

#### Equivalence Relations (Section 9.5)

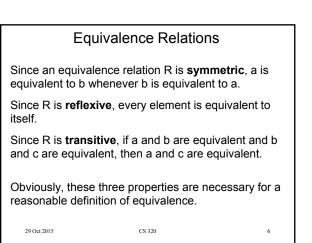
**Equivalence relations** are used to relate objects that are similar in some way. (section 9.5)

**Definition:** A relation on a set A is called an equivalence relation if it is reflexive, symmetric, and transitive.

Two elements that are related by an equivalence relation R are called **equivalent** under that relation.

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#### Equivalence Relations Example: Suppose that R is the relation on the set of strings that consist of English letters such that aRb iff I(a) = I(b), where I(x) is the length of the string x. Is R an equivalence relation? Solution: • R is reflexive, because I(a) = I(a) and therefore aRa for any string a. • R is symmetric, because if I(a) = I(b) then I(b) = I(a), so if aRb then bRa. • R is transitive, because if I(a) = I(b) and I(b) = I(c), then I(a) = I(c), so aRb and bRc implies aRc. R is an equivalence relation. 29 Oct 2015 CS 320 7

#### Equivalence Classes **Definition:** Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the **equivalence** class of a. The equivalence class of a with respect to R is denoted by [a]<sub>R</sub>. When only one relation is under consideration, we will delete the subscript R and write [a] for this equivalence class. If $b \in [a]_R$ , b is called a **representative** of this equivalence class. 29 Oct 2015 CS 320 8

Equivalence Classes Example: In the previous example (strings of identical length), what is the equivalence class of the word mouse, denoted by [mouse]? Solution: [mouse] is the set of all English words containing five letters. For example, 'horse' would be a representative of this equivalence class. 29 Oct 2015 CS 320

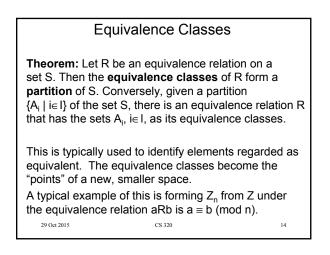
## **Equivalence Classes** Theorem: Let R be an equivalence relation on a set A. The following statements are equivalent: (i) aRb (meaning (a,b) $\in R$ ) (ii) [a] = [b] (iii) [a] $\cap$ [b] $\neq \emptyset$ Proof: we'll prove that (i) $\rightarrow$ (ii), (ii) $\rightarrow$ (iii), and (iii) $\rightarrow$ (i), when R is an equiv. relation 29 Oct 2015 CS 320 10

 $(i) \rightarrow (ii)$ Suppose aRb. If  $x \in [a]$  then xRa, so xRb by transitivity, and  $x \in [b]$ . By symmetry,  $\mathbf{x} \in [b] \rightarrow \mathbf{x} \in [a]$ (ii)  $\rightarrow$  (iii) if [a]=[b] then a  $\in$  [a]  $\cap$  [b].  $(iii) \rightarrow (i)$ Suppose  $x \in [a] \cap [b]$ . Then xRa and xRb, so by symmetry aRx and xRb, so aRb by transitivity. CS 320 11

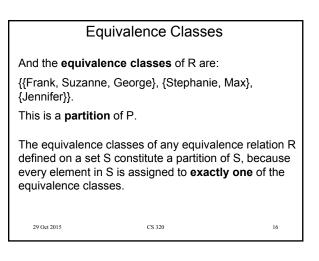
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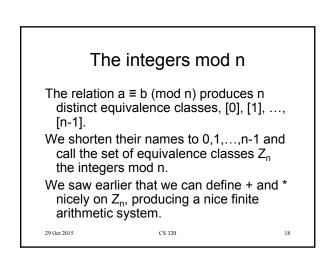
Equivalence Classes Definition: A partition of a set S is a collection of disjoint nonempty subsets of S that have S as their union. In other words, the collection of subsets A<sub>i</sub>, i∈ I, forms a partition of S if and only if (i)  $A_i \neq \emptyset$  for  $i \in I$ (ii)  $A_i \cap A_i = \emptyset$ , if  $i \neq j$ (iii)  $\cup_{i \in I} A_i = S$ 29 Oct 2015 CS 320 12

