their first element taken from A and their second element taken from B.

Since this is a relation between **two sets**, it is called a **binary relation**.

Definition: Let A and B be sets. A binary relation from A to B is a subset of A×B.

In other words, for a binary relation R we have $R \subseteq A \times B$.
We use the notation aRb to denote that $(a, b) \in R$ and $a \not R b$ to denote that $(a, b) \notin R$.

Relations

When (a, b) belongs to R, a is said to be **related** to b by R.

Example: Let P be a set of people, C be a set of cars, and D be the relation describing which person drives which car(s).

P = {Carl, Suzanne, Peter, Carla},

C = {Mercedes, BMW, tricycle}

D = {(Carl, Mercedes), (Suzanne, Mercedes),
(Suzanne, BMW), (Peter, tricycle)}

This means that Carl drives a Mercedes, Suzanne drives a Mercedes and a BMW, Peter drives a tricycle, and Carla does not drive any of these vehicles.

Functions as Relations

You might remember that a **function** f from a set A to a set B assigns a unique element of B to each element of A.

The **graph** of f is the set of ordered pairs (a, b) such that $b = f(a)$.

Since the graph of f is a subset of $A \times B$, it is a **relation** from A to B.

Moreover, for each element a of A, there is exactly one ordered pair in the graph that has a as its first element.

Functions as Relations

Conversely, if R is a relation from A to B such that every element in A is the first element of exactly one ordered pair of R, then a function can be defined with R as its graph.

This is done by assigning to an element $a \in A$ the unique element $b \in B$ such that $(a, b) \in R$.

Relations on a Set

Definition: A relation on the set A is a relation from A to A.

In other words, a relation on the set A is a subset of $A \times A$.

Example: Let $A = \{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) \mid a < b\}$?

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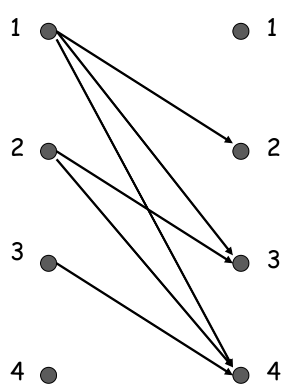
Relations on a Set

Solution: R = { (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)}



R	1	2	3	4
1		×	×	×
2			×	×
3				×
4				

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Relations on a Set

How many different relations can we define on a set A with n elements?

- A relation on a set A is a subset of $A \times A$.
- How many elements are in $A \times A$?
- There are n^2 elements in $A \times A$, so how many subsets (=relations on A) does $A \times A$ have?
- The number of subsets that we can form out of a set with m elements is 2^m. Therefore, 2^{n^2} subsets can be formed out of $A \times A$.

Answer: We can define 2^{n^2} different relations on A.

Properties of Relations

We will now look at some useful ways to classify relations.

Definition: A relation R on a set A is called **reflexive** if $(a, a) \in R$ for every element $a \in A$.

Are the following relations on {1, 2, 3, 4} reflexive?

R = {(1, 1), (1, 2), (2, 3), (3, 3), (4, 4)}	No.
R = {(1, 1), (2, 2), (2, 3), (3, 3), (4, 4)}	Yes.
R = {(1, 1), (2, 2), (3, 3)}	No.

Definition: A relation on a set A is called **irreflexive** if $(a, a) \notin R$ for every element $a \in A$.

Properties of Relations

Definitions:

- A relation R on a set A is called **symmetric** if $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$.
- A relation R on a set A is called **antisymmetric** if $a = b$ whenever $(a, b) \in R$ and $(b, a) \in R$.
- A relation R on a set A is called **asymmetric** if $(a, b) \in R$ implies that $(b, a) \notin R$ for all $a, b \in A$.

Properties of Relations

Definition: A relation R on a set A is called **transitive** if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ for all $a, b, c \in A$.

Are the following relations on $A = \{1, 2, 3, 4\}$ transitive?

R = {(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)}	Yes.
R = {(1, 3), (3, 2), (2, 1)}	No.
R = {(2, 4), (4, 3), (2, 3), (4, 1)}	No.
R = {(1, 1), (1, 2), (2, 2), (2, 1), (3,3)}	Yes.

Counting Relations

Example: How many different reflexive relations can be defined on a set A containing n elements?

Solution:

- Relations on R are subsets of $A \times A$, which contains n^2 elements.
- Therefore, different relations on A can be generated by choosing different subsets out of these n^2 elements, so there are 2^{n^2} relations.
- A **reflexive** relation, however, **must** contain the n elements (a, a) for every $a \in A$.
- Consequently, we can only choose among $n^2 - n = n(n-1)$ elements to generate reflexive relations, so there are $2^{n(n-1)}$ of them.

