

# CS724: Topics in Algorithms

## Set-Theoretical Preliminaries

### Slide Set 1

Prof. Dan A. Simovici



- 1 Sets and Set Operations
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# Set Notations

For a set  $S$  we denote by  $\mathcal{P}(S)$  the set of its subsets. The set of all finite non-empty subsets of  $S$  is denoted by  $\mathcal{F}(S)$ .

For a finite set  $S$  the number of elements of  $S$  is denoted by  $|S|$ . The empty set is denoted by  $\emptyset$ .

We write  $x \in S$  to denote the fact that  $x$  is an element of the set  $S$ . The usual symbols are used to denote set-theoretical operations:  $A \cup B$  is the union of the sets  $A$  and  $B$ ,  $A \cap B$  is the intersection of the sets  $A$  and  $B$ , and  $A - B$  is the difference of the sets  $A$  and  $B$ .

The *symmetric difference* of the sets  $A$  and  $B$  is denoted by  $A \oplus B$ . We have

$$A \oplus B = (A - B) \cup (B - A).$$



## Definition

Let  $S$  be a set. A *sequence of length  $n$  on  $S$*  is a mapping  $\mathbf{s} : \{1, \dots, n\} \rightarrow S$ . The set of sequences of length  $n$  on  $S$  is denoted by  $\text{Seq}_n(S)$ .

An *ordered pair on  $S$*  is a sequence of length 2 on  $S$ ; a *singleton* is a sequence of length 1.

If  $\mathbf{s}$  is a sequence of length  $n$  on  $S$  and  $\mathbf{s}(i) = x_i$  for  $1 \leq i \leq n$ , we write  $\mathbf{s} = (x_1, \dots, x_n)$ . The elements  $x_1, \dots, x_n$  are the *components* of  $\mathbf{s}$ .

The length of a sequence  $\mathbf{s}$  is denoted by  $|\mathbf{s}|$ .



## Example

A sequence of natural numbers of length 6 is  $\mathbf{s} = (6, 5, 2, 4, 9, 6)$ . Note that in a sequence the same element of  $S$  may occur on multiple positions.



# Counting Sequences

If  $S$  is a finite set containing  $m$  elements, then there are  $m^n$  sequences of length  $n$  for any  $n \geq 1$ . We extend the definition of sequences on  $S$  by defining the *null sequence on  $S$*  as the sequence  $\lambda$  that has no components,  $\lambda = ()$ . Note that there exists exactly one such sequence on  $S$  and this is consistent with the fact that  $m^0 = 1$  for every  $m \geq 1$ . The *set of sequences of elements of  $S$*  is the set

$$\mathbf{Seq}(S) = \bigcup \{ \mathbf{Seq}_n(S) \mid n \geq 0 \}.$$



# Operations with Sequences

If  $\mathbf{s} = (s_1, s_2, \dots, s_n)$  is a sequence in  $S$ , we refer to the sequence  $\tilde{\mathbf{s}} = (s_n, \dots, s_2, s_1)$  as the *reversal* of the sequence  $\mathbf{s}$ . Clearly  $\tilde{\tilde{\mathbf{s}}} = \mathbf{s}$ .

If  $\mathbf{s} = (s_1, \dots, s_n)$  and  $\mathbf{t} = (t_1, \dots, t_m)$  are two sequences on a set  $S$ , their *concatenation* is the sequence  $\mathbf{st} = (s_1, \dots, s_n, t_1, \dots, t_m)$ . For the null sequence we define  $\lambda\mathbf{s} = \mathbf{s}\lambda = \mathbf{s}$  for every  $\mathbf{s} \in \mathbf{Seq}(S)$ . Note that  $|\mathbf{st}| = |\mathbf{s}| + |\mathbf{t}|$  for all sequences  $\mathbf{s}, \mathbf{t} \in \mathbf{Seq}(S)$ .

Note that sequence concatenation is not a commutative operation in general.



## Example

Let  $\mathbf{s} = (1, 2, 3)$ ,  $\mathbf{t} = (4, 5)$ . We have

$$\mathbf{st} = (1, 2, 3, 4, 5) \text{ and } \mathbf{ts} = (4, 5, 1, 2, 3),$$

so  $\mathbf{st} \neq \mathbf{ts}$ .

