Pushdown Automata - II (part II)

Prof. Dan A. Simovici

UMB

1. An Alternative Method of Language Acceptance by PDAs

Another method for associating a language with a pda is to consider the language that consists of those input words for which there is a computation that leads to the emptying of the pushdown store. This is captured by the following definition.

Definition

Let $\mathcal{M} = (A, Z, Q, \delta, q_0, z_0, F)$ be a pda. The *language accepted by $\mathcal{M}$ with an empty store* is given by

$$N(\mathcal{M}) = \{x \in A^* \mid (z_0, q_0, x) \overset{*}{\underset{\mathcal{M}}{\vdash}} (\lambda, q, \lambda) \text{ for some } q \in Q\}.$$

The set F plays no role in the definition of $N(\mathcal{M})$.

Theorem

For every pda $\mathcal{M}$ *there is a pda* $\mathcal{M}'$ *such that* $L(\mathcal{M}) = N(\mathcal{M}')$.

Proof

Let $\mathcal{M} = (A, Z, Q, \delta, q_0, z_0, F)$. We have

$$L(\mathcal{M}) = \{x \in A^* \mid (z_0, q_0, x) \overset{*}{\underset{\mathcal{M}}{\vdash}} (w, q, \lambda) \text{ for some } w \in Z^* \text{ and } q \in F\}.$$

Define the pda $\mathcal{M}' = (A, Z \cup \{z'\}, Q \cup \{q', q_0'\}, \delta', q_0', z', \emptyset)$, where q', q_0' are two new states, and z' is a new initial pushdown symbol, where $z' \notin Z$. The transition function δ' is given by

$$\delta'(z, q, a) = \begin{cases} \{(z'z_0, q_0)\} & \text{if } (z, q, a) = (z', q_0', \lambda), \\ \delta(z, q, a) & \text{if } q \in Q - F, a \in A \cup \{\lambda\}, z \in Z, \\ \delta(z, q, a) \cup \{(\lambda, q')\} & \text{if } q \in F, z \in Z \cup \{z'\}, a \in A \cup \{\lambda\}, \\ \{(\lambda, q')\} & \text{if } q = q', z \in Z \cup \{z'\}, a = \lambda, \\ \emptyset & \text{in any other case.} \end{cases}$$

The symbol z' was introduced for the pda $\mathcal{M}'$ since some words in $A^* - L(\mathcal{M})$ may empty the pushdown store of $\mathcal{M}$. The presence of z' at the bottom of the pushdown store makes this impossible in $\mathcal{M}'$.
Since $\delta(z', q_0', \lambda) = \{(z'z_0, q_0)\}$, $\mathcal{M}'$ begins its work by entering the state q_0 and by placing z_0 at the top of the pushdown store. If $x \in L(\mathcal{M})$, then $(z_0, q_0, x) \overset{*}{\underset{\mathcal{M}}{\vdash}} (w, q, \lambda)$ for some $w \in Z^*$ and $q \in F$. Correspondingly, in $\mathcal{M}'$ we have

$$(z', q_0', x) \underset{\mathcal{M}'}{\vdash} (z'z_0, q_0, x) \overset{*}{\underset{\mathcal{M}'}{\vdash}} (z'w, q, \lambda) \overset{*}{\underset{\mathcal{M}'}{\vdash}} (\lambda, q', \lambda),$$

by the definition of δ'. This implies $L(\mathcal{M}) \subseteq N(\mathcal{M}')$.

(Proof cont'd)

To prove the converse inclusion, let $x \in N(\mathcal{M}')$, so $(z', q'_0, x) \overset{*}{\underset{\mathcal{M}'}{\vdash}} (\lambda, q, \lambda)$ for some state $q \in Q \cup \{q', q'_0\}$. The definition of δ' implies that this computation necessarily has the form

$$(z', q'_0, x) \underset{\mathcal{M}'}{\vdash} (z' z_0, q_0, x) \overset{*}{\underset{\mathcal{M}'}{\vdash}} (\lambda, q, \lambda),$$

since there exists only one transition for the triple (z', q', λ), namely $(z' z_0, q_0)$. Note that the symbol z' can be erased by $\mathcal{M}'$ only if this pda reaches a state $q \in F \cup \{q'\}$. Let u be the suffix of x that remains to be read when $\mathcal{M}'$ reached the state q' for the first time. Since $\mathcal{M}'$ enters q' only from a final state q_1 of $\mathcal{M}$ we have:

$$(z', q'_0, x) \underset{\mathcal{M}'}{\vdash} (z' z_0, q_0, x) \overset{*}{\underset{\mathcal{M}'}{\vdash}} (w, q_1, u) \underset{\mathcal{M}'}{\vdash} (w', q', u) \overset{*}{\underset{\mathcal{M}'}{\vdash}} (\lambda, q, \lambda),$$

