Finite Automata and Regular Languages (part I)

Prof. Dan A. Simovici

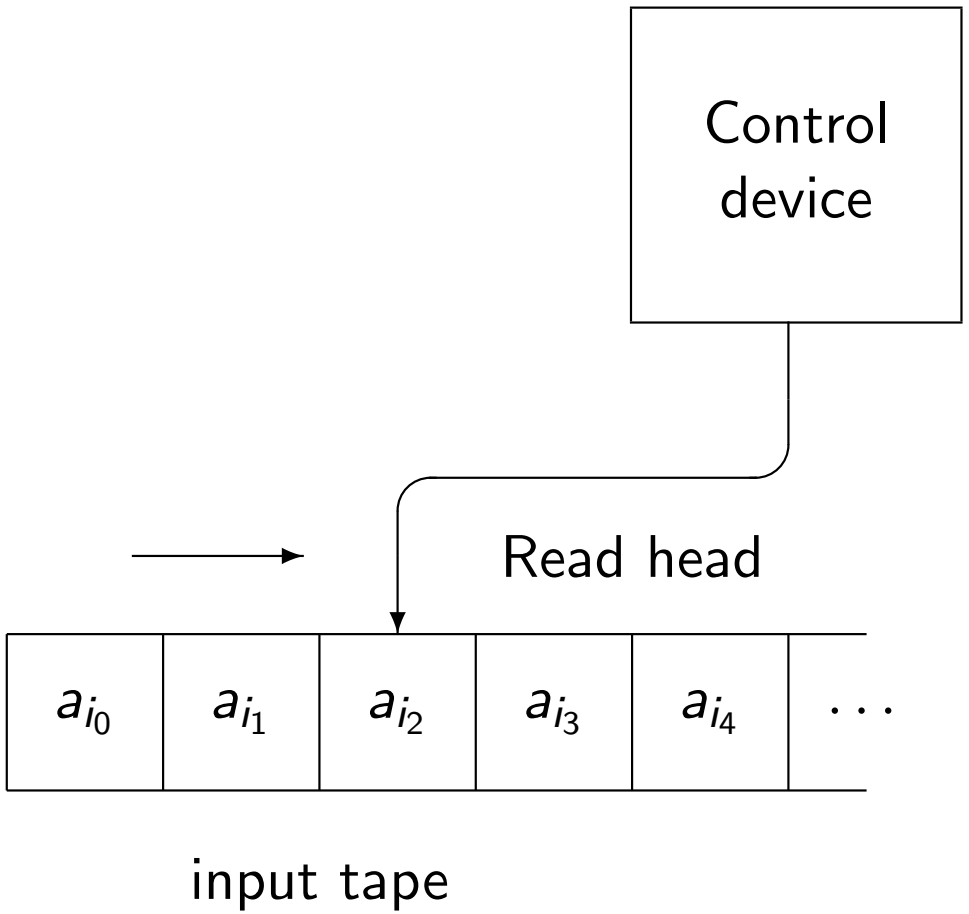
UMB

Informally, a deterministic finite automaton consists of:

- an input tape divided into cells;
- a control device equipped with a reading head that scans the input tape one cell at a time.

Each cell of the input tape contains a symbol $a \in A$, where A is an alphabet, called the input alphabet. The tape can accommodate words of arbitrary finite length. Thus, although the tape is thought of as being infinitely long, only a finite initial segment of it contains input symbols.

Main Components of a Finite Automaton



How a finite automaton works

- A dfa works discretely. Consider a clock that advances in discrete units; at any time on the clock, the automaton is resting in one of its states.
- Between two successive clock times, the automaton consumes its next available input and goes into a new state (which may happen to be the same state it was in at the previous time).
- The time scale of the automaton is the set $\mathbb{N}$ of natural numbers.

Definition

A deterministic finite automaton (dfa) is a quintuple

$$\mathcal{M} = (A, Q, \delta, q_0, F),$$

where A and Q are two finite, disjoint sets called the input alphabet of $\mathcal{M}$, and the set of states of $\mathcal{M}$, respectively, $\delta : Q \times A \longrightarrow Q$ is the transition function, q_0 is the initial state of $\mathcal{M}$, and $F \subseteq Q$ is the set of final states of $\mathcal{M}$.