Sequence concatenation is an associative operation on  $\mathbf{Seq}(S)$ , that is  $(\mathbf{st})\mathbf{u} = \mathbf{s}(\mathbf{tu})$  for every  $\mathbf{s}, \mathbf{t}, \mathbf{u} \in \mathbf{Seq}(S)$ .





# Operations with Collections of Sets

Let  $\mathcal{C} = \{S_i \mid i \in I\}$  be a collection of sets. Its union is the set  $U$  defined as

$$U = \bigcup_{i \in I} S_i.$$

Note that  $\mathcal{C} \subseteq \mathcal{C}'$  implies  $\bigcup \mathcal{C} \subseteq \bigcup \mathcal{C}'$ .

Unlike the union, the intersection is defined only for collections that consist of subsets of a set  $S$ .

If  $\mathcal{C}$  is a collection of subsets of  $S$ , that is, if  $\mathcal{C} \subseteq \mathcal{P}(S)$ , the intersection of  $\mathcal{C}$  is the set of all elements of  $S$  that belong to every set of  $\mathcal{C}$ . The intersection of  $\mathcal{C}$  is denoted by  $\bigcap \mathcal{C}$ .



If  $\mathcal{C}$  and  $\mathcal{C}'$  are two collections of subsets of a set  $S$  and  $\mathcal{C} \subseteq \mathcal{C}'$ , then  $\bigcap \mathcal{C}' \subseteq \bigcap \mathcal{C}$ . If  $\emptyset$  is the empty collection of subsets of  $S$ , we define  $\bigcap \emptyset = S$ .



## Definition

A *closure system on the set  $S$*  is a collection  $\mathcal{K}$  of subsets of  $S$  such that for every collection of subsets  $\mathcal{C}$  such that  $\mathcal{C} \subseteq \mathcal{K}$  we have  $\bigcap \mathcal{C} \in \mathcal{K}$ .

Note that if  $\mathcal{K}$  is a closure system on a set  $S$ , then  $S \in \mathcal{K}$  because  $S$  is the intersection of the empty collection of subsets of  $\mathcal{K}$ .



## Definition

Let  $\mathcal{K}$  be a closure system on a set  $S$  and let  $T$  be a subset of  $S$ . The *closure of  $T$  relative to the closure system  $\mathcal{K}$*  is the set

$$\mathbf{K}(T) = \bigcap \{U \in \mathcal{K} \mid T \subseteq U\}.$$

For every set  $T$  the collection  $\mathcal{C}_T = \{U \in \mathcal{K} \mid T \subseteq U\}$  is non-empty because it includes at least  $S$ . The set  $\bigcap \mathcal{C}_T$  is denoted by  $\mathbf{K}(T)$  and is referred to as the *closure* of  $T$ .

To emphasize that the closure of  $T$  is computed relative to the closure system  $\mathcal{K}$  we may denote this closure by  $\mathbf{K}_{\mathcal{K}}(T)$ .



## Example

A subset  $E$  of  $\mathbb{R}$  is said to be symmetric if  $x \in E$  if and only if  $-x \in E$ . Let  $\{E_i \mid i \in I\}$  be a collection of symmetric subsets of  $\mathbb{R}$ . It is easy to see that  $\bigcap\{E_i \mid i \in I\}$  is a symmetric set. Note that  $\mathbb{R}$  itself is symmetric. Thus, the collection  $\mathcal{E}$  of symmetric subsets of  $\mathbb{R}$  is a closure system. For a subset  $T$  of  $\mathbb{R}$  the set  $K_{\mathcal{E}}(T)$  is the smallest symmetric set that includes  $T$ .



Let  $A$  and  $B$  be two finite sets. It is easy to verify that

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

In this section we discuss a generalization of this Equality known as the *inclusion-exclusion principle*.