(Proof cont'd)

Once $\mathcal{M}'$ enters the state q' no symbol is read from the input, so we have $u = \lambda$. This allows us to write the previous computation as

$$(z', q_0', x) \underset{\mathcal{M}'}{\vdash} (z'z_0, q_0, x) \overset{*}{\underset{\mathcal{M}'}{\vdash}} (w, q_1, \lambda) \underset{\mathcal{M}'}{\vdash} (w', q', \lambda) \overset{*}{\underset{\mathcal{M}'}{\vdash}} (\lambda, q, \lambda),$$

and this implies the existence of the computation

$$(z_0, q_0, x) \overset{*}{\underset{\mathcal{M}}{\vdash}} (w, q_1, \lambda),$$

which, in turn, implies $x \in L(\mathcal{M})$. This proves the needed inclusion $N(\mathcal{M}') \subseteq L(\mathcal{M})$.

Theorem

For every context-free grammar G there is a one-state pda $\mathcal{M}$ such that $L = N(\mathcal{M})$.

Suppose that $L = L(G)$, where $G = (A_N, A_T, S, P)$ is a context-free grammar. Let $\mathcal{M} = (A_T, A_N \cup A_T, \{q_0\}, \delta, q_0, S, \emptyset)$ be a pda whose transition function is given by

$$\begin{aligned} \delta(X, q_0, \lambda) &= \{(\alpha^R, q_0) \mid X \to \alpha \in P\}, \\ \delta(a, q_0, a) &= \{(\lambda, q_0)\}, \end{aligned}$$

for every $a \in A_T$, $X \in A_N$, and $\delta(s, q_0, a) = \emptyset$ in all other cases.
Let

$$S = \gamma_0 \underset{G}{\Rightarrow} \gamma_1 \underset{G}{\Rightarrow} \cdots \underset{G}{\Rightarrow} \gamma_n = u\alpha$$

be a leftmost derivation of $u\alpha$ in G, where $u \in A_T^*$ and $\alpha \in (A_N \cup A_T)^*$ is either the null word or a word that begins with a nonterminal symbol. We claim that $(S, q_0, uw) \overset{*}{\underset{\mathcal{M}}{\vdash}} (\alpha^R, q_0, w)$ for every $w \in A_T^*$. The argument is by induction on n.

For $n = 0$, we have $u = \lambda$ and $\alpha = S$. Thus, the claim is simply

$$(S, q_0, w) \overset{*}{\underset{\mathcal{M}}{\vdash}} (S, q_0, w),$$

which follows from the definition of $\overset{*}{\underset{\mathcal{M}}{\vdash}}$.

For the induction step suppose that

$$S = \gamma_0 \underset{G}{\Rightarrow} \cdots \underset{G}{\Rightarrow} \gamma_n \underset{G}{\Rightarrow} \gamma_{n+1} = u\alpha$$

is a leftmost derivation, where $\gamma_n = u'X\theta$ and $\gamma_{n+1} = u'u''\beta\theta$. In other words, the last step of the derivation uses the production $X \to u''\beta$, where $u'' \in A_T^*$ and $\beta \in (A_N \cup A_T)^*$ is either the null word or a word that begins with a nonterminal symbol.

Thus, the derivation above may be written

$$S \overset{n}{\underset{G}{\Rightarrow}} u'X\theta \underset{G}{\Rightarrow} u'u''\beta\theta,$$

and we have the following computation of $\mathcal{M}$:

$$\begin{aligned}
(S, q_0, u'u''w) &\overset{*}{\vdash} ((X\theta)^R, q_0, u''w) = (\theta^R X, u''w) \\
&\quad \text{(by the inductive hypothesis)} \\
&\vdash (\theta^R \beta^R u''^R, q_0, u''w) \\
&\quad \text{(since } (\beta^R u''^R, q_0) \in \delta(X, q_0, \lambda)) \\
&\overset{*}{\vdash} (\theta^R \beta^R, q_0, w).
\end{aligned}$$

The last line follows from the observation that $\delta(a, q_0, a) = \{(\lambda, q_0)\}$ for each $a \in A_T$ implies that $(x^R, q_0, x) \overset{*}{\vdash} (\lambda, q_0, \lambda)$ for every $x \in A_T^*$. Since $\theta^R\beta^R = (\beta\theta)^R$, we have completed the induction step. Therefore, if $u \in L(G)$ we have $S \overset{*}{\underset{G}{\Rightarrow}} u$, and this implies $(S, q_0, u) \overset{*}{\vdash} (\lambda, q_0, \lambda)$, which shows that $u \in N(\mathcal{M})$, hence $L(G) \subseteq N(\mathcal{M})$.

