Problem Session 4

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Problem 1

Prove that the language

$$L = \{a^n b^m c^k \mid n + m = k\}$$

is context-free.

Solution 1

Let $G = (\{S, X\}, \{a, b, c\}, S, P)$ be the context-free grammar defined by

$$P = \{S \to aSc, S \to X, S \to \lambda, X \to bXc, X \to \lambda\}.$$

It is easy to see that $L = L(G)$.

Problem 2

Prove that the language

$$L = \{a^i b^j \mid i, j \in \mathbb{N} \text{ and } i \neq j\}$$

is context-free.

Solution 2

Consider the cfg $G = (\{S, A, B, C\}, \{a, b\}, S, P)$ where P consists of the following productions:

$$S \to A, S \to B$$
$$A \to aA, A \to aC$$
$$B \to Bb, B \to Cb$$
$$C \to \lambda, C \to aCb.$$

If $C \overset{*}{\underset{G}{\Rightarrow}} w$, then $n_a(w) = n_b(w)$, so the numbers of as and bs in w are equal. If $A \overset{*}{\underset{G}{\Rightarrow}} u$, then $n_a(u) > n_b(u)$, and if $B \overset{*}{\underset{G}{\Rightarrow}} v$, then $n_a(v) < n_b(v)$. Derivations fall into one of two cases: either $S \underset{G}{\Rightarrow} A \overset{*}{\underset{G}{\Rightarrow}} t$ or $S \underset{G}{\Rightarrow} B \overset{*}{\underset{G}{\Rightarrow}} t$. In the first case $n_a(t) > n_b(t)$; in the second case $n_a(t), n_b(t)$.

Linear Grammars

Let $G = (A_N, A_T, S, P)$ be a context-free grammar. G is *linear* if all its productions have the form $X \to uYv$ or $X \to u$, where $X, Y \in A_N$ and $u, v \in A_T^*$.
G is *left-linear* (*right-linear*) if its productions have the form $X \to Yv$ or $X \to v$ ($X \to uY$ or $X \to u$, respectively).
If G is a linear grammar we will refer to $L = L(G)$ as a *linear language*. Clearly, a type-3 grammar is right-linear, so a type-3 language is a right-linear language.

Problem 3

Prove that the intersection of two linear context-free languages is not necessarily a context-free language.

Solution 3

Consider the linear context-free languages $L = \{a^n b^n c^m \mid m, n \in \mathbb{N}\}$ and $K = \{a^n b^m c^m \mid m, n \in \mathbb{N}\}$. Their intersection is not a context-free language.

Problem 4

Prove that the language

$$L = \{a^n b^m c^p \mid n \neq m \text{ or } m \neq k\}$$

is context-free.

Solution 4

There are four cases to consider for a word $w \in L$:

1. $n_a(w) > n_b(w)$ and any number of cs,
2. $n_b(w) > n_a(w)$ and any number of cs,
3. $n_b(w) > n_c(w)$ and any number of as, and
4. $n_c(w) > n_b(w)$ and any number of as, and

Solution 4 (cont'd)

Let G be the context-free grammar

$$G = (\{S, S_1, S_2, S_3, S_4, S_5, S_6\}, \{a, b, c\}, S, P),$$

where P consists of the following productions:

$$S \to S_1 S_3, S \to S_2 S_3, S \to S_4 S_5, S \longrightarrow S_4 S_6$$

and

$$\begin{array}{l} S_1 \to aS_1 b, S_1 \to aS_1, S \to a, \\ S_2 \to aS_2 b, S_2 \to S_2 b, S_2 \to b, \\ S_3 \to S_3 c, S \to \lambda, \\ S_4 \to aS_4, S_4 \to \lambda, \\ S_5 \to bS_5 c, S_5 \to bS_6, S_5 \to b, \\ S_6 \to bS_6 c, S_6 \to S_6 c, S \to c. \end{array}$$

Case (i) is dealt with by begining a derivation with the production $S \to S_1 S_3$, case (ii) corresponds to a derivation that begins with $S \longrightarrow S_2 S_3$, etc.

Problem 5

We saw that the language $H = \{a^n b^n c^n \mid n \in \mathbb{N}\}$ is not context-free. Prove that its complement is context-free.

Solution 5

Note that a word x belongs to $\overline{H} = \{a, b, c\}^* - H$ if either it does not belong to $a^*b^*c^*$, or if it does belong to $a^*b^*c^*$ either the number of as is different from the number of bs, or the number of bs is different from the number of cs. Thus, we can write

$$\overline{H} = \overline{a^*b^*c^*} \cup L,$$

where L is the context-free language defined in Problem 4. Thus, $\overline{H}$ is a context-free language.