# Problem Session 1

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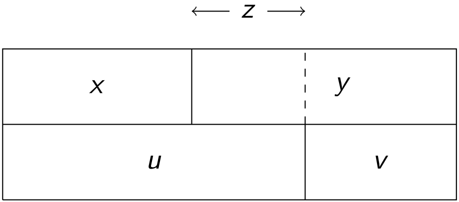
# Problem 1

Let $x, y, u, v$ be words in $A^+$ such that $xy = uv$ and $|x| \leqslant |u|$. Show that there exists $z \in A^*$ such that $u = xz$ and $y = zv$.

# Solution 1

Note that none of the words $x, y, u, v$ is null because they belong to $A^+$. Since $|x| \leqslant |u|$, $x$ is a prefix of $u$ and we can write $u = xz$. Substituting $u$ in the equality $u = xz$ we obtain $xy = xzv$, which implies $y = zv$.



# Problem 2

Let $u, v$ be two words from $A^*$. Prove that if $uv = vu$, then there exists a word $z \in A^*$ and two numbers $p, q \in \mathbb{N}$ such that $u = z^p$ and $v = z^q$.

# Solution 2

The argument is by induction on $n = |u| + |v|$.
*Basis step:* For $n = 0$ we have $u = v = \lambda$ and the statement holds.
*Inductive step:* Suppose that the statement holds for words whose total length is less than $n$ and let $u, v$ be words whose total length is $n$.
If $|u| = |v|$, then $uv = vu$ implies $u = v$ and, thus, $u = u^1$ and $v = u^1$.
If $|u| \neq |v|$ suppose that $|u| > |v|$. Then $uv = vu$ implies that $u = vt$ for some $t \in A^*$. Therefore, $vtv = vvt$, hence $tv = vt$.
Since $|tv| < |uv|$, by the inductive hypothesis, there is a word $w$ such that $t = w^r$ and $v = w^s$. This implies $u = vt = w^{r+s}$ which shows that $u$ and $v$ are both powers of $w$.
The case when $|u| < |v|$ is similar.

# Problem 3

Let $x, y, z$ be three words from $A^*$ such that $xy = yz$ and $x \neq \lambda$. Prove that there exist $u, v \in A^*$ such that $x = uv$, $y = (uv)^n u$ and $z = vu$ for some $n \in \mathbb{N}$.

## Solution 3

Note that if $xy = yz$, then $|x| = |z|$.
If $|y| \leqslant |x|$, then $y$ is a prefix of $x$, hence $x = yt$. This implies $yty = yz$, which in turn, yields $z = ty$. This allows us to write:

$$x = yt, y = (yt)^0 y, z = ty$$

and the claim is proven with $u = y$ and $v = t$.
If $|y| > |x|$, then $y = xw$ and we have $xxw = xwz$, which implies $xw = wz$. If $|w| \leqslant |x|$, by the previous case we have $x = uv$, $w = (uv)^n u$ and $z = vu$ for some $n \in \mathbb{N}$, hence $y = xw = uvw = (uv)^{n+1} u$ and $z = vu$.
If $|w| < |x|$ we are again in the first case.

# Problem 4

Let $x, y$ be two words in $A^*$. Prove that the following statements are equivalent:

1. there exist $m, n \in \mathbb{N}$ such that the words $x^m$ and $y^n$ have a common prefix of length $|x| + |y|$.
2. $xy = yx$;
3. there exist $z \in A^*$ and $p, q \in \mathbb{N}$ such that $x = z^p$ and $y = z^q$;

# Solution 4

`(1) implies (2)` Suppose that there exist $m, n \in \mathbb{N}$ such that the words $x^m$ and $y^n$ have a common prefix of length $|x| + |y|$. Then, the words $yx^m$ and $y^{n+1}$ have a common prefix of length $|x| + 2|y|$. Therefore, $yx^m$ and $y^n$ have a common prefix of length $|x| + |y|$. In a similar manner, $xy^n$ and $x^m$ have a common prefix of length $|x| + |y|$. This implies that $xy^n$ and $yx^m$ have a common prefix of length $|x| + |y|$, so $xy = yx$.

# Solution 4 (cont'd)

`(2) implies (3)` The argument is by induction on $k = |x| + |y|$. If $k = 0$, then $x = y = \lambda$ and the conclusion follows immediately. Suppose that the statement holds for $k < n$ and let $x, y \in A^*$ be such that $xy = yx$. We need to consider three cases: $|x| = |y|$, $|x| < |y|$, and $|x| > |y|$. In the first case, $xy = yx$ implies $x = y$, so the statement obviously holds. Suppose now that $|x| < |y|$. Then, there exists $t \in A^*$ such that $y = xt$, which implies $xxt = xtx$. This, in turn, implies $xt = tx$. By the inductive hypothesis, there exists a word $z \in A^*$ such that $x = z^k$ and $t = z^\ell$. Consequently, $y = xt = z^{k+\ell}$, which concludes the argument for the second case. The third case is entirely similar to the second.

`(3) implies (1)` This implication is immediate.

# Problem 5

Let $A = \{0, 1\}$ be an alphabet and let $L = A^* 1 A^n$. In other words, $L$ consists of strings of bits that begin with an arbitrary sequence of bits, and end with an 1 followed by a sequence of $n$ bits. Compute the collection of languages $\{x^{-1}L \mid x \in A^*\}$.

# Solution 5

We have:

$$\begin{aligned} 0^{-1}L &= 0^{-1}((A^*)(1A^n)) = (0^{-1}A^*)(1A^n) \cup 0^{-1}(1A^n) \\ &= A^*1A^n \\ 1^{-1}L &= 1^{-1}((A^*)(1A^n)) = (1^{-1}A^*)(1A^n) \cup 1^{-1}(1A^n) \\ &= A^*1A^n \cup A^n \end{aligned}$$

Set of derivatives: $A^*1A^n$ and $A^*1A^n \cup A^n$.

# Solution 5 cont'd

$$
\begin{aligned}
0^{-1}(A^*1A^n) &= A^*1A^n \\
1^{-1}(A^*1A^n) &= A^*1A^n \cup A^n \\
0^{-1}(A^*1A^n \cup A^n) &= A^*1A^n \cup A^{n-1} \\
1^{-1}(A^*1A^n \cup A^n) &= A^*1A^n \cup A^{n-1}
\end{aligned}
$$

Set of derivatives: $A^*1A^n$, $A^*1A^n \cup A^n$, $A^*1A^n \cup A^{n-1}$.

# Solution 5 cont'd

$0^{-1}(A^*1A^n) = A^*1A^n 0^{-1}(A^*1A^n \cup A^n) = A^*1A^n \cup A^{n-1} 0^{-1}(A^*1A_n \cup A^{n-}$

Set of derivatives: $A^*1A^n$, $A^*1A^n \cup A^n$, $A^*1A^n \cup A^{n-1}$, $A^*1A_n \cup A^{n-2}$, $A^*1A^n \cup A^{n-3}$, etc.