To prove that $N(\mathcal{M}) \subseteq L(G)$, we show that $(X, q_0, u) \overset{*}{\underset{\mathcal{M}}{\vdash}} (\lambda, q_0, \lambda)$ implies $X \overset{*}{\underset{G}{\Rightarrow}} u$ for $X \in A_N$ and $u \in A_T^*$.
We factor the input word u into a series of subwords $u = u_0 u_1 \cdots u_{k-1}$, each corresponding to a certain change in the pushdown store. Specifically, the top symbol of the pushdown store of each step of the computation can be either a terminal or a nonterminal symbol. Any step at which a nonterminal is at the top determines the boundary between a u_i and its successor u_{i+1} in the input. Thus, u_i could be empty (when a nonterminal at the top is replaced by a nonterminal) or could contain several symbols (when there are terminal symbols at the top that are popped off by transitions of the form $(\lambda, q_0) \in \delta(a, q_0, a)$).

Thus, we can write $u = u_0 u_1 \cdots u_{k-1}$, where $u_i \in A_T^*$ for $0 \leq i \leq k-1$, and

$$\begin{aligned}
(X, q_0, u) &= (\gamma_0, q_0, u_0 u_1 \cdots u_{k-1}) \\
&\vdash^* (\gamma_1, q_0, u_1 \cdots u_{k-1}) \\
&\vdots \\
&\vdash^* (\gamma_{k-1}, q_0, u_{k-1}) \\
&\vdash^* (\lambda, q_0, \lambda),
\end{aligned}$$

where each γ_i has the form $\gamma_i' X$ for $\gamma_i' \in (A_T \cup A_N)^*$ and $X \in A_N$.

The definition of $\mathcal{M}$ implies that the computation

$$(\gamma_i, q_0, u_i \cdots u_{k-1}) \overset{*}{\underset{\mathcal{M}}{\vdash}} (\gamma_{i+1}, q_0, u_{i+1} \cdots u_{k-1})$$

can be written as

$$\begin{array}{rcl}(\gamma_i, q_0, u_i \cdots u_{k-1}) & = & (\gamma_i' X_{p_i}, q_0, u_i \cdots u_{k-1}) \\ & \vdash & (\gamma_i' {\alpha_{p_i}}^R, q_0, u_i \cdots u_{k-1}) \\ & \overset{*}{\vdash} & (\gamma_{i+1}, q_0, u_{i+1} \cdots u_{k-1}),\end{array}$$

where $X_{p_i} \to \alpha_{p_i} = u_i \beta_{p_i}$ is a production of G such that $\beta_{p_i} \in (A_N \cup A_T)^*$ is the null word or a word that begins with a nonterminal symbol, and $\gamma_{i+1} = \gamma_i' {\beta_{p_i}}^R$ for $0 \leq i \leq k-1$.
We prove by induction on ℓ that we have the leftmost derivation

$${\gamma_{k-1-\ell}}^R \overset{*}{\underset{G}{\Rightarrow}} u_{k-1-\ell} \cdots u_{k-1}.$$

For $\ell = 0$ we have

$$(\gamma_{k-1}, q_0, u_{k-1}) \underset{\mathcal{M}}{\vdash} ({u_{k-1}}^R, q_0, u_{k-1}) \overset{*}{\underset{\mathcal{M}}{\vdash}} (\lambda, q_0, \lambda),$$

because γ_{k-1} is the last content of the pushdown store that may contain a nonterminal, which means that $\gamma_{k-1} = X \in A_N$ and $X \to u_{k-1} \in P$. Therefore, ${\gamma_{k-1}}^R = X \overset{*}{\underset{G}{\Rightarrow}} u_{k-1}$.

