Problem Session 3

Prof. Dan A. Simovici

UMB

Problem 1:

Let $A = \{a, b\}$ be an alphabet. Define the language L as follows:

- $\lambda \in L$.
- if $x \in L$, then $axb \in L$.
- if $x \in L$ and $y \in L$, then $xy \in L$.

1. Show that if $x \in L$, then x is a word that has exactly as many a's as b's, that is, $n_a(x) = n_b(x)$.
2. Show that L is not regular.

Solution 1:

The proof is by induction of words.
The base step, $x = \lambda$ is immediate because $n_a(\lambda) = n_b(\lambda) = 0$.
The inductive step has two subcases.
Suppose that $x \in L$ and $n_a(x) = n_b(x)$. Then, $axb \in L$ and $n_a(axb) = 1 + n_a(x) = 1 + n_b(x) = n_b(axb)$, which completes the proof for this subcase.
Suppose that x and y belong to L, so $n_a(x) = n_b(x)$ and $n_a(y) = n_b(y)$. Then,
$n_a(xy) = n_a(x) + n_a(y) = n_b(x) + n_b(y) = n_b(xy)$, which completes the proof for the second subcase.

Solution 1 (cont'd):

Suppose that L were regular. Since the language $\{a\}^*\{b\}^*$ is regular, the intersection $L \cap \{a\}^*\{b\}^*$ would be regular. But, we have:

$$L \cap \{a\}^*\{b\}^* = \{a^n b^n \mid n \in N\}$$

and we know that the language $\{a^n b^n \mid n \in N\}$ is not regular.

Problem 2:

Prove that the language $\{x \in \{a, b\}^* \mid n_a(x) \leq n_b(x)\}$ is not regular.

Solution 2:

Suppose that L were regular and let n_0 be a pumping threshold for L. Choose $p > n_0$. Then, if $x = a^p b^q$ with $p \leq q$, x can be factored as $x = uvw$ with $|uv| \leq n_0 < p$ such that $uv^n w \in L$ for every n.

By the choice of p, the words u and v consist only of as, so we can write $u = a^k$, $v = a^\ell$ and $w = a^h b^q$, where $k + \ell + h = p$ and $\ell \geq 1$. Thus,

$$uv^n w = a^{k+n\ell+h} b^q \in L \text{ for every } n \in \mathbb{N}.$$

By choosing n such that $p + (n-1)\ell > q$, that is $n > \frac{q-p}{\ell} + 1$, we obtain a word that violates the definition of L, which requires every word of L to contain more bs than as.

Problem 3:

Prove that the language $L = \{x \in \{a, b\}^* \mid n_a(x) | n_b(x)\}$ is not regular.

Solution 3:

Suppose that L were regular. In this case, $L \cap \{a\}^*\{b\}^* = \{a^n b^{kn} \mid k, n \in \mathbb{N}\}$ would be regular.
This would imply that a number n_0 exists such that if $n + kn > n_0$ we could write

$$a^n b^{kn} = uvw$$

with $|uv| < n_0$ such that $uv^n w \in L$ for every $n \in \mathbb{N}$.
Let $n < n_0$. Then, v consists only of as and pumping v would increase only the as but will not affect the bs; thus, the number of as would exceed the number of bs.