### Definition

Let  $S$  be a set and let  $U$  be a subset of  $S$ . The *indicator* of  $U$  is the function  $I_U : S \rightarrow \{0, 1\}$  given by

$$I_U(x) = \begin{cases} 1 & \text{if } x \in U, \\ 0 & \text{if } x \notin U. \end{cases}$$



If  $U$  and  $V$  are two subsets of a finite set  $S$  such that  $V \subseteq U$ , then the function  $I$  defined by  $I(x) = I_U(x) - I_V(x)$  for  $x \in S$  is an indicator function, namely the indicator function of the subset  $U - V$  of  $S$ .

Let  $a$  and  $b$  be two numbers that belong to the set  $\{-1, 1\}$  such that the function  $I_{ab}$  defined by

$$I_{ab}(x) = aI_U(x) + bI_V(x)$$

for  $x \in S$  is the indicator function of a subset  $W$  of the set  $S$ . We need to examine conditions under which  $I_{ab}$  is the indicator function of a set.



Since  $I_{ab}(x) \in \{0, 1\}$  and  $I_{ab}(x) = aI_U(x) + bI_V(x)$ , the following cases are possible:

- If  $a = b = 1$ , then we have  $U \cap V = \emptyset$ ; otherwise (that is, if  $x \in U \cap V$ ) we would have  $aI_U(x) + bI_V(x) = 2$  and this would prevent  $I_{ab}$  from being an indicator function. Clearly, in this case,  $W = U \cup V$ .
- If  $a = 1$  and  $b = -1$ , we must have  $I_V(x) \leq I_U(x)$  for every  $x \in S$ , which implies  $V \subseteq U$ . Thus,  $W = U - V$ .
- The case where  $a = -1$  and  $b = 1$  is similar to the previous case, and we have  $W = V - U$ .
- The case when  $a = -1$  and  $b = -1$  is possible only if  $U = V = \emptyset$ . In this case,  $W = \emptyset$ .

Note that in all these cases we have  $|W| = a|U| + b|V|$ . This observation is generalized by the next statement.





## Theorem

Let  $U_0, \dots, U_{n-1}$  be  $n$  subsets of a finite set  $S$ , where  $n \geq 2$ , and let  $(a_0, \dots, a_{n-1}) \in \mathbf{Seq}_n(\{-1, 1\})$  be a sequence of  $n$  numbers such that the function  $I : S \rightarrow \{0, 1\}$  defined by

$$I(x) = a_0 I_{U_0}(x) + \dots + a_{n-1} I_{U_{n-1}}(x)$$

for  $x \in S$  is the indicator function of a subset  $W$  of  $S$ . Then,

$$|W| = a_0 |U_0| + \dots + a_{n-1} |U_{n-1}|.$$



# Proof

If  $W$  is a subset of  $S$ , then  $\sum_{x \in S} I_W(x) = |W|$  because for each  $x \in S$  its contribution to the sum  $\sum_{x \in S} I_W(x)$  is equal to 1 if and only if  $x \in W$ . Therefore, if  $I_W(x) = \sum_{i=0}^{n-1} a_i I_{U_i}(x)$  for  $x \in S$ , we have

$$\begin{aligned} |W| &= \sum_{x \in S} I_W(x) = \sum_{x \in S} \sum_{i=0}^{n-1} a_i I_{U_i}(x) = \sum_{i=0}^{n-1} \sum_{x \in S} a_i I_{U_i}(x) \\ &= \sum_{i=0}^{n-1} a_i \sum_{x \in S} I_{U_i}(x) = \sum_{i=0}^{n-1} a_i |U_i|. \end{aligned}$$



## Corollary

**(Principle of Inclusion-Exclusion)** Let  $A_0, \dots, A_{n-1}$  be  $n$  finite sets, where  $n \geq 2$ . We have

$$\left| \bigcup_{i=0}^{n-1} A_i \right| = \sum_{0 \leq i \leq n-1} |A_i| - \sum_{0 \leq i_1 < i_2 \leq n-1} |A_{i_1} \cap A_{i_2}| + \sum_{0 \leq i_1 < i_2 < i_3 \leq n-1} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| - \dots + (-1)^{n+1} |A_0 \cap \dots \cap A_{n-1}|.$$