Suppose that ${\gamma_{i+1}}^R \overset{*}{\underset{G}{\Rightarrow}} u_{i+1} \cdots u_{k-1}$; that is, $\beta_{p_i} {\gamma'_i}^R \overset{*}{\underset{G}{\Rightarrow}} u_{i+1} \cdots u_{k-1}$.
This implies $u_i \beta_{p_i} {\gamma'_i}^R \overset{*}{\underset{G}{\Rightarrow}} u_i u_{i+1} \cdots u_{k-1}$, so $\alpha_{p_i} {\gamma'_i}^R \overset{*}{\underset{G}{\Rightarrow}} u_i u_{i+1} \cdots u_{k-1}$.
The existence of the production $X_i \to \alpha_{p_i}$ allows us to write

$$X_i {\gamma'_i}^R \underset{G}{\Rightarrow} \alpha_{p_i} {\gamma'_i}^R \overset{*}{\underset{G}{\Rightarrow}} u_i u_{i+1} \cdots u_{k-1},$$

and $X_i {\gamma'_i}^R = (\gamma'_i X_i)^R = {\gamma_i}^R$.
Choosing $X = S$ we conclude that $x \in N(\mathcal{M})$ implies
$(S, q_0, u) \overset{*}{\underset{\mathcal{M}}{\vdash}} (\lambda, q_0, \lambda)$, which in turn, implies $S \overset{*}{\underset{G}{\Rightarrow}} u$ and $u \in L(G)$.

Note that a computation of the pda $\mathcal{M}$ that leads to the acceptance of a word u uniquely defines a leftmost derivation in the grammar G.
Consider the nonambiguous context-free grammar

$$G_{ae} = (\{X_e, X_t, X_f\}, \{+, -, *, /, \mathbf{(}, \mathbf{)}, \boldsymbol{v}, \boldsymbol{n}\}, X_e, P)$$

introduced before which generates the language of parenthesized arithmetic expressions. The transitions in the pda are:

Production in G_{ae}	Transitions in $\delta(X, q_0, \lambda)$ where X is the left member of production
$\pi_0 : X_e \rightarrow X_e + X_t$	$(X_t + X_e, \lambda)$
$\pi_1 : X_e \rightarrow X_e - X_t$	$(X_t - X_e, \lambda)$
$\pi_2 : X_e \rightarrow X_t$	(X_t, λ)
$\pi_3 : X_t \rightarrow X_t * X_f$	$(X_f * X_t, \lambda)$
$\pi_4 : X_t \rightarrow X_t / X_f$	$(X_f / X_t, \lambda)$
$\pi_5 : X_t \rightarrow X_f$	(X_f, λ)
$\pi_6 : X_f \rightarrow \boldsymbol{v}$	$(\boldsymbol{v}, \lambda)$
$\pi_7 : X_f \rightarrow \boldsymbol{n}$	$(\boldsymbol{n}, \lambda)$
$\pi_8 : X_f \rightarrow \mathbf{(}X_e\mathbf{)}$	$(\mathbf{)}X_e\mathbf{(}, \lambda)$

The pda that accepts the language $L(G_{ae})$ with an empty pushdown store is:

$$\begin{aligned}\mathcal{M} \;=\; & (\{+,-,*,/,\mathbf{(},\mathbf{)},\boldsymbol{v},\boldsymbol{n}\},\{X_e,X_t,X_f,+,-,*,/,\mathbf{(},\mathbf{)},\boldsymbol{v},\boldsymbol{n}\},\\ & \{q_0\},\delta,q_0,X_e,\emptyset),\end{aligned}$$

where δ is specified by the table in the next slide.

(Example cont'd)

Top	State	Input	Transition Function
z	q	a	$\delta(z, q, a)$
X_e	q_0	λ	$\{(X_t + X_e, q_0), (X_t - X_e, q_0), (X_t, q_0)\}$
X_t	q_0	λ	$\{(X_f * X_t, q_0), (X_f / X_t, q_0), (X_f, q_0)\}$
X_f	q_0	λ	$\{(\boldsymbol{v}, q_0), (\boldsymbol{n}, q_0), (\mathbf{)}X_e\mathbf{(}, q_0)\}$
a	q_0	a	$\{(\lambda, q_0)\}$

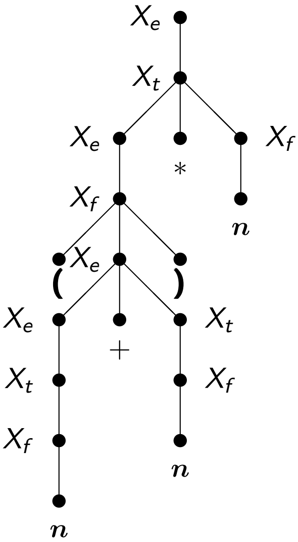
The last line of the table applies to every symbol $a \in \{+, -, *, /, (,), \boldsymbol{v}, \boldsymbol{n}\}$.
If $\delta(z, q_0, a)$ is not mentioned in the table, then $\delta(z, q_0, a) = \emptyset$.

