Finite Automata and Regular Languages (Preliminaries)

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1. Equivalences
2. Partitions

Definition of Equivalences

Definition

An equivalence on a set S is a relation $\rho \subseteq S \times S$ that satisfies the following conditions:

- **Reflexivity:** $(x, x) \in \rho$ for every $x \in S$;
- **Symmetry:** $(x, y) \in \rho$ if and only if $(y, x) \in \rho$;
- **Transitivity:** if $(x, y) \in \rho$ and $(y, z) \in \rho$ then $(x, z) \in \rho$.

Example

Define the relation ρ_m on $\mathbb{N}$ as

$$\rho_m = \{(p, q) \mid p, q \in \mathbb{N} \text{ and } p - q = km \text{ for some } k \in \mathbb{Z}\}.$$

- $(p, p) \in \rho_m$ because $p - p = 0 \cdot m$, so ρ_m is reflexive;
- if $(p, q) \in \rho_m$, then $p - q = km$, hence $q - p = (-k)m$, which means that $(q, p) \in \rho_m$;
- if $(p, q) \in \rho_m$ and $(q, r) \in \rho_m$ then $p - q = km$ and $q - r = hm$; therefore, $p - r = p - q + q - r = (k + h)m$, so $(p, r) \in \rho_m$, hence ρ_m is transitive.

Thus, ρ_m is an equivalence relation on $\mathbb{N}$.

Example

Let L be the set of lines in a plane Π. Define $\ell \parallel \ell'$ if ℓ is parallel to ℓ'.

- we have $\ell \parallel \ell$, so $\parallel$ is reflexive;
- if $\ell \parallel \ell'$, then $\ell' \parallel \ell$ hence $\parallel$ is symmetric;
- if $\ell \parallel \ell'$ and $\ell' \parallel \ell''$, then $\ell \parallel \ell''$, so parallel is transitive.

Thus, "$\parallel$" is an equivalence relation on the set L.

Definition

Let S be a set and let ρ be an equivalence on S.
The ρ-equivalence class of an element x of S is the set $[x]$ defined by

$$[x] = \{z \in S \mid (x, z) \in \rho\}.$$

The quotient set S/ρ is the set of all equivalence classes defined by ρ on the set S.

Note that:

- we have $x \in [x]$ because $(x, x) \in \rho$, so none of the equivalence classes is empty;
- if $y \in [x]$, then $x \in [y]$ because ρ is symmetric; thus, in this case, $[y] = [x]$.
- if two equivalence classes are distinct, they are disjoint.

Suppose that $[y] \neq [x]$ and there exists $t \in [x] \cap [y]$. Then $(x, t) \in \rho$ and $(y, t) \in \rho$, which means that $(t, y) \in \rho$. By transitivity $(x, y) \in \rho$, which implies $[x] = [y]$, contradicting the initial assumption.

Definition

Let S be a non-empty set. A partition on S is a collection of sets $\pi = \{B_i \mid B_i \subseteq S \text{ for } i \in I\}$ such that

- $B_i \neq \emptyset$ for all $i \in I$;
- $B_i \cap B_j = \emptyset$ for $i, j \in I$ and $i \neq j$;
- $\bigcup_{i \in I} B_i = S$.

Example

If ρ is an equivalence on a set S, its set of classes

$$\{[x] \mid x \in S\}$$

is a partition of the set S. This shows how we can "move" from equivalences to partitions.

Conversely, if $\pi = \{B_i \mid B_i \subseteq S \text{ for } i \in I\}$ is a partition of S, an equivalence ρ_π is defined by

$$(x, y) \in \rho_\pi \text{ if and only if there is a block } B \text{ of } \pi \text{ such that } \{x, y\} \subseteq B.$$

Definition

Let π be a partition of a set S. A subset U of S is π-saturated if it equals the union of a collection of blocks of π.

Example

Let $S = \{1, 2, \ldots, 9\}$ be a set and let

$$\pi = \{\{1, 2, 7\}, \{4, 6\}, \{3, 5, 8, 9\}\}.$$

The set S has $2^9 = 512$ subsets. The following 8 subsets of S are π-saturated:

$$\begin{array}{l} \emptyset \\ \{1, 2, 7\}, \{4, 6\}, \{3, 5, 8, 9\} \\ \{1, 2, 7, 4, 6\}, \{1, 2, 7, 3, 5, 8, 9\}, \{4, 6, 3, 5, 8, 9\} \\ \{1, 7, 4, 6, 3, 5, 8, 9\}. \end{array}$$