Combining Relations

Relations are sets, and therefore, we can apply the usual **set operations** to them.

If we have two relations R_1 and R_2, and both of them are from a set A to a set B, then we can combine them to $R_1 \cup R_2$, $R_1 \cap R_2$, or $R_1 - R_2$.

In each case, the result will be **another relation from A to B**.

Combining Relations

… and there is another important way to combine relations.

Definition: Let R be a relation from a set A to a set B and S a relation from B to a set C. The **composite** of R and S is the relation consisting of ordered pairs (x, z), where $x \in A$, $z \in C$, and for which there exists an element $y \in B$ such that $(x, y) \in R$ and $(y, z) \in S$. We denote the composite of R and S by $S \circ R$.

In other words, if relation R contains a pair (x, y) and relation S contains a pair (y, z), then $S \circ R$ contains a pair (x, z).

Combining Relations

Example: Let D and S be relations on $A = \{1, 2, 3, 4\}$.

$D = \{(a, b) \mid b = 5 - a\}$ “b equals (5 – a)”

$S = \{(a, b) \mid a < b\}$ “a is smaller than b”

$D = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$

$S = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$

S∘D = {(2, 4), (3, 3), (3, 4), (4, 2), (4, 3), (4, 4)}

D maps an element a to the element $(5 - a)$, and afterwards S maps $(5 - a)$ to all elements larger than (5 – a), resulting in :

$S \circ D = \{(a, b) \mid b > 5 - a\}$ or $S \circ D = \{(a, b) \mid a + b > 5\}$.

Combining Relations

We already know that functions are just special cases of relations (namely those that map each element in the domain onto exactly one element in the codomain).

If we formally convert two functions into relations, that is, write them down as sets of ordered pairs, the composite of these relations will be exactly the same as the composite of the functions (as defined earlier).

Combining Relations

Definition:

Let R be a relation on the set A. The powers R^n, $n = 1, 2, \ldots$, are defined inductively by:

$$R^1 = R$$

$$R^{n+1} = R^{\boldsymbol{n}} \circ R$$

In other words:

$$R^n = R \circ R \circ \ldots \circ R \text{ (n times the letter } R\text{)}$$

Combining Relations

Theorem: The relation R on a set A is **transitive** if and only if $R^n \subseteq R$ for all positive integers n.

<u>Proof:</u>

We know that a relation R on a set A is transitive if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ for $a, b, c \in A$. The composite of R with itself with contains these pairs (a, c).

Therefore, for a transitive relation R, the $R \circ R$ does not contain any pairs that are not in R, so $R \circ R \subseteq R$.

Since $R \circ R$ does not introduce any pairs that are not already in R, it must also be true that $(R \circ R) \circ R \subseteq R$, and so on, so that $R^{\boldsymbol{n}} \subseteq R$

Representing Relations

We already know different ways of representing relations. We will now take a closer look at two ways of representation: Zero-one matrices and directed graphs.

If R is a relation from $A = \{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$, then R can be represented by the zero-one matrix $M_R = [m_{ij}]$ with

- $m_{ij} = 1$, if $(a_i, b_j) \in R$, and
- $m_{ij} = 0$, if $(a_i, b_j) \notin R$.

Note that for creating this matrix we first need to list the elements in A and B in a particular, but arbitrary order.

Representing Relations

Example: How can we represent the relation R from the set $A = \{1, 2, 3\}$ to the set $B = \{1, 2\}$ with $R = \{(2, 1), (3, 1), (3, 2)\}$ as a zero-one matrix?

Solution: The matrix M_R is given by

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Representing Relations

What do we know about the matrices representing a **relation on a set** (a relation from A to A) ?

They are **square** matrices.

What do we know about matrices representing **reflexive** relations?

All the elements on the **diagonal** of such matrices M_{ref} must be **1s**.

$$M_{ref} = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & 1 \end{bmatrix}$$

Representing Relations

What do we know about the matrices representing **symmetric relations**?

These matrices are symmetric, that is, $M_R = (M_R)^t$.

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

symmetric matrix,
symmetric relation.

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

non-symmetric matrix,
non-symmetric relation.

Representing Relations

The Boolean operations **join** and **meet** (you remember?) can be used to determine the matrices representing the **union** and the **intersection** of two relations, respectively.