Example

Let $\mathcal{M} = (\{a, b\}, \{q_0, q_1, q_2, q_3\}, \delta, q_0, \{q_3\})$ be the dfa defined by the following table:

Input	State			
	q_0	q_1	q_2	q_3
a	q_1	q_0	q_3	q_3
b	q_2	q_3	q_0	q_3

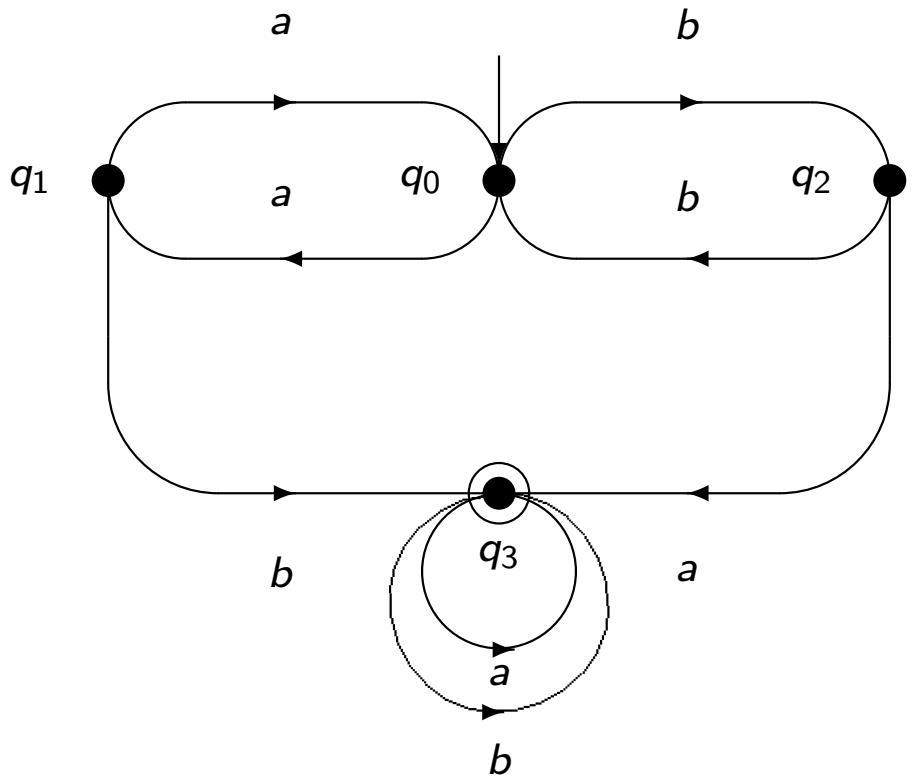
The entry that corresponds to the input line labeled i and the state column labeled q gives the value of $\delta(q, i)$.

Directed Graphs and Deterministic Finite Automata

- The graph of the deterministic finite automaton $\mathcal{M} = (A, Q, \delta, q_0, F)$ is the graph $\mathcal{G}(\mathcal{M})$ whose set of vertices is the set of states Q.
- The set of edges of $\mathcal{G}(\mathcal{M})$ consists of all pairs (q, q') such that there is a transition from q to q'; an edge (q, q') is labeled by the symbol a if $\delta(q, a) = q'$.
- The initial state q_0 is denoted by an incoming arrow with no source, and the final states are circled.

Example

The graph of the previous dfa is:



The Work of a dfa

- the symbols of a word $x = a_{i_0} \cdots a_{i_{n-1}}$ are read by the automaton one at a time;
- to compute the state reached by the dfa after the application of x, the function δ must be extended from single symbols to a function δ^* defined for words.

Extending the Transition Function

Starting from a function $\delta : Q \times A \longrightarrow Q$ we define the function $\delta^* : Q \times A^* \longrightarrow Q$ by:

$$\begin{aligned} \delta^*(q, \lambda) &= q \\ \delta^*(q, xa) &= \delta(\delta^*(q, x), a), \end{aligned}$$

for every $x \in A^*$ and $a \in A$.
Note that for single character words, e.g., $y = a$, where $a \in A$, $\delta^*(q, y) = \delta(q, a)$. This follows from by setting $x = \lambda$ and noticing that $y = \lambda a$. Thus,

$$\delta^*(q, a) = \delta(q, a) \text{ for all } q \in Q \text{ and } a \in A,$$

justifying our observation that δ^* extends δ.

Theorem

Let $\delta : Q \times A \longrightarrow Q$ be a function, and let δ^* be its extension to $Q \times A^*$. Then

$$\delta^*(q, xy) = \delta^*(\delta^*(q, x), y)$$

for every $x, y \in A^*$.

Proof.

The argument is by induction on $|y|$. The basis step, $|y| = 0$, is immediate since the equality of the theorem amounts to

$$\delta^*(q, x\lambda) = \delta^*(\delta^*(q, x), \lambda) = \delta^*(q, x).$$

□

Proof (cont'd)

For the induction step, suppose that the equality holds for words of length less or equal to n, and let y be a word of length $n+1$, $y = za$, where $z \in A^*$ and $a \in A$. We have

$$\begin{aligned}
\delta^*(q, xy) &= \delta^*(q, xza) \\
&= \delta(\delta^*(q, xz), a) \text{ (since } \delta^* \text{ extends } \delta) \\
&= \delta(\delta^*(\delta^*(q, x), z), a) \text{ (ind. hyp.)} \\
&= \delta^*(\delta^*(q, x), za) \text{ (since } \delta^* \text{ extends } \delta) \\
&= \delta^*(\delta^*(q, x), y).
\end{aligned}$$