## Proof

Suppose that  $A_i \subseteq S$  for  $0 \leq i \leq n-1$ , where  $S$  is a finite set. For  $x \in S$ , we have  $x \notin A = \bigcup_{i=0}^{n-1} A_i$  if and only if  $x \notin A_i$  for  $0 \leq i \leq n-1$ . This is equivalent to writing

$$1 - I_A(x) = (1 - I_{A_0}(x)) \cdots (1 - I_{A_{n-1}}(x))$$

for every  $x \in S$ . This equality is, in turn, equivalent to

$$\begin{aligned} I_A(x) &= \sum_{i=0}^{n-1} I_{A_i}(x) - \sum_{0 \leq i_1 < i_2 \leq n-1} I_{A_{i_1}}(x) I_{A_{i_2}}(x) \\ &\quad + \sum_{0 \leq i_1 < i_2 < i_3 \leq n-1} I_{A_{i_1}}(x) I_{A_{i_2}}(x) I_{A_{i_3}}(x) - \cdots + (-1)^{n+1} I_{A_0}(x) \cdots I_{A_{n-1}}(x) \\ &= \sum_{i=0}^{n-1} I_{A_i}(x) - \sum_{0 \leq i_1 < i_2 \leq n-1} I_{A_{i_1} \cap A_{i_2}}(x) \\ &\quad + \sum_{0 \leq i_1 < i_2 < i_3 \leq n-1} I_{A_{i_1} \cap A_{i_2} \cap A_{i_3}}(x) - \cdots + (-1)^{n+1} I_{A_0 \cap \cdots \cap A_{n-1}}(x). \end{aligned}$$



## Corollary

Let  $A_0, \dots, A_{n-1}$  be  $n$  finite sets, where  $n \geq 2$ , and let  $S = \bigcup_{i=0}^{n-1} A_i$ . We have

$$\left| \bigcap_{i=0}^{n-1} A_i \right| = |S| - \sum_{0 \leq i \leq n-1} |A_i| + \sum_{0 \leq i_1 < i_2 \leq n-1} |A_{i_1} \cap A_{i_2}| - \sum_{0 \leq i_1 < i_2 < i_3 \leq n-1} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| + \dots + (-1)^n |A_0 \cap \dots \cap A_{n-1}|$$

**Proof:** This follows immediately from the previous Corollary by observing that

$$\left| \bigcap_{i=0}^{n-1} A_i \right| = |S| - \left| \bigcup_{i=0}^{n-1} A_i \right|.$$



## Definition

A *relation* on the set  $S$  is a set of ordered pairs of  $S$ .

The set of relations on  $S$  is denoted by  $\text{rel}(S)$ .

Since relations on  $S$  are sets of pairs on  $S$  they can be involved in the usual set-theoretical operations: union, intersection, difference, etc. If  $\rho, \sigma \in \text{rel}(S)$ , the union, intersection, and difference of  $\rho$  and  $\sigma$  are denoted by  $\rho \cup \sigma$ ,  $\rho \cap \sigma$ , and  $\rho - \sigma$ , respectively.

Also,  $\rho \subseteq \sigma$  denotes the inclusion of the set of pairs  $\rho$  into the set of pairs  $\sigma$ .



Two important relations on  $S$  are the *diagonal relation*

$$\iota_S = \{(x, x) \mid x \in S\},$$

and the *total relation*

$$\theta_S = \{(x, y) \mid x, y \in S\}.$$

### Definition

Let  $\rho, \sigma \in \text{rel}(S)$ . The *product* of  $\rho$  and  $\sigma$  is the relation  $\rho\sigma$  given by

$$\rho\sigma = \{(x, z) \in \mathbf{Seq}_2(S) \mid (x, y) \in \rho \text{ and } (y, z) \in \sigma\}.$$



## Definition

A relation  $\rho \in \text{rel}(S)$  is:

- *reflexive*, if  $\iota_S \subseteq \rho$ ;
- *symmetric*, if  $(x, y) \in \rho$  is equivalent to  $(y, x) \in \rho$ ;
- *antisymmetric*, if  $(x, y) \in \rho$  and  $(y, x) \in \rho$  implies  $x = y$ ;
- *transitive*, if  $(x, y), (y, z) \in \rho$  implies  $(x, z) \in \rho$ ,

for all  $x, y, z \in S$ .

