Trees

Section 11 in the textbook

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Trees

Definition: A **tree** is a connected undirected graph with no simple circuits.

Since a tree cannot have a simple circuit, a tree cannot contain multiple edges or loops. Therefore, any tree must be a **simple graph**.

Theorem: An undirected graph is a tree if and only if there is a **unique simple path** between any of its vertices.

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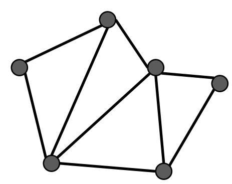
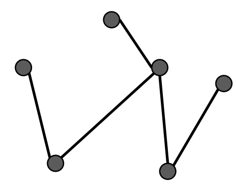
Trees

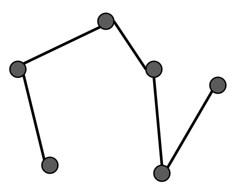
Example: Are the following graphs trees?



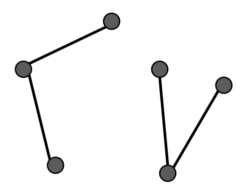
No.



Yes.



Yes.



No.

Trees

Definition: An undirected graph that does not contain simple circuits and is not necessarily connected is called a **forest**.

In general, we use trees to represent **hierarchical structures**.

We often designate a particular vertex of a tree as the **root**. Since there is a unique path from the root to each vertex of the graph, we direct each edge away from the root.

Thus, a tree together with its root produces a **directed graph** called a **rooted tree**.

Characteristic of trees

If T is a graph with n vertices, the following are equivalent:

a) T is a tree
b) T is connected and acyclic (having no cycles)
c) T is connected and has n-1 edges
d) T is acyclic and has n-1 edges

Tree Terminology

If v is a vertex in a rooted tree other than the root, the **parent** of v is the unique vertex u such that there is a directed edge from u to v.

When u is the parent of v, v is called the **child** of u.

Vertices with the same parent are called **siblings**.

The **ancestors** of a vertex other than the root are the vertices in the path from the root to this vertex, excluding the vertex itself and including the root.

Tree Terminology

The **descendants** of a vertex v are those vertices that have v as an ancestor.

A vertex of a tree is called a **leaf** if it has no children.

Vertices that have children are called **internal vertices**.

If a is a vertex in a tree, then the **subtree** with a as its root is the subgraph of the tree consisting of a and its descendants and all edges incident to these descendants.

Tree Terminology

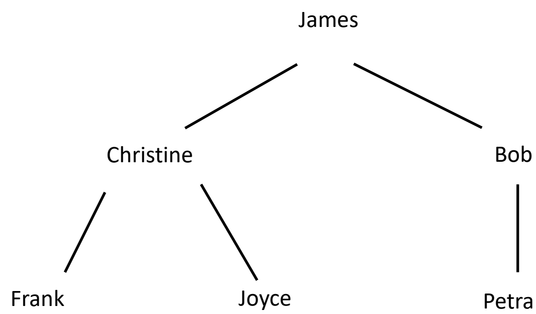
The **level** of a vertex v in a rooted tree is the length of the unique path from the root to this vertex.

The level of the root is defined to be zero.

The **height** of a rooted tree is the maximum of the levels of vertices.

Trees

Example I: Family tree



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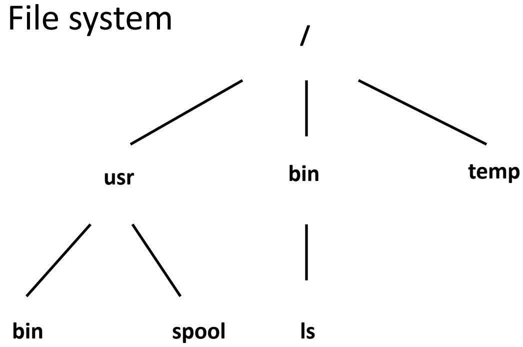
Trees

Example II: File system



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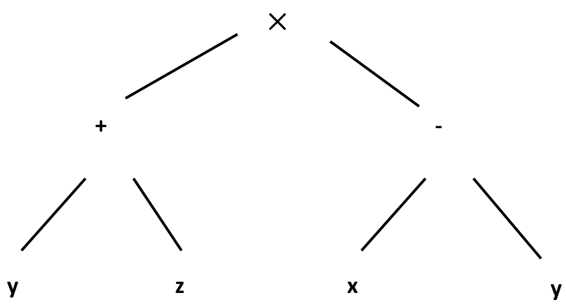
Trees

Example III: Arithmetic expressions



This tree represents the expression $(y + z)\times(x - y)$.