Dfa as Language Acceptors

Definition

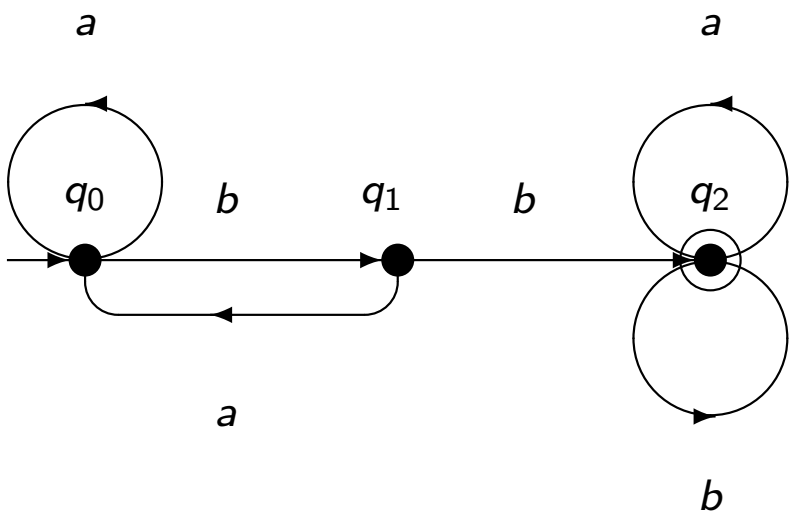
The language accepted by the dfa $\mathcal{M} = (A, Q, \delta, q_0, F)$ is the set

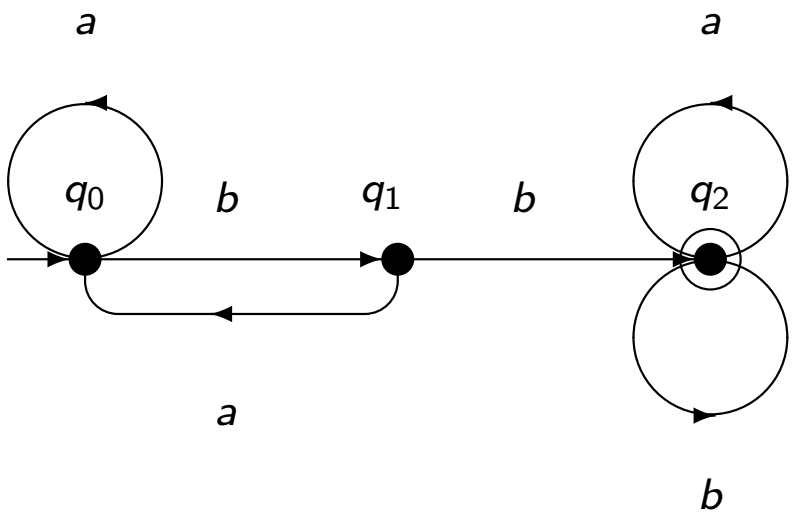
$$L(\mathcal{M}) = \{x \in A^* \mid \delta^*(q_0, x) \in F\}.$$

A language $L \subseteq A^*$ is regular if it is accepted by some finite automaton $\mathcal{M}$ whose input alphabet is A.

Example

Let $\mathcal{M} = (A, Q, \delta, q_0, F)$ be the dfa whose graph is given below, where $A = \{a, b\}$ and $Q = \{q_0, q_1, q_2\}$.





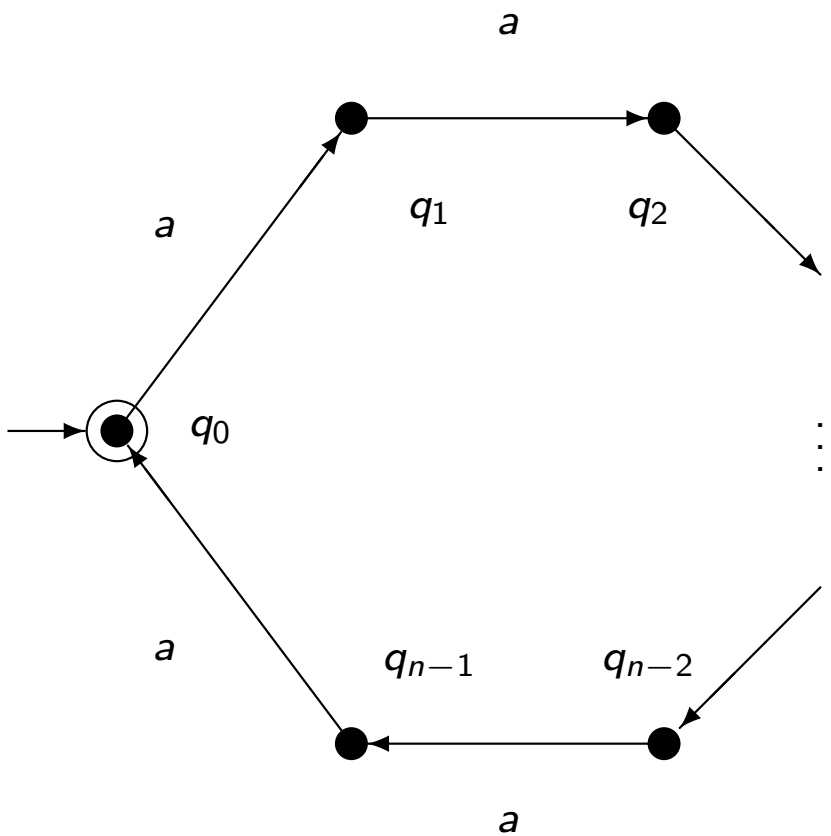
The language accepted by $\mathcal{M}$ consists of all words over A that contain at least two consecutive b symbols; in other words, $L(\mathcal{M}) = A^* bbA^*$.