If  $\rho \in \text{rel}(S)$ , the *inverse* of  $\rho$  is the relation

$$\rho^{-1} = \{(y, x) \in S \times S \mid (x, y) \in \rho\}.$$





The  $n^{\text{th}}$  *power of a relation*  $\rho$ , where  $\rho \subseteq S \times S$  is defined inductively as

$$\begin{aligned}\rho^0 &= \iota_S, \\ \rho^{n+1} &= \rho^n \rho\end{aligned}$$

for  $n \geq 0$ .

If  $\rho$  is a relation on  $S$ , then  $(x, x) \in \rho^0$  for every  $x \in S$ . An easy argument by induction on  $n \in \mathbb{N}$  shows that  $(x, y) \in \rho^n$  if and only if there exists a sequence  $\mathbf{z} = (z_0, z_1, \dots, z_n)$  of length  $n + 1$  such that  $x = z_0$ ,  $(z_i, z_{i+1}) \in \rho$  for  $0 \leq i \leq n - 1$  and  $z_n = y$ .



Properties of relations can be expressed using the operations just introduced. For example, a relation  $\rho$  on a set  $S$  is symmetric if and only if  $\rho^{-1} = \rho$ ; a relation  $\rho$  is transitive if  $\rho^2 \subseteq \rho$ .

## Definition

An *equivalence relation* on a set  $S$  is a relation  $\rho$ ,  $\rho \subseteq S \times S$  that is reflexive, symmetric, and transitive.

The set of equivalence relations on  $S$  is denoted by  $\text{EQ}(S)$ .



## Example

Both  $\iota_S$  and  $\theta_S$  are equivalence relations on  $S$ ; moreover, for any equivalence  $\rho \in \text{EQ}(S)$  we have  $\iota_S \subseteq \rho \subset \theta_S$ .



## Example

Let  $m$  be a positive integer. Define the relation  $\equiv_m$  on  $\mathbb{Z}$  as consisting of those pairs  $(p, q) \in \mathbb{Z} \times \mathbb{Z}$  if  $p - q = km$  for some  $k \in \mathbb{Z}$ . In other words, we have  $(p, q) \in \equiv_m$  if  $p - q$  is divisible by  $m$ .

Note that  $(r, r) \in \equiv_m$  because  $r - r = 0$  is divisible by  $m$ . If  $p - q = km$  for some  $k \in \mathbb{Z}$ , then  $q - p = (-k)m$ , so  $(p, q) \in \equiv_m$  implies  $(q, p) \in \equiv_m$ . Finally suppose that  $(p, q) \in \equiv_m$  and  $(q, s) \in \equiv_m$ . Since  $p - q = km$  and  $q - s = hm$ , we have  $p - s = (k + h)m$ , hence  $(p, s) \in \equiv_m$ . Thus,  $\equiv_m$  is an equivalence relation on  $\mathbb{Z}$ .



Following common practice, for an equivalence  $\rho$  on a set  $S$  and for  $x, y \in S$  we write  $x\rho y$  for  $(x, y) \in \rho$ .

## Definition

Let  $\rho$  be an equivalence relation on a set  $S$ . The *equivalence class* of an element  $x$  of  $S$  is the set

$$[x]_{\rho} = \{u \in S \mid (x, u) \in \rho\}.$$

By the reflexivity of  $\rho$ ,  $(x, x) \in \rho$  for every  $x \in S$ . Thus,  $x \in [x]_{\rho}$ , hence each equivalence class is non-empty.



## Lemma

*Let  $\rho$  be an equivalence relation on a set  $S$ . We have  $y \in [x]_\rho$  if and only if  $[y]_\rho = [x]_\rho$ .*



# Proof

Suppose that  $y \in [x]_\rho$  and that  $u \in [y]_\rho$ . Then, we have  $(x, y) \in \rho$  and  $(y, u) \in \rho$ . By transitivity,  $(x, u) \in \rho$ , that is,  $u \in [x]_\rho$ , which implies  $[y]_\rho \subseteq [x]_\rho$ .