Trees

Definition: A rooted tree is called an **m-ary tree** if every internal vertex has no more than m children.

The tree is called a **full m-ary tree** if every internal vertex has exactly m children.

An m-ary tree with m = 2 is called a **binary tree**.

Theorem: A tree with n vertices has (n – 1) edges.

Theorem: A full m-ary tree with i internal vertices contains n = mi + 1 vertices.

Binary Search Trees

If we want to perform a large number of searches in a particular list of items, it can be worthwhile to arrange these items in a **binary search tree** to facilitate the subsequent searches.

A binary search tree is a binary tree in which each child of a vertex is designated as a **right or left child**, and each vertex is labeled with a **key**, which is one of the items.

When we construct the tree, vertices are assigned keys so that the key of a vertex is both larger than the keys of all vertices in its left subtree and smaller than the keys of all vertices in its right subtree.

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Binary Search Trees

Example: Construct a binary search tree for the strings **math, computer, power, north, zoo, dentist, book**.

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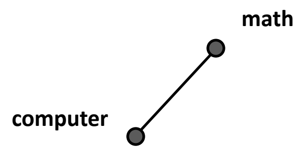

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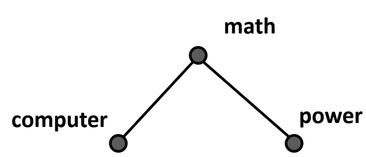

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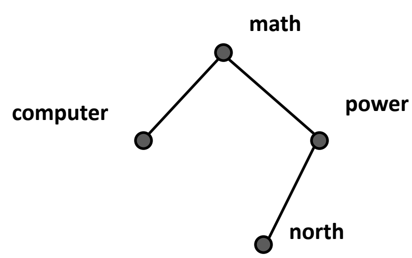

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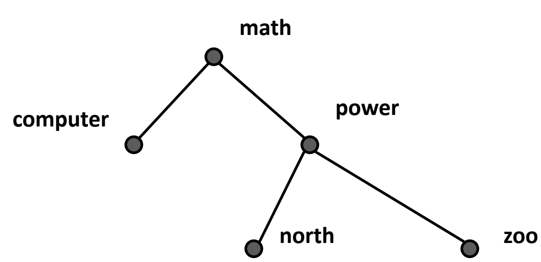

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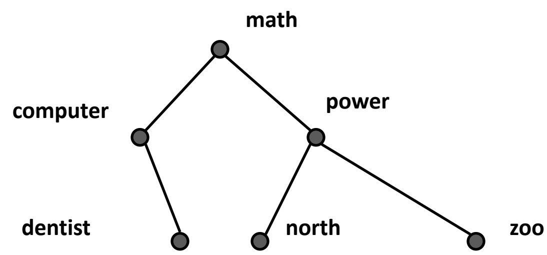

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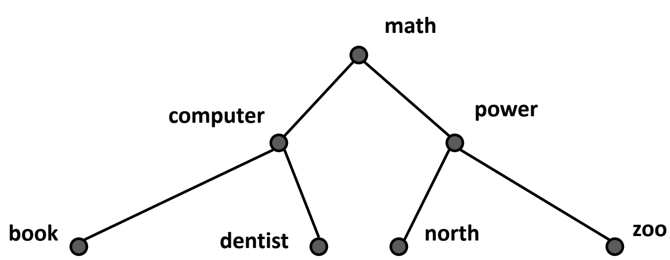


Binary Search Trees

To perform a search in such a tree for an item x, we can start at the root and compare its key to x. If x is **less** than the key, we proceed to the **left** child of the current vertex, and if x is **greater** than the key, we proceed to the **right** one.

This procedure is repeated until we either found the item we were looking for, or we cannot proceed any further.

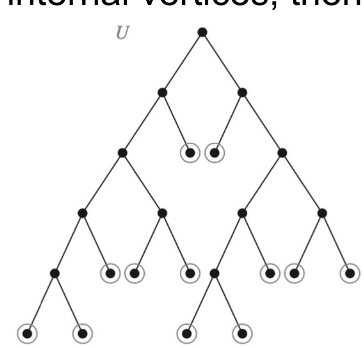
In a balanced tree representing a list of n items, search can be performed with a maximum of $\lceil \log(n + 1) \rceil$ steps (compare with binary search).