- if $x \in L(\mathcal{M})$, then x contains two consecutive b symbols since q_2 cannot be reached otherwise from q_0 using the symbols of x;
- conversely, suppose that x contains two consecutive b symbols; we can decompose $x = ubbv$, where bb is the leftmost occurrence of bb in x.

The definition of $\mathcal{M}$ implies that $\delta^*(q_0, u) = q_0$, $\delta^*(q_0, bb) = q_2$ and $\delta^*(q_2, v) = q_2$. Thus, $\delta^*(q_0, x) = q_2$, and this implies $x \in L(\mathcal{M})$. We conclude that $L(\mathcal{M}) = A^*bbA^*$.

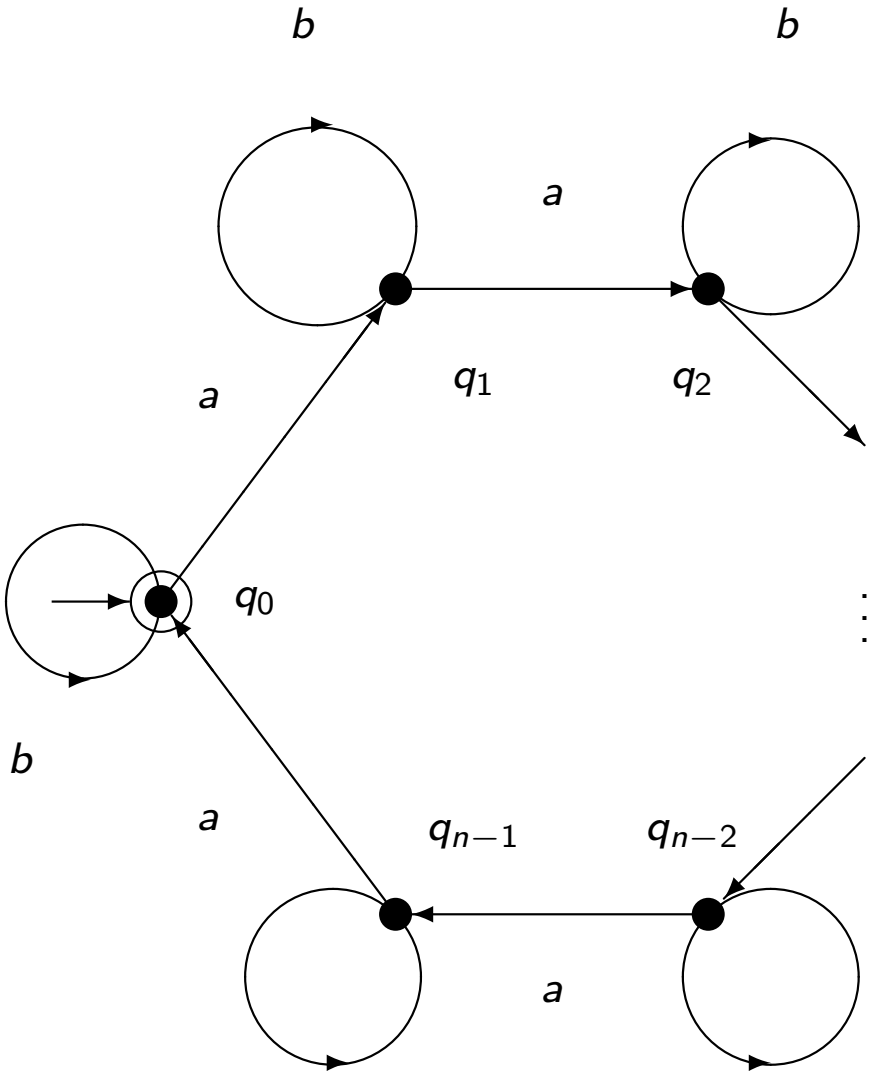
Counting Numbers

The dfa with n states shown in below accepts only inputs whose length is 0 (mod n), that is, an integral multiple of n.



Example

The dfa given below accepts those words in $\{a, b\}^*$ that have $0 \pmod{n}$ a's, regardless of how many b's are in the input.



