Closures of Relations

Let us finally solve **Example III** by finding the **transitive closure** of the relation R = {(1, 3), (1, 4), (2, 1), (3, 2)} on the set A = {1, 2, 3, 4}.

R can be represented by the following matrix M_R:

$$M_R = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Closures of Relations

$$M_R = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad M_R^{[2]} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_R^{[3]} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad M_R^{[4]} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]} \vee M_R^{[4]} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Closures of Relations

Solution: The transitive closure of the relation R = {(1, 3), (1, 4), (2, 1), (3, 2)} on the set A = {1, 2, 3, 4} is given by the relation
{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4)}

Warshall's Algorithm

A more efficient way of computing the transitive closure of a relation with digraph on vertices $\{v_1, v_2, \ldots, v_n\}$:

Theorem (p. 606). Let $W_k = (w_{ij}^{[k]})$ be the 0,1 matrix $w_{ij}^{[k]} = 1$ iff there is a path from v_i to v_j with any interior vertices in the set $\{v_1, v_2, \ldots v_k\}$. Then
$w_{ij}^{[k]} = w_{ij}^{[k-1]} \vee (w_{ik}^{[k-1]} \wedge w_{kj}^{[k-1]})$
$W_0 = W_R$, $W_n = W_{R^*}$.

Proof: We'll use induction.

Base case: k=0. $W_0 = W_R$ because there can be no interior vertices, so just a single edge.

Induction step: If true for k-1, show
$w_{ij}^{[k]} = w_{ij}^{[k-1]} \vee (w_{ik}^{[k-1]} \wedge w_{kj}^{[k-1]})$
because there is a path from v_i to v_j using interior vertices from $\{v_1, v_2, \ldots, v_k\}$ iff

- There is a path without v_k as an interior vertex (so $w_{ij}^{[k-1]} = 1$) or
- There is path with v_k as an interior vertex, in which case both $w_{ik}^{[k-1]}$ and $w_{kj}^{[k-1]}$ are 1. (there must be a k-1path from v_i to v_k and from v_k to v_j)

Using Warshall's Algorithm

As shown in the book, the formula giving Warshall's Algorithm easily translates to computer code.

If you do it by hand, just note that in $w_{ij}^{[k]} = w_{ij}^{[k-1]} \vee (w_{ik}^{[k-1]} \wedge w_{kj}^{[k-1]})$ you go from W_{k-1} to W_k by looking at the matrix for W_{k-1}. If you can go from v_i to v_k in W_{k-1} then in W_k you can add an entry ij if v_k goes to v_j in W_{k-1}. (this is easier than it sounds)

Transitive Closure via Warshall's Algorithm

$$W_0 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad W_1 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$W_2 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad W_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_{R^*} = W_3 = W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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