If  $v \in [x]_\rho$ , then  $(x, v) \in \rho$ . Since  $(x, y) \in \rho$ , by the symmetry and transitivity of  $\rho$  we obtain  $(y, v) \in \rho$ , hence  $v \in [y]_\rho$ , so  $[x]_\rho \subseteq [y]_\rho$ . This implies  $[x]_\rho = [y]_\rho$ .

Conversely, if  $[y]_\rho = [x]_\rho$ , we have  $y \in [x]_\rho$  because  $y \in [y]_\rho$ .



## Theorem

Let  $\rho$  be an equivalence relation on a set  $S$ . If  $[x]_\rho \neq [y]_\rho$ , then  $[x]_\rho \cap [y]_\rho = \emptyset$ .

## Proof.

Let  $x, y \in S$  be such that  $[x]_\rho \neq [y]_\rho$  and suppose that  $z \in [x]_\rho \cap [y]_\rho$ . Since  $z \in [x]_\rho$  we have  $[z]_\rho = [x]_\rho$ ; similarly, since  $z \in [y]_\rho$  we have  $[z]_\rho = [y]_\rho$ , which means that  $[x]_\rho = [y]_\rho$ . This contradicts the hypothesis. □





## Definition

Let  $S$  be a non-empty set. A *partition* on  $S$  is a non-empty collection

$\pi = \{B_i \mid i \in I\}$  such that

- $B_i \neq \emptyset$  for  $i \in I$ ;
- $i, j \in I$  and  $i \neq j$  implies  $B_i \cap B_j = \emptyset$ ;
- $\bigcup_{i \in I} B_i = S$ .

The sets  $B_i$  are the *blocks* of the partition  $\pi$ .

The set of partitions of a set  $S$  is denoted by  $PART(S)$ ; the set of partitions of  $S$  that have  $k$  blocks, where  $1 \leq k \leq |S|$  is denoted by  $PART_k(S)$ .

The partitions in  $PART_2(S)$  are referred to as *bipartitions*.

Clearly,  $PART(S) = \bigcup_{k=1}^{|S|} PART_k(S)$ .



## Example

The partition of a set  $S$  that consists of all singletons  $\{x\}$ , where  $x \in S$  is denoted by  $\alpha_S$ ; the partition of  $S$  that contains one block, namely  $S$ , is denoted by  $\omega_S$ . We have  $PART_{|S|}(S) = \{\alpha_S\}$  and  $PART_1(S) = \{\omega_S\}$ .



## Example

Let  $\rho$  be an equivalence relation on a set  $S$ . The set of equivalence classes of  $\rho$  is a partition of the set  $S$ . Indeed, we saw that  $S = \bigcup_{x \in S} [x]_{\rho}$ , no equivalence class is empty and, as we saw, any two equivalence classes are disjoint.

The set of equivalence classes of an equivalence relation is known as the *quotient set* of  $S$  by  $\rho$  and is denoted by  $S/\rho$ . The partition generated by the equivalence relation is also denoted by  $\pi_{\rho}$ .



## Example

Let  $m \in \mathbb{P}$  and let  $B_i$  be the set of all members  $n$  of  $\mathbb{P}$  such that the remainder of the division of  $n$  by  $m$  equals  $i$ , where  $0 \leq i \leq m - 1$ . It is immediate that the collection  $\{B_0, B_1, \dots, B_{m-1}\}$  is a partition of the set  $\mathbb{P}$ . For instance, if  $m = 3$ , we have  $B_0 = \{3, 6, 9, 12, \dots\}$ ,  $B_1 = \{1, 4, 7, 10, \dots\}$ , and  $B_2 = \{2, 5, 8, 11, \dots\}$ .



## Theorem

Let  $\pi = \{B_i \mid i \in I\}$  be a partition of the set  $S$ . The relation  $\rho_\pi$  defined by

$$\rho_\pi = \{(x, y) \in S \times S \mid \{x, y\} \subseteq B_i \in \pi\}$$

is an equivalence on  $S$ .



# Proof

Each  $x$  belongs to a block  $B_i$  of  $\pi$ , so  $(x, x) \in \rho_\pi$  for every  $x \in S$ , which means that  $\rho_\pi$  is reflexive.