Example

Next, we present a dfa that accepts words over the alphabet $\{0,1\}$ only when their binary equivalents are multiples of a fixed integer, say $m \in \mathbb{N}$. Let $B = \{0,1\}$. A word $x \in B^*$ can be regarded as a binary number as follows. Define the function $f : B^* \longrightarrow \mathbb{N}$ by

$$\begin{aligned} f(\lambda) &= 0 \\ f(xb) &= \begin{cases} 2f(x)+0 & \text{if } b = 0 \\ 2f(x)+1 & \text{if } b = 1, \end{cases} \end{aligned}$$

for every $x \in B^*$ and $b \in B$. Note that $f(x)$ is the value represented by x regarded as a binary number.

Let $m \in \mathbb{N}$ be a number such that $m > 1$. Note that for every $x \in B^*$, there exists a number k, $0 \leq k \leq m-1$, such that $f(x) \equiv k (\text{mod } m)$. Of course, if $f(x) \equiv 0 (\text{mod } m)$, then $f(x)$ is a multiple of m, so x will be accepted by the automaton that we intend to define.
We design an automaton $\mathcal{M}_m$ that accepts the set of words x such that $f(x)$ is a multiple of a fixed number m. The states of $\mathcal{M}_m$ are defined such that $\delta^*(q_0, x) = q_h$ if and only if $f(x) \equiv h (\text{mod } m)$. In other words, if $\mathcal{M}_m$ reaches the state q_h after reading the symbols of x, then $f(x)$ is congruent to h modulo m. Therefore, after reading the symbol b, $\mathcal{M}$ enters the state q_ℓ, where $2h + b \equiv \ell (\text{mod } m)$. This allows us to define the transition function by $\delta(q_h, b) = q_\ell$.

The dfa $\mathcal{M}_3 = (B, \{q_0, q_1, q_2\}, \delta, q_0, \{q_0\})$ that recognizes the set of multiples of 3 is defined by the table:

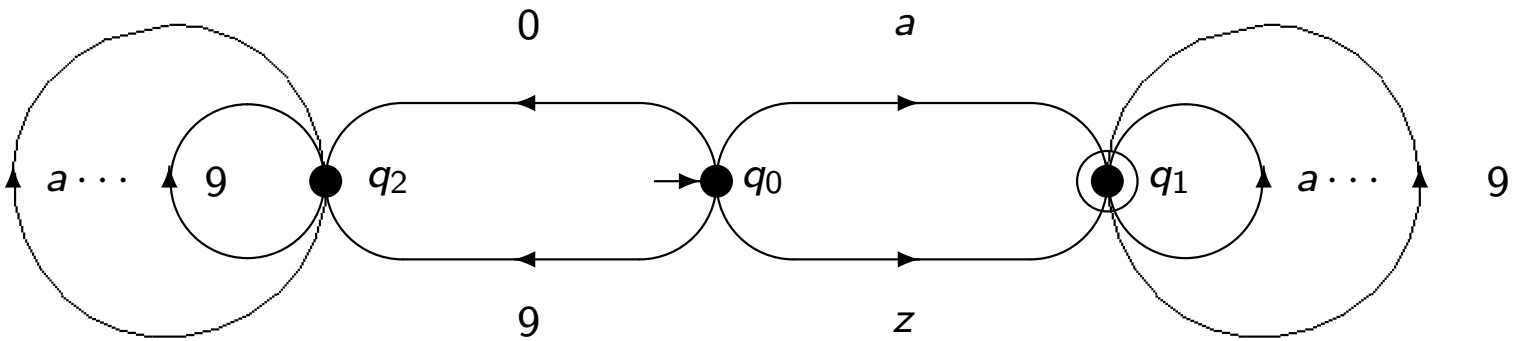
Input	State		
	q_0	q_1	q_2
0	q_0	q_2	q_1
1	q_1	q_0	q_2

Therefore, the language $L = \{x \in B^* \mid f(x) \equiv 0 (\text{mod } 3)\}$ is regular.

Example

Let $A = \{a, b, \ldots, z, 0, \ldots, 9\}$. The automaton

$$\mathcal{M} = \{A, \{q_0, q_1, q_2\}, \delta, q_0, \{q_1\}\}$$



accepts those words in A^* that begin with a letter and contain a sequence of letters and digits. In other words, $L(\mathcal{M}) = \{a, \ldots, z\}A^*$

The finiteness of the set of states Q of a dfa $\mathcal{M} = (A, Q, \delta, q_0, F)$ is essential for the definition of regular languages. If this assumption is dropped we obtain a weaker type of device.

Definition

A *deterministic automaton* (da) is a quintuple

$$\mathcal{M} = (A, Q, \delta, q_0, F),$$

where A is an alphabet, called the *input alphabet*; Q is a set that is disjoint from A, called the *set of states*, $\delta : Q \times A \longrightarrow Q$ is the *transition function of the da*, q_0 is the *initial state*, and $F \subseteq Q$ is the *set of final states*.

The transition function δ can be extended to $Q \times A^*$ in exactly the same way as for the deterministic finite automata. Again, we denote this extension by δ^*.