Full Binary tree

Definition: A *full* binary tree is a binary tree in which each vertex has two or no children.

Theorem: If T is a *full binary tree* with k internal vertices, then:

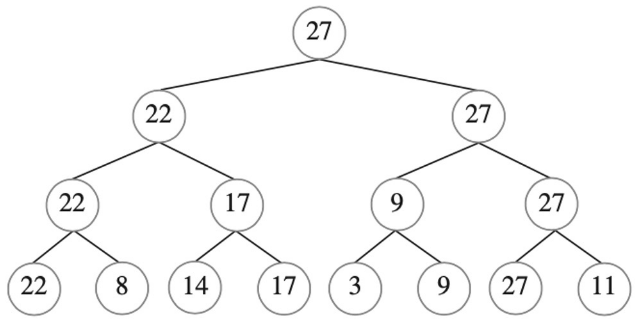
- T has k+1 leaves
- The total vertices is 2k+1



Full Binary tree

Theorem: If a binary tree of height h has t leaves, then $t \leq 2^h$

If all leaves t of a full binary tree T have the same level h = height of T, then $t = 2^h$



Spanning Trees

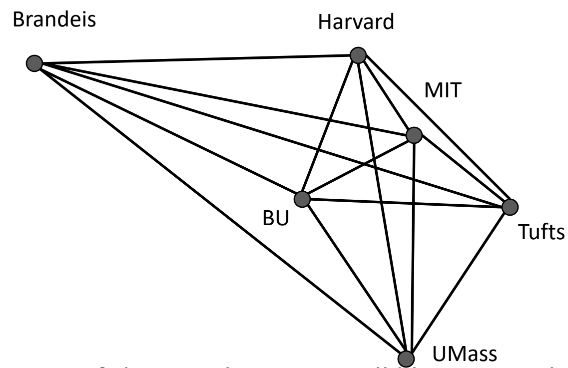
Definition: Let G be a simple graph. A spanning tree of G is a subgraph of G that is a tree containing every vertex of G.

Note: A spanning tree of G = (V, E) is a connected graph on V with a minimum number of edges (|V| - 1).

Example: Since winters in Boston can be very cold, six universities in the Boston area decide to build a tunnel system that connects their libraries.

Spanning Trees

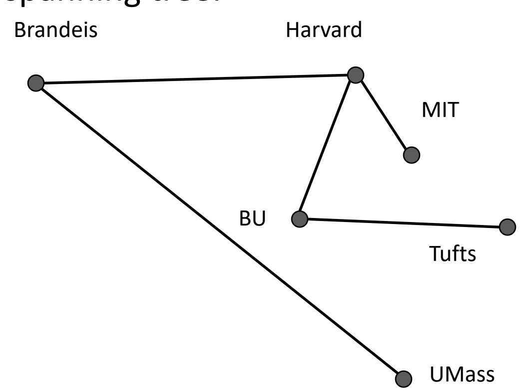
The complete graph including all possible tunnels:



The spanning trees of this graph connect all libraries with a minimum number of tunnels.

Spanning Trees

Example for a spanning tree:



Since there are 6 libraries, 5 tunnels are sufficient to connect all of them.

Spanning Trees

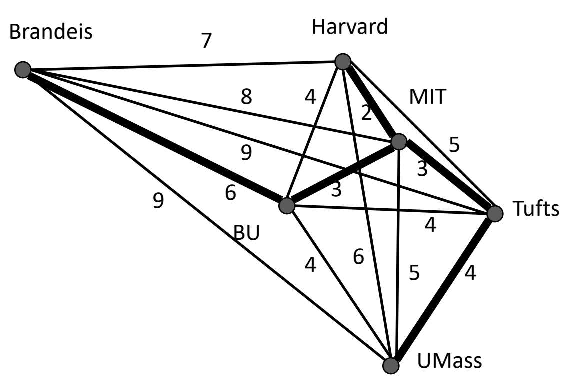
Now imagine that you are in charge of the tunnel project. How can you determine a tunnel system of **minimal cost** that connects all libraries?