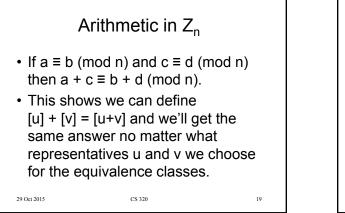
Equivalence Classes		
<b>Examples:</b> Let S be the set {u, m, b, r, o, c, k, s}. Do the following collections of sets partition S ?		
{{m, o, c, k}, {r, u, b, s}}	yes.	
{{c, o, m, b}, {u, s}, {r}}	no (k is missing).	
{{b, r, o, c, k}, {m, u, s, t}}	no (t is not in S).	
{{u, m, b, r, o, c, k, s}}	yes.	
{{b, o, o, k}, {r, u, m}, {c, s}}	yes $(\{b, o, o, k\} = \{b, o, k\})$ .	
{{u, m, b}, {r, o, c, k, s}, Ø}	no (Ø not allowed).	

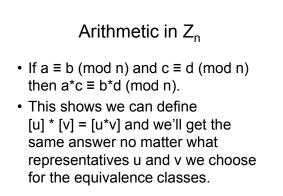


Equ	ivalence Class	ses
George live in Bos	assume that Frank, ston, Stephanie and ifer lives in Sydney	d Max live in
	valence relation { on the set P = {Frar e, Max, Jennifer}.	
	ank), (Frank, Suzanne zanne, Frank), (Suzar (George, Frank), (George, George),	
(Stephanie, Stephan (Max, Max),	ie), (Stephanie, Max),	(Max, Stephanie),
(Jennifer, Jennifer)}.		
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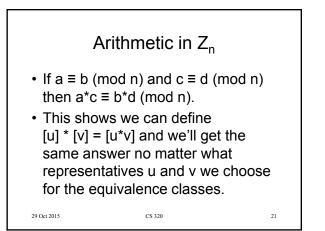


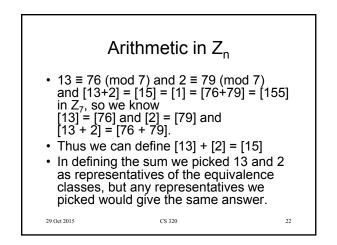


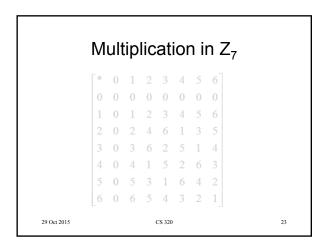
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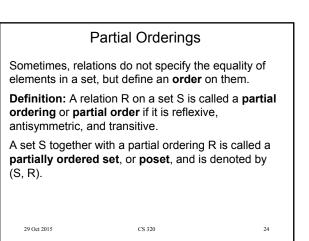
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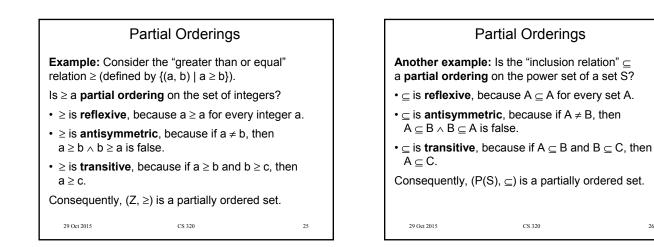
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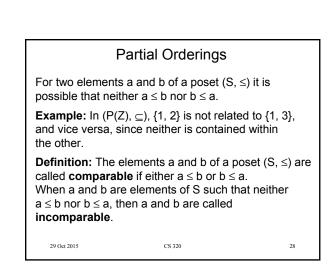












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### Partial Orderings

Partial Orderings

In a poset the notation  $a \le b$  denotes that  $(a, b) \in \mathbb{R}$ .

in any poset, not just the usual "less than or equal"

If a < b we say "a is less than b" or "b is greater than

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The notation a < b denotes that  $a \le b$ , but  $a \ne b$ .

relation in numbers.

a".

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Note that the symbol  $\leq$  is used to denote the relation

For some applications, we require all elements of a set to be comparable.

For example, if we want to write a dictionary, we need to define an order on all English words (alphabetic order).

**Definition:** If  $(S, \leq)$  is a poset and every two elements of S are comparable, S is called a totally ordered or linearly ordered set, and ≤ is called a total order or linear order. A totally ordered set is also called a chain.