To obtain the **join** of two zero-one matrices, we apply the Boolean “or” function to all corresponding elements in the matrices.

To obtain the **meet** of two zero-one matrices, we apply the Boolean “and” function to all corresponding elements in the matrices.

Representing Relations

Example: Let the relations R and S be represented by the matrices

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad M_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

What are the matrices representing $R \cup S$ and $R \cap S$?

Solution: These matrices are given by

$$M_{R \cup S} = M_R \vee M_S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad M_{R \cap S} = M_R \wedge M_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Representing Relations Using Matrices

Do you remember the **Boolean product** of two zero-one matrices?

Let $A = [a_{ij}]$ be an $m \times k$ zero-one matrix and $B = [b_{ij}]$ be a $k \times n$ zero-one matrix.

Then the **Boolean product** of A and B, denoted by $A \circ B$, is the $m \times n$ matrix with (i, j)th entry $[c_{ij}]$, where:

$$c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \cdots \vee (a_{ik} \wedge b_{kj}).$$

$c_{ij} = 1$ if and only if at least one of the terms $(a_{in} \wedge b_{nj}) = 1$ for some n; otherwise $c_{ij} = 0$.

Representing Relations Using Matrices

Let us now assume that the zero-one matrices
$M_A = [a_{ij}]$, $M_B = [b_{ij}]$ and $M_C = [cij]$ represent relations A, B, and C, respectively.

Remember: For $M_C = M_A \circ M_B$ we have:
$c_{ij} = 1$ if and only if at least one of the terms $(a_{in} \wedge b_{nj}) = 1$ for some n;
otherwise $c_{ij} = 0$.

In terms of the **relations**, this means that C contains a pair (x_i, z_j) if and only if there is an element y_n such that (x_i, y_n) is in relation A and (y_n, z_j) is in relation B.

Therefore, $C = B \circ A$ (**composite** of A and B).

Representing Relations Using Matrices

This gives us the following rule:

$$M_{B \circ A} = M_A \circ M_B$$

In other words, the matrix representing the **composite** of relations A and B is the **Boolean product** of the matrices representing A and B.

Analogously, we can find matrices representing the **powers of relations**:

$M_{R^n} = M_R^{[n]}$ (n-th **Boolean power**).

Representing Relations Using Matrices

Example: Find the matrix representing R^2, where the matrix representing R is given by

$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Solution: The matrix for R^2 is given by

$$M_{R^2} = M_R^{[2]} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Representing Relations Using Digraphs

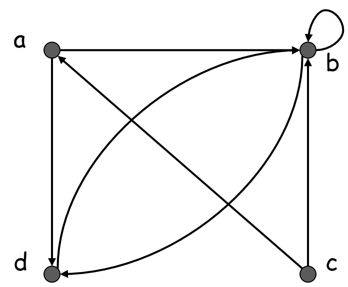
Definition: A **directed graph**, or **digraph**, consists of a set V of **vertices** (or **nodes**) together with a set E of ordered pairs of elements of V called **edges** (or **arcs**).

The vertex a is called the **initial vertex** of the edge (a, b), and the vertex b is called the **terminal vertex** of this edge.

We can use arrows to display graphs.

Representing Relations Using Digraphs

Example: Display the digraph with $V = \{a, b, c, d\}$,
$E = \{(a, b), (a, d), (b, b), (b, d), (c, a), (c, b), (d, b)\}$.



An edge of the form (b, b) is called a **loop.**

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Representing Relations Using Digraphs

Obviously, we can represent any relation R on a set A by the digraph with A as its vertices and all pairs $(a, b) \in R$ as its edges.

Vice versa, any digraph with vertices V and edges E can be represented by a relation on V containing all the pairs in E.

This **one-to-one correspondence** between relations and digraphs means that any statement about relations also applies to digraphs, and vice versa.

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Closures of Relations

What is the **closure** of a relation?

Definition: Let R be a relation on a set A. R may or may not have some **property P**, such as reflexivity, symmetry, or transitivity.

If there is a relation S that contains R and has property P, and S is a subset of **every** relation that contains R and has property P, then S is called the **closure** of R with respect to P.

Note that the closure of a relation with respect to a property may not exist.

Closures of Relations

Example 1: Find the **reflexive closure** of relation $R = \{(1,1),(1,2),(2,1),(3,2)\}$ on the set $A = \{1,2,3\}$.

Solution: We know that any reflexive relation on A must contain the elements (1, 1), (2, 2), and (3, 3).