If  $(x, y) \in \rho_\pi$ , then  $\{x, y\} \subseteq B_i$ , which obviously implies  $(y, x) \in \rho_\pi$ , so  $\rho_\pi$  is symmetric.

Finally, if  $(x, y) \in \rho_\pi$  and  $(y, z) \in \rho_\pi$ , there exist  $B_i, B_j \in \pi$  such that  $\{x, y\} \subseteq B_i$  and  $\{y, z\} \subseteq B_j$ . Thus,  $B_i \cap B_j \neq \emptyset$  (because both contain  $y$ ), which implies  $B_i = B_j$ . Therefore,  $\{x, z\} \subseteq B_i = B_j$ , hence  $(x, z) \in \rho_\pi$ , which allows us to conclude that  $\rho_\pi$  is an equivalence relation.



## Corollary

Let  $\pi \in PART(S)$  and let  $\rho \in EQ(S)$   $\rho = \rho_{\pi\rho}$  and  $\pi = \pi_{\rho\pi}$ .

## Proof.

The equalities follow easily from the definitions of  $\pi\rho$  and  $\rho\pi$ . □



## Example

Note that  $\pi_{\iota_S} = \alpha_S$ ,  $\pi_{\theta_S} = \omega_S$  and  $\rho_{\alpha_S} = \iota_S$ ,  $\rho_{\omega_S} = \theta_S$ .

We write  $x \equiv y(\pi)$  to denote that  $(x, y) \in \rho_\pi$ .





Denote by  $(x)_n$  the  $n$ -degree polynomial

$$(x)_n = x(x-1)\cdots(x-n+1).$$

The coefficients of this polynomial

$$(x)_n = s(n, n)x^n + s(n, n-1)x^{n-1} + \cdots + s(n, i)x^i + \cdots + s(n, 0)$$

are known as *the Stirling numbers of the first kind*.



## Theorem

We have:

$$\begin{aligned}s(n, 0) &= 0, \\s(n, n) &= 1, \\s(n + 1, k) &= s(n, k - 1) - ns(n, k).\end{aligned}$$

## Proof.

The verification of the first two equalities is immediate. The third equality follows by observing that  $\binom{x}{n+1} = \binom{x}{n}(x - n)$  and seeking the coefficient of  $x^k$  on both sides. □



Let  $S$  be a set having  $n$  elements. We are interested in the number of partitions of  $S$  that have  $k$  blocks. We begin by counting the number of onto functions of the form  $f : A \rightarrow B$ , where  $|A| = n$ ,  $|B| = k$ , and  $n \geq k$ .

### Lemma

*Let  $A$  and  $B$  be two sets, where  $|A| = n$ ,  $|B| = k$ , and  $n \geq k$ . The number of surjective functions from  $A$  to  $B$  is given by*

$$\sum_{j=0}^{k-1} (-1)^j \binom{k}{j} (k-j)^n.$$



# Proof

There are  $k^n$  functions of the form  $f : A \rightarrow B$ .

We begin by determining the number of functions that are not surjective.

Suppose that  $B = \{b_1, \dots, b_k\}$ , and let  $F_j = \{f : A \rightarrow B \mid b_j \notin f(A)\}$  for  $1 \leq j \leq k$ . A function is not surjective if it belongs to one of the sets  $F_j$ . Thus, we need to evaluate  $|\bigcup_{j=1}^k F_j|$ . By using the inclusion-exclusion principle, we can write:

$$\begin{aligned} \left| \bigcup_{j=1}^k F_j \right| &= \sum_{j_1=1}^k |F_{j_1}| - \sum_{j_1, j_2=1}^k |F_{j_1} \cap F_{j_2}| \\ &\quad + \sum_{j_1, j_2, j_3=1}^k |F_{j_1} \cap F_{j_2} \cap F_{j_3}| - \dots - + (-1)^k |F_1 \cap F_2 \cap \dots \cap F_k|. \end{aligned}$$



## Proof (cont'd)

Note that the set  $|F_{j_1} \cap F_{j_2} \cap \dots \cap F_{j_p}|$  is actually the set of functions defined on  $A$  with values in the set  $B - \{y_{j_1}, y_{j_2}, \dots, y_{j_p}\}$ , and there are  $(k - p)^n$  such functions. Since there are  $\binom{k}{p}$  choices for the set  $\{j_1, j_2, \dots, j_p\}$ , it follows that there are

$$\binom{k}{1}(k-1)^n - \binom{k}{2}(k-2)^n + \binom{k}{3}(k-3)^n - \dots + (-1)^k \binom{k}{k-1}$$

functions that are not surjective.