(Example cont'd)

The word $(\boldsymbol{n}+\boldsymbol{n})*\boldsymbol{n}$ can be generated in G_{ae} using the leftmost derivation

$$\begin{aligned} X_e &\underset{\pi_2}{\Rightarrow} X_t \underset{\pi_3}{\Rightarrow} X_t * X_f \underset{\pi_5}{\Rightarrow} X_f * X_f \\ &\underset{\pi_8}{\Rightarrow} \boldsymbol{(}X_e\boldsymbol{)} * X_f \underset{\pi_0}{\Rightarrow} \boldsymbol{(}X_e + X_t\boldsymbol{)} * X_f \\ &\underset{\pi_2}{\Rightarrow} \boldsymbol{(}X_t + X_t\boldsymbol{)} * X_f \underset{\pi_5}{\Rightarrow} \boldsymbol{(}X_f + X_t\boldsymbol{)} * X_f \underset{\pi_7}{\Rightarrow} \boldsymbol{(n} + X_t\boldsymbol{)} * X_f \\ &\underset{\pi_5}{\Rightarrow} \boldsymbol{(n} + X_f\boldsymbol{)} * X_f \underset{\pi_7}{\Rightarrow} \boldsymbol{(n+n)} * X_f \underset{\pi_7}{\Rightarrow} \boldsymbol{(n+n)} * \boldsymbol{n}. \end{aligned}$$

that corresponds to the derivation tree shown next:


X_e
X_t
X_e
$*$
X_f
X_f
$\boldsymbol{n}$
(X_e
)
X_e
$+$
X_t
X_t
X_f
X_f
$\boldsymbol{n}$
$\boldsymbol{n}$

The computation that leads to the acceptance of the word $\mathbf{(}n+n\mathbf{)}*n$ in $\mathcal{M}$ is

$$
\begin{aligned}
&(X_e, q_0, \mathbf{(}n+n\mathbf{)}*n) \underset{\mathcal{M}}{\vdash} (X_t, q_0, \mathbf{(}n+n\mathbf{)}*n)\\
&\underset{\mathcal{M}}{\vdash} (X_f * X_t, q_0, \mathbf{(}n+n\mathbf{)}*n) \underset{\mathcal{M}}{\vdash} (X_f * X_f, q_0, \mathbf{(}n+n\mathbf{)}*n)\\
&\underset{\mathcal{M}}{\vdash} (X_f * \mathbf{)}X_e\mathbf{(}, q_0, \mathbf{(}n+n\mathbf{)}*n) \underset{\mathcal{M}}{\vdash} (X_f * \mathbf{)}X_e, q_0, n+n\mathbf{)}*n)\\
&\underset{\mathcal{M}}{\vdash} (X_f * \mathbf{)}X_t + X_e, q_0, n+n\mathbf{)}*n) \underset{\mathcal{M}}{\vdash} (X_f * \mathbf{)}X_t + X_t, q_0, n+n\mathbf{)}*n)\\
&\underset{\mathcal{M}}{\vdash} (X_f * \mathbf{)}X_t + X_f, q_0, n+n\mathbf{)}*n) \underset{\mathcal{M}}{\vdash} (X_f * \mathbf{)}X_t + n, q_0, n+n\mathbf{)}*n)\\
&\underset{\mathcal{M}}{\vdash} (X_f * \mathbf{)}X_t+, q_0, +n\mathbf{)}*n) \underset{\mathcal{M}}{\vdash} (X_f * \mathbf{)}X_t, q_0, n\mathbf{)}*n)\\
&\underset{\mathcal{M}}{\vdash} (X_f * \mathbf{)}X_f, q_0, n\mathbf{)}*n) \underset{\mathcal{M}}{\vdash} (X_f * \mathbf{)}n, q_0, n\mathbf{)}*n) \underset{\mathcal{M}}{\vdash} (X_f * \mathbf{)}, q_0, \mathbf{)}*n)\\
&\underset{\mathcal{M}}{\vdash} (X_f *, q_0, *n) \underset{\mathcal{M}}{\vdash} (X_f, q_0, n) \underset{\mathcal{M}}{\vdash} (n, q_0, n) \underset{\mathcal{M}}{\vdash} (\lambda, q_0, \lambda)
\end{aligned}
$$

For every pda $\mathcal{M}$ the language $N(\mathcal{M})$ is context-free.

We need the following technical result showing that whenever there is a pda that accepts a language with an empty store, then there is a way to construct a pda that accepts the same language both with an empty store and by entering a final accepting state.