By adding (2, 2) and (3, 3) to R, we obtain the reflexive relation S, which is given by

$$S = \{(1,1),(1,2),(2,1),(2,2),(3,2),(3,3)\}.$$

S is reflexive, contains R, and is contained within every reflexive relation that contains R.

Therefore, S is the **reflexive closure** of R.

Closures of Relations

Example 2: Find the **symmetric closure** of the relation $R = \{(a, b) \mid a > b\}$ on the set of positive integers.

Solution: The symmetric closure of R is given by

$$R \cup R^{-1} = \{(a, b) \mid a > b\} \cup \{(b, a) \mid a > b\} = \{(a, b) \mid a \neq b\}$$

Closures of Relations

Example 3: Find the **transitive closure** of the relation $R = \{(1, 3), (1, 4), (2, 1), (3, 2)\}$ on the set $A = \{1, 2, 3, 4\}$.

Solution: R would be transitive, if for all pairs (a, b) and (b, c) in R there were also a pair (a, c) in R.

If we add the missing pairs $(1, 2), (2, 3), (2, 4)$, and $(3, 1)$, will R be transitive?

No, because the extended relation R contains $(3, 1)$ and $(1, 4)$, but does not contain $(3, 4)$.

By adding new elements to R, we also add **new requirements** for its transitivity. We need to look at **paths in digraphs** to solve this problem.

Closures of Relations

Imagine that we have a relation R that represents all **train connections** in the US.

For example, if (Boston, Philadelphia) is in R, then there is a **direct** train connection from Boston to Philadelphia.

If R contains (Boston, Philadelphia) and (Philadelphia, Washington), there is an **indirect** connection from Boston to Washington.

Because there are indirect connections, it is not possible by just looking at R to determine which cities are connected by trains.

The transitive closure of R contains exactly those pairs of cities that are connected, **either directly or indirectly**.

Closures of Relations

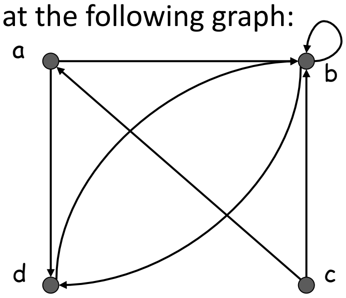
Definition: A **path** from a to b in the directed graph G is a sequence of one or more edges $(x_0, x_1), (x_1, x_2), (x_2, x_3), \dots, (x_{n-1}, x_n)$ in G, where $x_0 = a$ and $x_n = b$.

In other words, a path is a **sequence of edges** where the terminal vertex of an edge is the same as the initial vertex of the next edge in the path.

This path is denoted by $x_0, x_1, x_2, \dots, x_n$ and has **length** n.
A path that begins and ends at the same vertex is called a **circuit** or **cycle**.

Closures of Relations

Example: Let us take a look at the following graph:



Is **c,a,b,d,b** a path in this graph? Yes.

Is **d,b,b,b,d,b,d** a circuit in this graph? Yes.

Is there any circuit including **c** in this graph? No.

Closures of Relations

Due to the one-to-one correspondence between graphs and relations, we can transfer the definition of path from graphs to relations:

Definition: There is a path from a to b in a relation R, if there is a sequence of elements $a, x_1, x_2, \ldots, x_{n-1}, b$ with $(a, x_1) \in R$, $(x_1, x_2) \in R$, … , and $(x_{n-1}, b) \in R$.

Theorem: Let R be a relation on a set A. There is a path from a to b if and only if $(a, b) \in R^n$ for some positive integer n.

Closures of Relations

According to the train example, the transitive closure of a relation consists of the pairs of vertices in the associated directed graph that are connected by a path.

Definition: Let R be a relation on a set A. The connectivity relation R^* consists of the pairs (a, b) such that there is a path between a and b in R.

We know that R^n consists of the pairs (a, b) such that a and b are connected by a path of length n.

Therefore, R^* is the union of R^n across all positive integers n:

$$R^* = \bigcup_{n=1}^{\infty} R^n = R^1 \cup R^2 \cup R^3 \cup \ldots$$

Closures of Relations

Theorem: The transitive closure of a relation R equals the connectivity relation R^*.

But how can we compute R^* ?

Lemma: Let A be a set with n elements, and let R be a relation on A. If there is a path in R from a to b, then there is such a path with length not exceeding n.

Moreover, if $a \neq b$ and there is a path in R from a to b, then there is such a path with length not exceeding $(n - 1)$.

Closures of Relations

This lemma is based on the observation that if a path from a to b visits any vertex more than once, it must include at least one **circuit**.