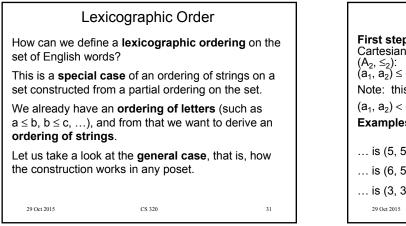
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### Partial Orderings

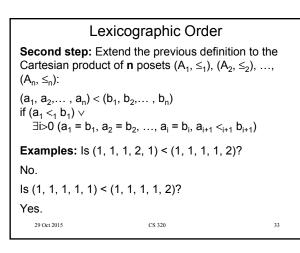
**Example I:** Is  $(Z, \leq)$  a totally ordered poset? Yes, because  $a \le b$  or  $b \le a$  for all integers a and b.

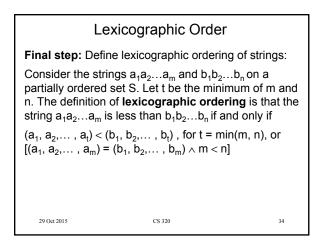
Example II: Is (Z<sup>+</sup>, |) a totally ordered poset? No, because it contains incomparable elements such as 5 and 7.

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raphic Order	
partial ordering on the posets, $(A_1, \leq_1)$ and	
$(\mathbf{b}_1) \vee [(\mathbf{a}_1 = \mathbf{b}_1) \land (\mathbf{a}_2 \leq_2 \mathbf{b}_2)]$	]
:	
$b_1) \vee [(a_1 = b_1) \land (a_2 <_2 b_2)]$	]
(Z×Z, ≤),	
yes.	
no.	
no.	
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	partial ordering on the posets, $(A_1, \leq_1)$ and $b_1) \lor [(a_1 = b_1) \land (a_2 \leq_2 b_2)]$ $b_1) \lor [(a_1 = b_1) \land (a_2 <_2 b_2)]$ $(Z \times Z, \leq), \dots$ yes. no. no. no.

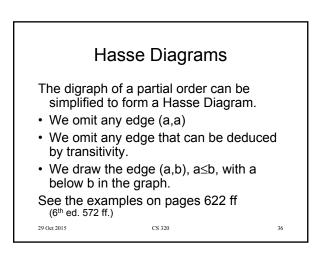




## Lexicographic Order Examples: If we apply this concept to lowercase English letters, ... ... is discreet < discrete ? Yes, because in the 7<sup>th</sup> position, e < t. ... is discreetness < discreet ? No, because discreet is a prefix of discreetness. ... is discrete < discretion ? Yes, because in the 8<sup>th</sup> position, e < i.

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# Maximal & Minimal elements

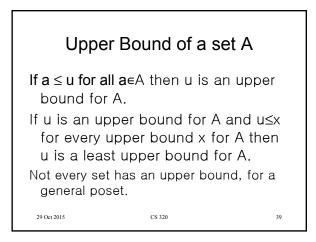
- An element a is minimal in a poset  $(S, \leq)$  if there is no b with b<a.
- An element a is maximal in a poset  $(S,\leq)$  if there is no b with b>a.
- Maximal (and minimal) elements are easy to spot in a Hasse diagram.
- They are elements with nothing above (or below) them.

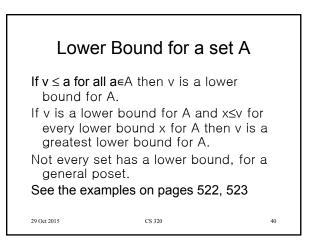
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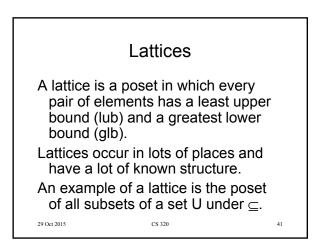
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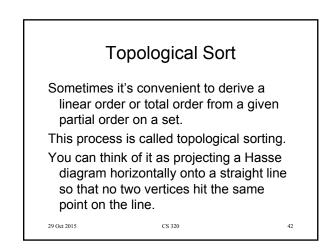
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Maximal & Minimal elements
a is the greatest element of a poset (S,≤) if b≤a for all b ∈ S.
c is the least element of a poset (S,≤) if c≤b for all b ∈ S.
If a greatest or least element exists it must be unique.
(Make sure you can prove this fact).









Т	opological Sort	
noting that has a minir	truct an algorithm to do every non empty subse nal element. truct a linear order on a	t in a poset
(S,⊆) by su	inccessively choosing a norm the elements left.	
These eleme the linear o	nts form an increasing s rder ≤.	equence in
	<b>der is compatible in that</b> s that a≤b in the linear	
The reverse	is guaranteed only if $\subseteq$	is linear.
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