The role of the finiteness of the set of states of a dfa is highlighted by the next theorem.

Theorem

For every language $L \subseteq A^*$, there is a deterministic automaton $\mathcal{M} = (A, Q, \delta, q_0, F)$ such that $L = L(\mathcal{M})$.

Proof.

Consider the da $\mathcal{M} = (A, Q, \delta, q_\lambda, \{q_u \mid u \in L\})$, where $Q = \{q_x \mid x \in A^*\}$ and $\delta(q_x, a) = q_{xa}$ for every $x \in A^*$ and $a \in A$. It is easy to verify that $\delta^*(q_x, y) = q_{xy}$ for every $x, y \in A^*$. Therefore, $L(\mathcal{M}) = \{y \in A^* \mid \delta^*(q_\lambda, y) = q_y \text{ and } y \in L\} = L$, which means that L is the language accepted by $\mathcal{M}$. □

Definition

Let $\mathcal{M} = (A, Q, \delta, q_0, F)$ be an automaton. The set of accessible states is the set

$$\text{acc}(\mathcal{M}) = \{q \in Q \mid \delta^*(q_0, x) = q \text{ for some } x \in A^*\}.$$

The automaton $\mathcal{M}$ is *accessible* if $\text{acc}(\mathcal{M}) = Q$.

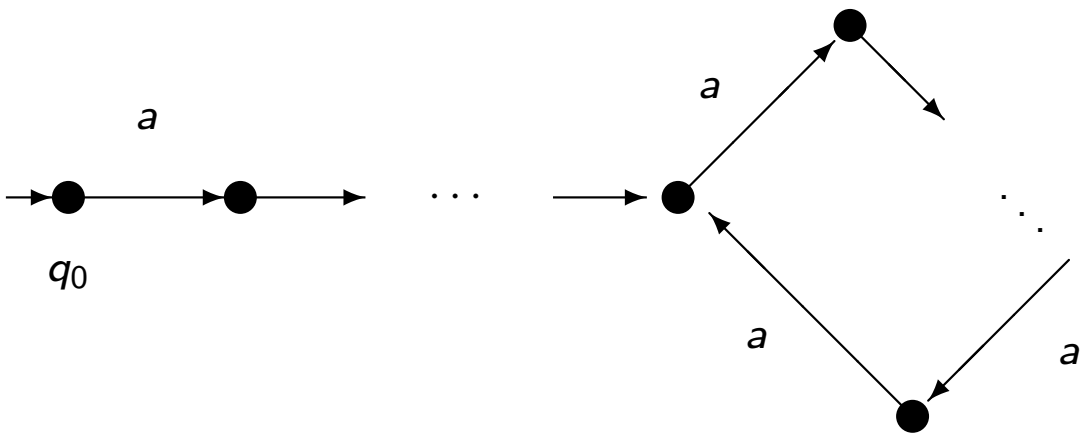
Only the set of accessible states plays a role in defining the language accepted by the automaton.

- If δ' is the restriction of δ to $\mathrm{acc}(\mathcal{M}) \times A$, then the automata $\mathcal{M}$ and $\mathcal{M}' = (A, \mathrm{acc}(\mathcal{M}), \delta', q_0, F \cap \mathrm{acc}(\mathcal{M}))$ accept the same language.
- If $x \in L(\mathcal{M})$, then $\delta^*(q_0, x) \in F$ and $\delta^*(q_0, y) \in \mathrm{acc}(\mathcal{M})$ for every prefix y of x (including x). Therefore, $(\delta')^*(q_0, x) = \delta^*(q_0, x) \in F$, so $x \in L(\mathcal{M}')$.
- it is immediate that $x \in L(\mathcal{M}')$ implies $x \in L(\mathcal{M})$, so $L(\mathcal{M}) = L(\mathcal{M}')$.

$\mathcal{M}'$ is denoted by $\mathrm{ACC}(\mathcal{M})$ and we refer to it as the accessible component of $\mathcal{M}$.

Example

Consider an automaton $\mathcal{M} = (\{a\}, Q, \delta, q_0, F)$ having a one-symbol input alphabet. We have $\text{acc}(\mathcal{M}) = \{\delta(q_0, a^n) \mid n \in \mathbb{N}\}$. Therefore, the subgraph of the accessible states in the graph of $\mathcal{M}$ consists of a path attached to a circuit, as shown:



Theorem

Let $\mathcal{M} = (A, Q, \delta, q_0, F)$ be an accessible automaton. For every state $q \in Q$ there is a word $x \in A^*$ such that $|x| < |Q|$ and $\delta^*(q_0, x) = q$.

Proof.

