# Problem Session 2

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# Problem 1:

Let $A = \{0, 1\}$ be an alphabet that consists of two binary digits. Denote by $f(x)$ the numerical equivalent of $x$. Design a dfa that accepts the set of words $\{x \in \{0,1\}^* \mid f(x) \text{ is a multiple of } 5\}$.

# Solution 1:

Define the funcion $f : A^* \longrightarrow \mathbb{N}$ that maps words into binary numbers as

$$
\begin{aligned}
f(\lambda) &= 0 \\
f(xb) &= \begin{cases} 2f(x) + 0 & \text{if } b = 0, \\ 2f(x) + 1 & \text{if } b = 1. \end{cases}
\end{aligned}
$$

We have

$$f(\lambda) = 0, f(0) = 0, f(1) = 1, f(00) = 0, f(01) = 1, f(10) = 2, f(11) = 3$$

The desired automaton has the states $\{q_0, q_1, q_2, q_3, q_4\}$ corresponding to the possible remainders of the division of a natural number by 5.
Suppose that the initial state is $q_0$.
The dfa is designed such that if it is currently in state $q_h$ after reading the symbols of the word $x$ and the next symbol is $b$ the device will enter the state $q_\ell$, where

$$\delta(q_h, b) = q_\ell, \text{ where } 2h + b \equiv \ell \pmod 5$$

We have:

$$\delta(q_0, 0) = q_0, \delta(q_0, 1) = q_1,$$
$$\delta(q_1, 0) = q_2, \delta(q_1, 1) = q_3,$$
$$\delta(q_2, 0) = q_4, \delta(q_2, 1) = q_0,$$
$$\delta(q_3, 0) = q_1, \delta(q_3, 1) = q_2,$$
$$\delta(q_4, 0) = q_3, \delta(q_4, 1) = q_4.$$

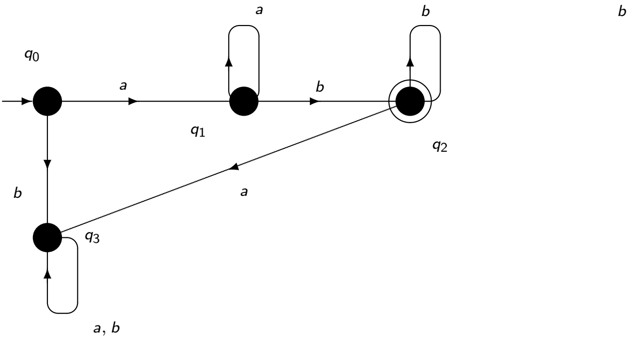
The initial and final state is $q_0$.
Note that $\delta^*(q_0, 101) = q_0$ and $\delta * (q_0, 1111) = q_0$.

# Problem 2:

Design deterministic finite automata that will recognize the languages $L_1 = a^+b^+$, $L_2 = a^*b^+$, and $L_3 = a^*b^*$ defined over the alphabet $A = \{a, b\}$, respectively.

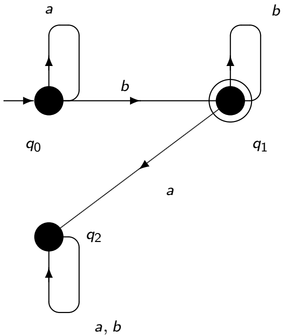
# Solution 2:

Important observation: a dfa must have a successor state $q'$ for every state $q$ and every input symbol!
For $L_1 = a^+b^+$ the dfa is:



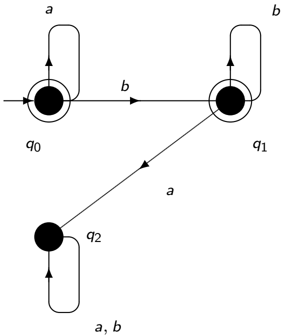
# Solution 2 cont'd

For $L_2 = a^* b^+$ the dfa is:



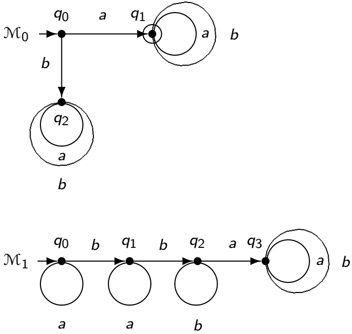
# Solution 2 (cont'd)

For $L_3 = a^* b^*$ the dfa is:



# Problem 3

Determine the language accepted by each of the dfas $\mathcal{M}_0$ and $\mathcal{M}_1$ shown below.



# Problem 4

Design transition systems that recognize the languages $L_1 = a^+ b^+$, $L_2 = a^* b^+$, and $L_3 = a^* b^*$.

# Problem 5

Let $L = \{a^n b_n \mid n \in \mathbb{N}\}$. Compute the set of left derivatives of the form $x^{-1}L$

# Solution 5

Note that

$$\begin{aligned} a^{-1}L &= \{a^{n-1}b^n \mid n \geqslant 1\}, \\ b^{-1}L &= \{b^n \mid n \geqslant 0\} = \{b\}^*. \end{aligned}$$

Next,

$$\begin{aligned} (aa)^{-1}L &= \{a^{n-2}b^n \mid n \geqslant 2\}, \\ (ab)^{-1}L &= \{\lambda\} \\ (ba)^{-1}L &= \emptyset \\ (bb)^{-1}L &= \emptyset. \end{aligned}$$

# Solution 5 (cont'd)

$$
\begin{aligned}
(aaa)^{-1}L &= \{a^{n-3}b^n \mid n \geqslant 3\} \\
(aab)^{-1}L &= \{b\} \\
(aba)^{-1}L &= \emptyset \\
(abb)^{-1}L &= \emptyset \\
(baa)^{-1}L &= \emptyset \\
(bab)^{-1}L &= \emptyset \\
(bba)^{-1}L &= \emptyset \\
(bbb)^{-1}L &= \emptyset
\end{aligned}
$$

# Solution 5 (cont'd)

In general,

$$\begin{aligned} (a^k)^{-1}L &= \{a^{n-k}b^n \mid n \geqslant 3\} \\ (a^h b^p)^{-1}L &= \{b^{h-p}\}\text{if } p \leqslant h \\ (a^h b^p)^{-1}L &= \emptyset\text{if } p \leqslant h. \end{aligned}$$