These circuits can be **eliminated** from the path, and the reduced path will still connect a and b.

Theorem: For a relation R on a set A with n elements, the transitive closure R^* is given by:

$$R^* = R \cup R^2 \cup R^3 \cup \ldots \cup R^n$$

For matrices representing relations we have:

$$M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]} \vee \ldots \vee M_R^{[n]}$$

Closures of Relations

Let us finally solve **Example 3** by finding the **transitive closure** of the relation $R = \{(1,3), (1,4), (2,1), (3,2)\}$ on the set $A = \{1, 2, 3, 4\}$.

R can be represented by the following matrix M_R:

$$M_R = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Closures of Relations

$$M_R = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad M_R^{[2]} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_R^{[3]} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad M_R^{[4]} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]} \vee M_R^{[4]} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Closures of Relations

$$M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]} \vee M_R^{[4]} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution:

The transitive closure of the relation $R = \{(1,3), (1,4), (2,1), (3,2)\}$ on the set $A = \{1,2,3,4\}$ is given by the relation:

$\{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4)\}$

Partial Orderings

Sometimes, relations define an **order** on the elements in a set.

Definition: A relation R on a set S is called a **partial ordering** or **partial order** if it is reflexive, antisymmetric, and transitive.

A set S together with a partial ordering R is called a **partially ordered set**, or **poset**, and is denoted by (S, R).

Partial Orderings

Example: Consider the "greater than or equal" relation ≥ (defined by $\{(a, b) \mid a \geq b\}$).

Is ≥ a **partial ordering** on the set of integers?

- ≥ is **reflexive**, because a ≥ a for every integer a.
- ≥ is **antisymmetric**, because if a ≠ b, then a ≥ b ∧ b ≥ a is false.
- ≥ is **transitive**, because if a ≥ b and b ≥ c, then a ≥ c.

Consequently, $(Z, \geq)$ is a partially ordered set.

Partial Orderings

Another example: Is the "inclusion relation" $\subseteq$ a **partial ordering** on the power set of a set S?

$\subseteq$ is **reflexive**, because $A \subseteq A$ for every set A.

$\subseteq$ is **antisymmetric**, because if $A \neq B$, then $A \subseteq B \wedge B \subseteq A$ is false.

$\subseteq$ is **transitive**, because if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Consequently, $(P(S), \subseteq)$ is a partially ordered set.

Partial Orderings

In a poset the notation $a \leq b$ denotes that $(a, b) \in R$.

Note that the symbol $\leq$ is used to denote the relation in **any** poset, not just the "less than or equal" relation.

The notation $a < b$ denotes that $a \leq b$, but $a \neq b$.

If $a < b$ we say "a is less than b" or "b is greater than a".

Partial Orderings

For two elements a and b of a poset $(S, \leq)$ it is possible that neither $a \leq b$ nor $b \leq a$.

Example: In $(P(Z), \subseteq)$, $\{1, 2\}$ is not related to $\{1, 3\}$, and vice versa, since neither is contained within the other.

Definition: The elements a and b of a $poset$ $(S, \leq)$ are called **comparable** if either $a \leq b$ or $b \leq a$. When a and b are elements of S such that neither $a \leq b$ nor $b \leq a$, then a and b are called **incomparable**.

Partial Orderings

For some applications, we require **all** elements of a set to be comparable.

For example, if we want to write a dictionary, we need to define an order on **all** English words (alphabetic order).

Definition: If $(S, \leq)$ is a poset and **every two elements** of S are comparable, S is called a **totally ordered** or **linearly ordered set**, and $\leq$ is called a **total order** or **linear order**. A totally ordered set is also called a **chain**.

Partial Orderings

Example 1: Is $(Z, \leq)$ a totally ordered set?

Yes, because $a \leq b$ or $b \leq a$ for all integers a and b.

Example 2: Is $(Z^+, |)$ a totally ordered set?

No, because it contains incomparable elements such as 5 and 7.

Equivalence Relations

Equivalence relations are used to relate objects that are similar in some way.

Definition: A relation on a set A is called an equivalence relation if it is **reflexive**, **symmetric**, and **transitive**.

Two elements that are related by an equivalence relation R are called **equivalent**.

Equivalence Relations

Since R is **symmetric**, a is equivalent to b whenever b is equivalent to a.

Since R is **reflexive**, every element is equivalent to itself.

Since R is **transitive**, if a and b are equivalent and b and c are equivalent, then a and c are equivalent.