Since $\mathcal{M}$ is an accessible automaton, for every state $q \in Q$ there is a word y such that $\delta^*(q_0, y) = q$. Let x be a word of minimal length that allows $\mathcal{M}$ to reach the state q. We claim that $|x| < |Q|$. Let $x = a_{i_0} \cdots a_{i_p}$, and let $q_1, \ldots, q_{p+1}$ be the sequence of states reached while processing x, i.e.,

$$\begin{aligned} q_1 &= \delta(q_0, a_{i_0}) \\ &\vdots \\ q_{p+1} &= \delta(q_p, a_{i_p}) = q, \end{aligned}$$

that is, the sequence of states assumed by $\mathcal{M}$ when the symbols of x are applied starting from the state q_0. □

Proof (cont'd)

If $p+1 \geq |Q|$, then the sequence $(q_0, q_1, \ldots, q_{p+1})$ must contain two equal states because its length exceeds the number of elements of Q. If, say, $q_c = q_d$, we can write $x = uvw$, where $\delta^*(q_0, u) = q_c$, $\delta^*(q_c, v) = q_d$, $\delta^*(q_d, w) = q_{p+1}$ and $|v| > 0$. Since $q_d = q_c$, we have $\delta^*(q_0, uw) = q_{p+1} = q$, and this contradicts the minimality of x. Therefore, $|x| < |Q|$.

Computing The Accessible States

Input: A dfa $\mathcal{M} = (A, Q, \delta, q_0, F)$.
Output: The set $\text{acc}(\mathcal{M})$.
Method: Define the sequence $Q_0, Q_1, \ldots, Q_n, \ldots$ by
$Q_0 = \{q_0\}$ and $Q_{i+1} = Q_i \cup \{s = \delta(q, a) \mid q \in Q_i \text{ and } a \in A\}$.
$\text{acc}(\mathcal{M}) = Q_k$, where k is the least number such that $Q_k = Q_{k+1}$.