Definition: A **minimum spanning tree** in a connected weighted graph is a spanning tree that has the smallest possible sum of weights of its edges.

How can we find a minimum spanning tree?

Spanning Trees

The complete graph with cost labels (in billion $):



The least expensive tunnel system costs $20 billion.

Prim's algorithm

Step 0: Pick any vertex as a starting vertex (call it *a*). T = {a}.

Step 1: Find the edge with smallest weight incident to *a*. Add it to T Also include in T the next vertex and call it *b*.

Step 2: Find the edge of smallest weight incident to either *a* or *b.* Include in T that edge and the next incident vertex. Call that vertex *c*.

Step 3: Repeat Step 2, choosing the edge of smallest weight that does not form a cycle until all vertices are in T. The resulting subgraph T is a minimum spanning tree.

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Prim's algorithm

ALGORITHM 1 **Prim's Algorithm.**

procedure *Prim*(G: weighted connected undirected graph with n vertices)
T := a minimum-weight edge
for $i := 1$ **to** $n - 2$
  e := an edge of minimum weight incident to a vertex in T and not forming a simple circuit in T if added to T
  $T := T$ with e added
return T {T is a minimum spanning tree of G}

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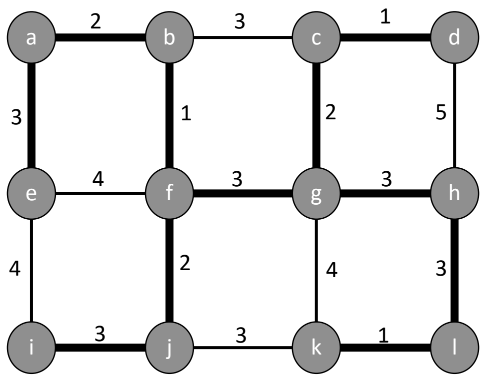
Prim's algorithm



Choice	Edge	Weight
1	{b, f}	1
2	{a, b}	2
3	{f, j}	2
4	{a, e}	3
5	{i, j}	3
6	{f, g}	3
7	{c, g}	2
8	{c, d}	1
9	{g, h}	3
10	{h, l}	3
11	{k, l}	1
		24

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Kruskal's algorithm

Step 1: Find the edge in the graph with smallest weight (if there is more than one, pick one at random). Mark it with any given color, say red.

Step 2: Find the next edge in the graph with smallest weight that doesn't close a cycle. Color that edge and the next incident vertex.

Step 3: Repeat Step 2 until you reach out to every vertex of the graph. The chosen edges form the desired minimum spanning tree.

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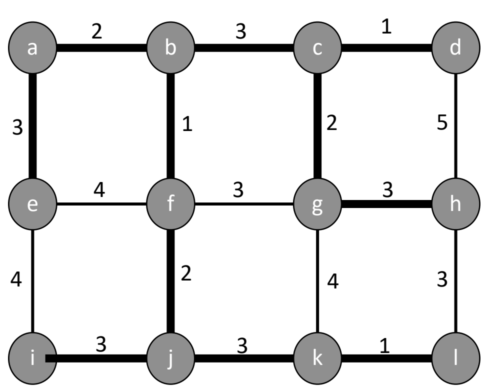
Kruskal's algorithm

ALGORITHM 2 Kruskal's Algorithm.

procedure *Kruskal*(G: weighted connected undirected graph with n vertices)
T := empty graph
for i := 1 **to** $n - 1$
  e := any edge in G with smallest weight that does not form a simple circuit
    when added to T
  T := T with e added
return T {T is a minimum spanning tree of G}

Kruskal's algorithm



Choice	Edge	Weight
1	{c, d}	1
2	{b, f}	1
3	{k, l}	1
4	{c, g}	2
5	{a, b}	2
6	{f, j}	2
7	{b, c}	3
8	{j, k}	3
9	{g, h}	3
10	{i, j}	3
11	{a, e}	3
		24

Spanning Trees

Prim vs. Kruskal:

- Both algorithms are **guaranteed** to produce a minimum spanning tree of a connected weighted graph.
- The two algorithms differ in the way they can be implemented and their efficiency under different conditions.
- As a rule of thumb, **Prim's algorithm** is more efficient when initially there are many **more edges** than vertices.
- For graphs with initially only **few edges** in comparison to the number of vertices, **Kruskal's algorithm** typically performs more efficiently.