Theorem

For every pda $\mathcal{M} = (A, Z, Q, \delta, q_0, z_0, F)$ *there exists a pda* $\mathcal{M}' = (A, Z', Q', \delta', q_0', z_0', \{q'\})$ *such that* $(z_0', q_0', x) \overset{*}{\underset{\mathcal{M}'}{\vdash}} (\lambda, q_1, \lambda)$ *implies* $q_1 = q'$ *and* $N(\mathcal{M}) = N(\mathcal{M}') = L(\mathcal{M}')$.

Proof

Pick $q_0', q' \notin Q$ and $z_0' \notin Z$. Define the pda

$$\mathcal{M}' = (A, Z', Q', \delta', q_0', z_0', \{q'\})$$

as $Q' = Q \cup \{q_0', q'\}$, $Z' = Z \cup \{z'\}$, $\delta'(z_0', q_0', \lambda) = \{(z_0' z_0, q_0)\}$, $\delta'(z_0', q, \lambda) = \{(\lambda, q')\}$ for every $q \in Q$, and $\delta'(z, q, a) = \delta(z, q, a)$ in every other case. In other words, $\mathcal{M}'$ begins by putting a marker, z_0', onto the pushdown store and then simulating $\mathcal{M}$ until $\mathcal{M}$ would have emptied its pushdown store. At this time $\mathcal{M}'$ removes the marker, thus emptying its store, and goes into a final state.

(Proof cont'd)

Let $x \in N(\mathcal{M})$. We have $(z_0, q_0, x) \overset{*}{\underset{\mathcal{M}}{\vdash}} (\lambda, q, \lambda)$ for some $q \in Q$. Therefore, in $\mathcal{M}'$ we have the computation

$$(z_0', q_0', x) \underset{\mathcal{M}'}{\vdash} (z_0' z_0, q_0, x) \overset{*}{\underset{\mathcal{M}'}{\vdash}} (z_0', q, \lambda) \underset{\mathcal{M}'}{\vdash} (\lambda, q', \lambda),$$

so $x \in N(\mathcal{M}')$ and $x \in L(\mathcal{M}')$, which shows that $N(\mathcal{M}) \subseteq N(\mathcal{M}')$ and $N(\mathcal{M}) \subseteq L(\mathcal{M}')$.

(Proof cont'd)

Conversely, suppose that $x \in N(\mathcal{M}')$ or that $x \in L(\mathcal{M}')$.

In the first case, $(z_0', q_0', x) \overset{*}{\underset{\mathcal{M}'}{\vdash}} (\lambda, \bar{q}, \lambda)$ for some state $\bar{q} \in Q'$. The definition of $\mathcal{M}'$ implies that this computation can be written as

$$(z_0', q_0', x) \underset{\mathcal{M}'}{\vdash} (z_0' z_0, q_0, x) \overset{*}{\underset{\mathcal{M}'}{\vdash}} (\lambda, \bar{q}, \lambda).$$

Note that in $\mathcal{M}'$ the symbol z_0' cannot be erased unless $\mathcal{M}'$ switches to the state q'. Therefore, in the previous computation we have $\bar{q} = q'$, and this computation can be written as

$$(z_0', q_0', x) \underset{\mathcal{M}'}{\vdash} (z_0' z_0, q_0, x) \overset{*}{\underset{\mathcal{M}'}{\vdash}} (z_0', q, \lambda) \underset{\mathcal{M}'}{\vdash} (\lambda, q', \lambda)$$

for some $q \in Q$. Thus, we must have $(z_0, q_0, x) \overset{*}{\underset{\mathcal{M}}{\vdash}} (\lambda, q, \lambda)$, that is $x \in N(\mathcal{M})$.

In the second case, $x \in L(\mathcal{M}')$ implies $(z_0', q_0', x) \overset{*}{\underset{\mathcal{M}'}{\vdash}} (w, q', \lambda)$. Observe that $\mathcal{M}'$ may enter its final state q' only by erasing the symbol z_0' located at the bottom of the pushdown store. This implies that the above computation has the form

$$(z_0', q_0', x) \underset{\mathcal{M}}{\vdash} (z_0' z_0, q_0, x) \overset{*}{\underset{\mathcal{M}'}{\vdash}} (z_0', q, \lambda) \underset{\mathcal{M}'}{\vdash} (\lambda, q', \lambda).$$

As before, this implies the existence of the computation $(z_0, q_0, x) \overset{*}{\underset{\mathcal{M}}{\vdash}} (\lambda, q, \lambda)$, so $x \in N(\mathcal{M})$.
We proved that $N(\mathcal{M}') \subseteq N(\mathcal{M})$ and $L(\mathcal{M}') \subseteq N(\mathcal{M})$. Thus, $N(\mathcal{M}) = N(\mathcal{M}') = L(\mathcal{M}')$, which is the desired conclusion.