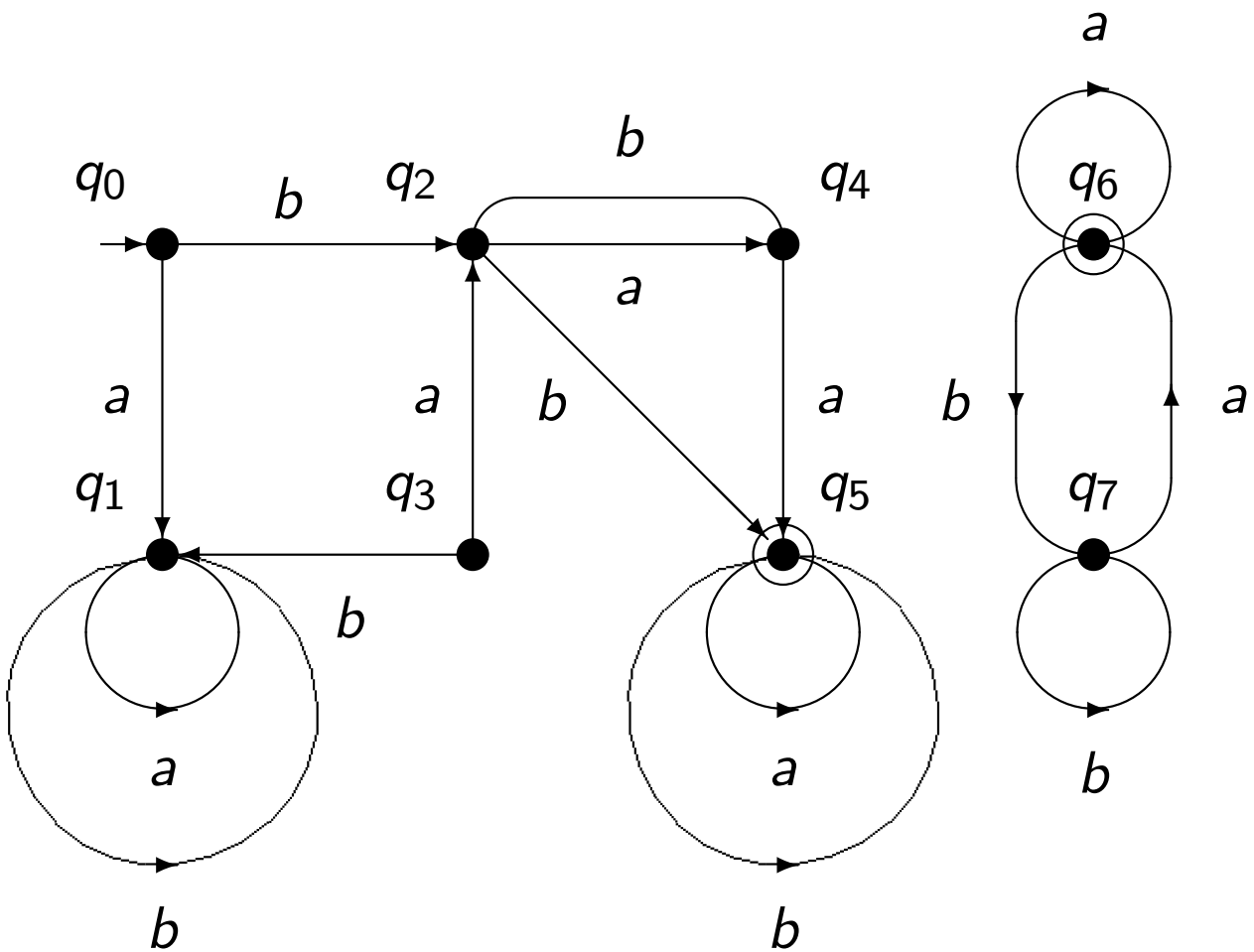
Proof of Correctness

Since $Q_0, \ldots, Q_i, \ldots$ is an increasing sequence and all sets Q_i are subsets of the finite set Q, there is a number k such that
$Q_0 \subset Q_1 \subset \cdots \subset Q_k = Q_{k+1} = \cdots$.
We claim that

$$Q_i = \{q \in Q \mid \delta^*(q_0, x) = q, \text{ for some } x \in A^*, |x| \leq i\},$$

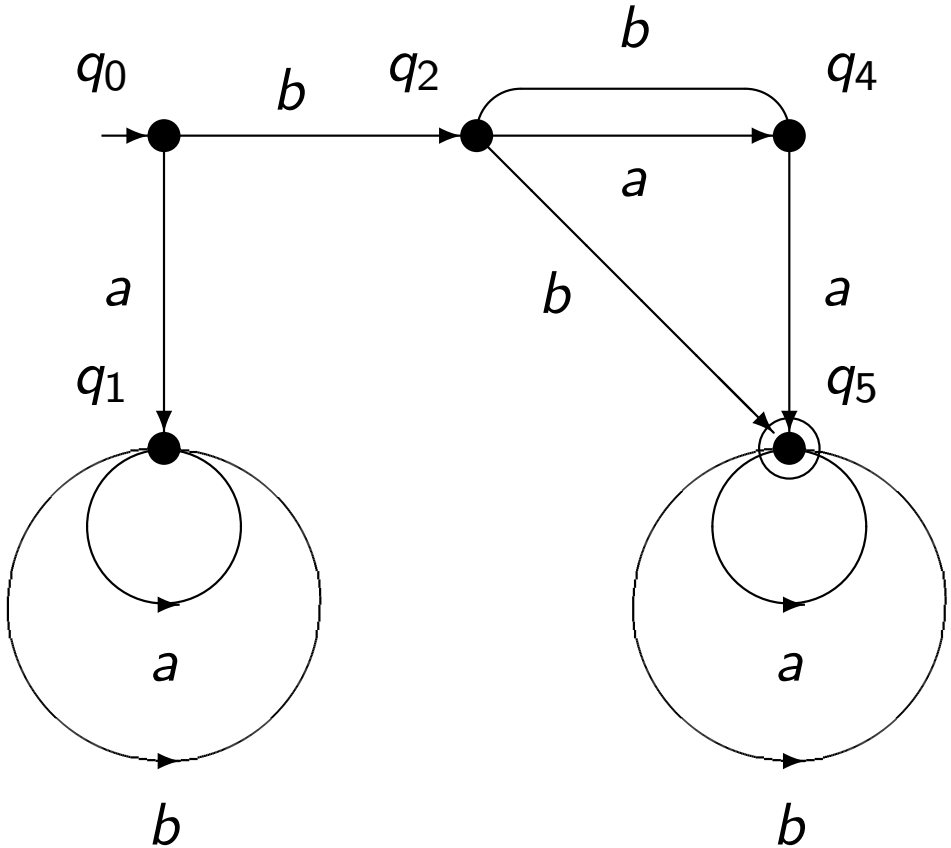
for every $i \in \mathbb{N}$. The argument is by induction on i and is left to the reader. Thus, every state in Q_k belongs to $\mathrm{acc}(\mathcal{M})$. Conversely, if $q \in \mathrm{acc}(\mathcal{M})$ there is a word x such that $|x| < |Q|$ and $\delta^*(q_0, x) = q$. Therefore, $q \in Q_{|x|} \subseteq Q_k$. We conclude that $\mathrm{acc}(\mathcal{M}) = Q_k$.

Let $\mathcal{M} = (\{a, b\}, \{q_i \mid 0 \leq i \leq 7\}, \delta, q_0, \{q_5, q_6\})$ be the dfa whose graph is shown:



$$\begin{aligned} Q_0 &= \{q_0\} \\ Q_1 &= \{q_0, q_1, q_2\} \\ Q_2 &= \{q_0, q_1, q_2, q_4, q_5\} \\ Q_3 &= \{q_0, q_1, q_2, q_4, q_5\} \end{aligned}$$

Thus, ACC($\mathcal{M}$) is the dfa $\mathcal{M}' = (\{a, b\}, \{q_0, q_1, q_2, q_4, q_5\}, \delta', q_0, \{q_5\})$ whose graph is given next.


q_0
q_1
q_2
q_4
q_5
b
b
a
a
b
a
a
a
b
b