Obviously, these three properties are necessary for a reasonable definition of equivalence.

Equivalence Relations

Example: Suppose that R is the relation on the set of strings that consist of English letters such that aRb if and only if $l(a) = l(b)$, where $l(x)$ is the length of the string x. Is R an equivalence relation?

Solution:

R is, because $l(a) = l(a)$ and therefore aRa for any string a.

R is symmreflexiveetric, because if $l(a) = l(b)$ then $l(b) = l(a)$, so if aRb then bRa.

R is transitive, because if $l(a) = l(b)$ and $l(b) = l(c)$, then $l(a) = l(c)$, so aRb and bRc implies aRc.

R is an equivalence relation.

Equivalence Classes

Definition: Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the **equivalence class** of a.

The equivalence class of a with respect to R is denoted by $[\boldsymbol{a}]_{\boldsymbol{R}}$

When only one relation is under consideration, we will delete the subscript R and write $[\boldsymbol{a}]$ for this equivalence class.

If $b \in [a]_R$, b is called a **representative** of this equivalence class.

Equivalence Classes

Example: In the previous example (strings of identical length), what is the equivalence class of the word mouse, denoted by [mouse] ?

Solution: [mouse] is the set of all English words containing five letters.

For example, ‘horse’ would be a representative of this equivalence class.

Equivalence Classes

Theorem: Let R be an equivalence relation on a set A. The following statements are equivalent:

(i) aRb

(ii) $[a] = [b]$

(iii) $[a] \cap [b] \neq \emptyset$

Reminder: A **partition** of a set S is a collection of disjoint nonempty subsets of S that have S as their union. In other words, the collection of subsets A_i, $i \in I$, forms a partition of S if and only if

(i) $A_i \neq \emptyset$ for $i \in I$

(ii) $A_i \cap A_j = \emptyset$, if $i \neq j$

(iii) $\cup_{i \in I} A_i = S$

Equivalence Classes

Theorem: Let R be an equivalence relation on a set S. Then the **equivalence classes** of R form a **partition** of S. Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S, there is an equivalence relation R that has the sets A_i, $i \in I$, as its equivalence classes.

Equivalence Classes

Example: Let us assume that Frank, Suzanne and George live in Boston, Stephanie and Max live in Lübeck, and Jennifer lives in Sydney.

Let R be the **equivalence relation** $\{(a, b)$ | a and b live in the same city} on the set P = {Frank, Suzanne, George, Stephanie, Max, Jennifer}.

Then R = {(Frank, Frank), (Frank, Suzanne), (Frank, George), (Suzanne, Frank), (Suzanne, Suzanne), (Suzanne, George), (George, Frank), (George, Suzanne), (George, George), (Stephanie, Stephanie), (Stephanie, Max), (Max, Stephanie), (Max, Max), (Jennifer, Jennifer)}.

Equivalence Classes

Then the **equivalence classes** of R are:

{{Frank, Suzanne, George}, {Stephanie, Max}, {Jennifer}}.

This is a **partition** of P.

The equivalence classes of any equivalence relation R defined on a set S constitute a partition of S, because every element in S is assigned to **exactly one** of the equivalence classes.

Equivalence Classes

Another example: Let R be the relation $\{(a, b) \mid a \equiv b \ (mod\ 3)\}$ on the set of integers.

Is R an equivalence relation?

Yes, R is reflexive, symmetric, and transitive.

What are the equivalence classes of R ?

$$\{\{\dots, -6, -3, 0, 3, 6, \dots\},$$
$$\{\dots, -5, -2, 1, 4, 7, \dots\},$$
$$\{\dots, -4, -1, 2, 5, 8, \dots\}\}$$

Again, these three classes form a partition of the set of integers.

n-ary Relations

In order to study an interesting application of relations, namely **databases**, we first need to generalize the concept of binary relations to **n-ary relations**.

Definition: Let $A_1, A_2, \dots, An$ be sets. An **n-ary relation** on these sets is a subset of $A_1 \times A_2 \times \dots \times An$.

The sets $A_1, A_2, \dots, An$ are called the **domains** of the relation, and n is called its **degree**.

n-ary Relations

Example:
Let $R = \{(a, b, c) \mid a = 2b \land b = 2c \text{ with } a, b, c \in \mathbb{Z}\}$

What is the degree of R?
The degree of R is 3, so its elements are triples.

What are its domains?
Its domains are all equal to the set of integers.

Is (2, 4, 8) in R?
No.

Is (4, 2, 1) in R?
Yes.