Tree Traversal

Let T be an ordered rooted tree with root r.

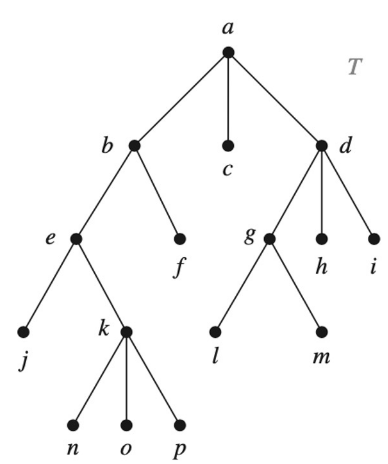
Definition 1: If T consists only of r, then r is the ***preorder traversal*** of T . Otherwise, suppose that $T_1, T_2, \ldots, T_n$ are the subtrees at r from left to right in T . The *preorder traversal* begins by visiting r. It continues by traversing T_1 in preorder, then T_2 in preorder, and so on, until T_n is traversed in preorder.

Defintion 2: If T consists only of r, then r is the ***inorder traversal*** of T . Otherwise, suppose that $T_1, T_2 \ldots, T_n$ are the subtrees at r from left to right. The *inorder traversal* begins by traversing T_1 in inorder, then visiting r. It continues by traversing T_2 in inorder, and so on until T_n is traversed in inorder.

Tree Traversal

Definition 3: Let T be an ordered rooted tree with root r. If T consists only of r, then r is the ***postorder traversal*** of T . Otherwise, suppose that $T_1, T_2, \ldots, T_n$ are the subtrees at r from left to right. The *postorder traversal* begins by traversing T_1 in postorder, then T_2 in postorder, . . . , then T_n in postorder, and ends by visiting r.

Pre-Order Traversal



Preorder traversal:

a-b-e-j-k-n-o-p-f-c-d-g-l-m-h-I

Inorder traversal:

j-e-n-k-o-p-b-f-a-c-l-g-m-d-h-i

Postorder traversal:

j-n-o-p-k-e-f-b-c-l-m-g-h-i-d-a

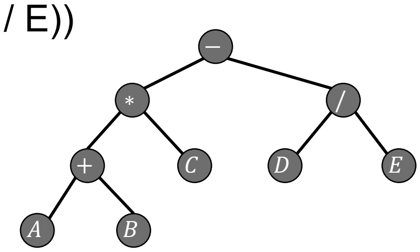
Arithmetic expressions

Standard: *infix* form

(A+B) ∗ C – D/ E

Fully parenthesized form (in-order & parenthesis):

(((A + B) ∗ C) – (D / E))



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Arithmetic expressions

Prefix form (Polish notation):

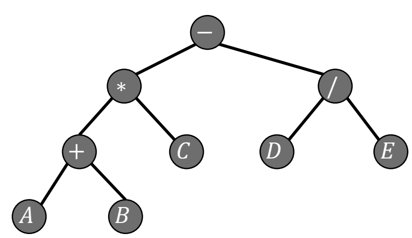
- ∗ + A B C / D E

→ Preorder traversal

Postfix form (reverse Polish notation):

A B + C ∗ D E / -

→ Postorder traversal



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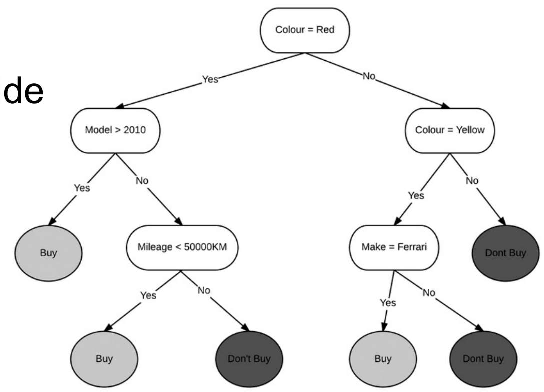
Decision Tree

A *decision tree* is a binary tree containing an algorithm to decide which course of action to take.



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END

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Final Exam policy

- **Camera on!** Webcams will be used for proctoring
- Showing Student ID at the start of the exam.
- Please login 15 minutes early (9:45am)
- Private message in the chatbox
- **Exactly 10 minutes** to submit the solution (Gradescope)