CS724: Topics in Algorithms
Problem Set 2

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Problem 1:

Let $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{n \times p}$ be two conformant matrices. Prove that:

- computing the matrix $G = AB$ using standard matrix multiplication requires mnp number multiplications;
- if $C \in \mathbb{C}^{p \times q}$ the computation of the matrix $D = (AB)C = A(BC)$ by the standard method, the first modality $D = (AB)C$ requires $mp(n + q)$ multiplications, while the second, $D = A(BC)$ requires $nq(m + p)$ multiplications.

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Solution 1:

Each element g_{ij} of $G \in \mathbb{C}^{mp}$ requires n multiplications because

$$g_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}.$$

Since there are mp such elements, the total number of multiplications is mpn.
Note that $AB \in \mathbb{R}^{m \times p}$ and $BC \in \mathbb{R}^{n \times q}$. Therefore, $(AB)C$ requires $mp(n+q)$ multiplications, and $A(BC)$ requires $nq(m+p)$ multiplications.
A judicious organization of computations would compare these numbers. If $mp(n+q) < nq(m+p)$, then $(AB)C$ is preferable to $A(BC)$ because it would involve fewer multiplications.

Problem 2:

Let A, B be two matrices in $\mathbb{C}^{n\times n}$. Suppose that $B = C + D$, where C is a Hermitian matrix and D is a skew-Hermitian matrix. Prove that if A is Hermitian and $AB = BA$, then $AC = CA$ and $AD = DA$.

Solution 2:

Since C is Hermitian we have $C^{\mathsf{H}} = C$. Since D is skew-Hermitian, $D^{\mathsf{H}} = -D$. Let A be such that $A^{\mathsf{H}} = A$ and $AB = BA$.
$AB = BA$ is equivalent to $A(C + D) = (C + D)A$.
Also, $AB = BA$ implies $B^{\mathsf{H}}A = AB^{\mathsf{H}}$ because A is Hermitian. Since $B^{\mathsf{H}} = C^{\mathsf{H}} + D^{\mathsf{H}} = C - D$, this amounts to $A(C - D) = (C - D)A$. Since we also have $A(C + D) = (C + D)A$ be adding the last two equalities we obtain $2AC = 2CA$ and, by subtracting then, we have $2AD = 2DA$, which yield the conclusion.

Problem 3:

Recall that $J_{n,n} \in \mathbb{R}^{n \times n}$ is the complete $n \times n$ matrix, that is the matrix having all components equal to 1. Prove that for every number $m \in \mathbb{N}$ and $m \geqslant 1$ we have

$$J_{n,n}^m = n^{m-1} J_{n,n}.$$

Solution 3:

Observe that $J_{n,n}^2 = nJ_{n,n}$.
The proof is by induction on m. The base step, $m = 1$ is immediate.
Suppose this holds for m. Then

$$
\begin{aligned}
J_{n,n}^{m+1} &= J_{n,n}J_{n,n}^{m} = J_{n,n} \cdot n^{m-1} J_{n,n} \\
&= \text{(by inductive hypothesis)} \\
&= n^m J_{n,n}.
\end{aligned}
$$

Problem 4:

Let $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times n}$ be two matrices. Prove that if AB is invertible, then both A and B are invertible.

Solution 4:

Since AB is invertible we have $AB(AB)^{-1} = I_n$. Thus, A is invertible and $A^{-1} = B(AB)^{-1}$. Similarly, since $(AB)^{-1}AB = I_n$ it follows that B is invertible and $B^{-1} = (AB)^{-1}A$.

Problem 5:

Let $A = (a_{ij})$ be an $(m \times n)$-matrix of real numbers. Prove that

$$\max_j \min_i a_{ij} \leqslant \min_i \max_j a_{ij}$$

(the *minimax inequality*).

Solution 5:

Observe that $a_{ij_0} \leqslant \max_j a_{ij}$ for every i and j_0, so $\min_i a_{ij_0} \leqslant \min_i \max_j a_{ij}$, again for every j_0. Thus, $\max_j \min_i a_{ij} \leqslant \min_i \max_j a_{ij}$.

Problem 6:

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ be two vectors such that $\mathbf{x}'\mathbf{y} \neq 0$ and $\mathbf{x}\mathbf{y}' \neq O_{n,n}$.
Prove that both matrices $\mathbf{x}'\mathbf{y}$ and $\mathbf{x}\mathbf{y}'$ have rank 1.
Note that if $\mathbf{x} \in \mathbb{R}^m$ (or $\mathbb{R}^{m\times 1}$) and $\mathbf{y} \in \mathbb{R}^n$ (or $\mathbb{R}^{n\times 1}$)and $m \neq n$ the multiplication $\mathbf{x}'\mathbf{y}$ can not be performed, but $\mathbf{x}\mathbf{y}'$ is feasible and the result is the same (matrix $\mathbf{x}\mathbf{y}'$ has rank 1).

Solution 6:

Note that $\boldsymbol{x}'\boldsymbol{y}$ is a number:

$$\boldsymbol{x}'\boldsymbol{y} = x_1 y_1 + \cdots + x_n y_n,$$

and therefore has rank 1.
On other hand,

$$\boldsymbol{x}\boldsymbol{y}' = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \boldsymbol{y}' = \begin{pmatrix} x_1 \boldsymbol{y}' \\ \vdots \\ x_n \boldsymbol{y}' \end{pmatrix},$$

which implies that $\boldsymbol{x}\boldsymbol{y}'$ also has rank 1 because every row of this matrix is a multiple of $\boldsymbol{y}'$.