Theorem

If L is a language such that $L = N(\mathcal{M})$ for some pda $\mathcal{M}$, then L is a context-free language.

Proof

Suppose that $L = N(\mathcal{M})$, where $\mathcal{M} = (A, Z, Q, \delta, q_0, F)$ is a pda. By Theorem 4 we can assume without loss of generality that $F = \{q_f\}$ and that $L = \{x \in A^* \mid (z_0, q_0, x) \overset{*}{\underset{\mathcal{M}}{\vdash}} (\lambda, q_f, \lambda)\}$.

Consider the alphabet $\hat{Z} = \{z^{q_i q_j} \mid z \in Z, q_i, q_j \in Q\}$ and the context-free grammar $G = (\hat{Z}, A, z_0^{q_0 q_f}, P)$, whose set of productions P is constructed as follows:

- If $(z_{i_k} \cdots z_{i_0}, p) \in \delta(z, q, a)$, then place the following productions into P:

$$z^{qq_{i_k}} \rightarrow a z_{i_0}^{pq_{i_0}} z_{i_1}^{q_{i_0} q_{i_1}} \cdots z_{i_k}^{q_{i_{k-1}} q_{i_k}},$$

  for every $q_{i_0}, \cdots, q_{i_k} \in Q$.
- If $(\lambda, p) \in \delta(z, q, a)$, then place the production $z^{qp} \rightarrow a$ into P.

Define the relation $\rho \subseteq \hat{Z} \times \hat{Z}$ by $(z_m^{q_i q_j}, z_n^{q_k q_h}) \in \rho$ if and only if $q_j = q_k$ and consider the regular language $H = L_\rho$. Let $d : \hat{Z}^* \longrightarrow Z^*$ be the morphism defined by $d(z^{q_i q_j}) = z$ for every $z^{q_i q_j} \in \hat{Z}$.

(Proof cont'd)

We prove that for $n \geq 1$, we have the leftmost derivation $z^{q_i q_j} \overset{n}{\underset{G}{\Rightarrow}} w\alpha$ (where $w \in A^*$ and $\alpha \in \hat{Z}^*$) if and only if $(z, q_i, wy) \overset{n}{\underset{\mathcal{M}}{\vdash}} (d(\alpha)^R, p, y)$ and one of the following conditions is satisfied:

- $\alpha \in H$, the first symbol of α has the form z^{pq}, and the last symbol of α has the form $z^{q q_j}$, or
- $\alpha = \lambda$ and $p = q_j$.

The argument is by induction on n. For the basis step, $n = 1$, suppose that $z^{q_i q_j} \underset{G}{\Rightarrow} w\alpha$. The production applied for this one-step derivation is either $z^{q_i q_j} \to a z_{i_0}^{p q_{i_0}} z_{i_1}^{q_{i_0} q_{i_1}} \cdots z_{i_k}^{q_{i_{k-1}} q_j}$ which implies

$$w = a \text{ and } \alpha = z_{i_0}^{p q_{i_0}} z_{i_1}^{q_{i_0} q_{i_1}} \cdots z_{i_k}^{q_{i_{k-1}} q_j},$$

or is $z^{qp} \to a$, which implies

$$w = a \text{ and } \alpha = \lambda,$$

respectively.

The first case may occur if and only if $(z_{i_k} \cdots z_{i_0}, p) \in \delta(z, q_i, a)$. Therefore, we have

$$(z, q_i, ay) \underset{\mathcal{M}}{\vdash} (z_{i_k} \cdots z_{i_0}, p, y) = (d(\alpha)^R, p, y)$$

Also, the second case takes place if and only if $(\lambda, p) \in \delta(z, q_i, a)$ which is equivalent to

$$(z, q_i, ay) \underset{\mathcal{M}}{\vdash} (\lambda, p, y).$$

This concludes the basis step.

For the inductive step assume that the statement holds for n and consider a leftmost derivation of length $n+1$: $z^{q_i q_j} \overset{n+1}{\underset{G}{\Rightarrow}} w'\alpha'$. Two cases may occur depending on form of the production applied in the last step of this derivation:

- If the production applied in the last step was

  $$z^{qq_{j_\ell}} \rightarrow a z_{j_0}^{rq_{j_0}} z_{j_1}^{q_{j_0} q_{j_1}} \cdots z_{j_\ell}^{q_{j_{\ell-1}} q_{j_\ell}},$$

  then the derivation can be written as

  $$z^{q_i q_j} \overset{n}{\underset{G}{\Rightarrow}} w z^{qq_{j_\ell}} \alpha \underset{G}{\Rightarrow} w a z_{j_0}^{rq_{j_0}} z_{j_1}^{q_{j_0} q_{j_1}} \cdots z_{j_\ell}^{q_{j_{k-1}} q_{j_\ell}} \alpha. \qquad (1)$$

This takes place if and only if $w' = wa$, $\alpha' = z_{j_0}^{rq_{j_0}} z_{j_1}^{q_{j_0}q_{j_1}} \cdots z_{j_\ell}^{q_{j_{\ell-1}}q_{j_\ell}} \alpha$. By the inductive hypothesis, the first part of the derivation takes place if and only if $z^{qq_{j_\ell}}\alpha \in H$ and

$$(z, q_i, way) \overset{n}{\underset{\mathcal{M}}{\vdash}} (d(z^{qq_{j_\ell}}\alpha)^R, q, y) = (d(\alpha)^R z, q, ay).$$

The last step of the derivation can be executed if and only if

$$(z_{j_\ell} \cdots z_{j_0}, r) \in \delta(z, q, a),$$

by the definition of the grammar G. Thus, the derivation (1) takes place if and only if

$$\begin{aligned}
&(z, q_i, w'y) = (z, q_i, way) \overset{n}{\underset{\mathcal{M}}{\vdash}} (d(\alpha)^R z, q, ay)\\
&\underset{\mathcal{M}}{\vdash} (d(\alpha)^R z_{j_\ell} \cdots z_{j_0}, r, y)\\
&= (d(z_{j_0}^{rq_{j_0}} z_{j_1}^{q_{j_0}q_{j_1}} \cdots z_{j_\ell}^{q_{j_{\ell-1}}q_{j_\ell}} \alpha)^R, r, y)\\
&= (d(\alpha')^R, r, y).
\end{aligned}$$

- If the production applied in the last step of the derivation was $z^{qq_{j_\ell}} \rightarrow a$, the derivation can be written

$$z^{q_i q_j} \overset{n}{\underset{G}{\Rightarrow}} wz^{qq_{j_\ell}}\alpha \underset{G}{\Rightarrow} wa\alpha. \tag{2}$$

Thus $w' = wa$ and $\alpha' = \alpha$. By the inductive hypothesis we have:

$$(z, q_i, w'y) = (z, q_i, way) \overset{n}{\underset{\mathcal{M}}{\vdash}} (d(z^{qq_{j_\ell}}\alpha)^R, q, ay) = (d(\alpha)^R z, q, ay).$$

- The existence of the production $z^{qq_{j_\ell}} \to a$ is equivalent to $(\lambda, q_{j_\ell}) \in \delta(z, q, a)$, so the existence of the derivation (2) is equivalent to the existence of the computation

$$(z, q_0, w'y) \overset{n}{\underset{\mathcal{M}}{\vdash}} (d(\alpha)^R z, q, ay) \underset{\mathcal{M}}{\vdash} (d(\alpha)^R, q_{j_\ell}, y) = (d(\alpha')^R, q_{j_\ell}, y).$$

By taking $q_i = q_0$, $\alpha = \lambda$, $z = z_0$, $y = \lambda$, and $p = q_f$ in the initial claim, we conclude that a leftmost derivation $z^{q_0 q_f} \overset{n}{\underset{G}{\Rightarrow}} w$ exists if and only if

$$(z_0, q_i, w) \overset{n}{\underset{\mathcal{M}}{\vdash}} (\lambda, q_f, \lambda).$$

This shows that $L(G) = N(\mathcal{M})$, so $N(\mathcal{M})$ is indeed a context-free language.

Theorem

Let $L \subseteq A^*$ be a language over the alphabet A. The following statements are equivalent:

- *There is a pda $\mathcal{M}$ such that $L = L(\mathcal{M})$.*
- *There is a pda $\mathcal{M}$ such that $L = N(\mathcal{M})$.*
- *There is a pda $\mathcal{M}$ (having a single final state) such that $L = N(\mathcal{M}) = L(\mathcal{M})$.*
- *There is an one-state pda $\mathcal{M}$ such that $L = N(\mathcal{M})$.*
- *L is a context-free language.*