## Proof (cont'd)

Thus, we can conclude that there are

$$\begin{aligned} & \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} (k-j)^n \\ &= k^n - \binom{k}{1} (k-1)^n + \binom{k}{2} (k-2)^n - \dots + (-1)^{k-1} \binom{k}{k-1} \end{aligned}$$

surjective functions from  $A$  to  $B$ .



## Theorem

The number of partitions of a set  $S$  that have  $k$  blocks ( $k \leq n$ ) is given by

$$\frac{1}{k!} \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} (k-j)^n.$$



# Proof

Note that there are  $k!$  distinct onto functions that have the same kernel partition. Indeed, given a surjective function  $f : A \rightarrow B$ , one can obtain a function  $g$  that has the same partition as  $f$  by defining  $g(a) = p(f(a))$ , where  $p$  is a permutation of the set  $B$ , that is, a bijection  $p : B \rightarrow B$ . Since there are  $k!$  such bijections, it follows that the number of partitions is  $\frac{1}{k!} \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} (k-j)^n$ .





The numbers  $S(n, k)$  defined by

$$S(n, k) = \begin{cases} \frac{1}{k!} \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} (k-j)^n & \text{if } n \geq k > 0, \\ 1 & \text{if } n = k = 0, \\ 0 & \text{in other cases.} \end{cases}$$

for  $n, k \in \mathbb{N}$  and are known as the *Stirling numbers of the second kind*.



## Example

Note that  $S(n, 1) = 1$  and  $S(n, n) = 1$  because only one partition of a set with  $n$  elements,  $\omega_S$ , has one block, and only one partition of a set with  $n$  elements,  $\alpha_S$  has  $n$  blocks which are singletons.

The number of partitions of a 4-element set having two blocks is

$$\begin{aligned} S(4, 2) &= \frac{1}{2!} \sum_{j=0}^1 \binom{2}{j} (2-j)^4 \\ &= \frac{1}{2!} \left( \binom{2}{0} \cdot 2^4 - \binom{2}{1} \cdot 1^4 \right) = 7. \end{aligned}$$

Namely, these partitions are:

$$\{\{1\}, \{2, 3, 4\}\}, \{\{2\}, \{1, 3, 4\}\}, \{\{3\}, \{1, 2, 4\}\}, \{\{4\}, \{1, 2, 3\}\}, \\ \{\{1, 2\}, \{3, 4\}\}, \{1, 3\}, \{2, 4\}\}, \{\{1, 4\}, \{2, 3\}\}.$$



We claim that

$$S(n, k) = kS(n - 1, k) + S(n - 1, k - 1).$$

Indeed, observe that a partition  $\pi$  of the set  $\{1, \dots, n - 1\}$  can be transformed into a partition of  $\{1, \dots, n\}$  by adjoining  $n$  to one of the blocks of  $\pi$  or by increasing the number of blocks by 1 and making  $\{n\}$  a block.



## Theorem

For every  $n \geq 1$  we have  $m^n = \sum_{j=1}^n S(n, j)(m)_j$ .



# Proof

Let  $A$  and  $B$  be two finite sets such that  $|A| = n$  and  $|B| = m$ . There are  $m^n$  functions  $f : A \rightarrow B$ . These functions can be classified depending on the size of their range  $f(A)$ . If  $g : A \rightarrow B$  is a function such that  $|g(A)| = j$ , then  $g$  can be regarded as a surjection from  $A$  to  $g(A)$ . Since there are  $j!S(n, j)$  such surjective functions and there are  $\binom{m}{j}$  subsets of  $B$  that have  $j$  elements, we can write

$$\begin{aligned} m^n &= \sum_{j=1}^m \binom{m}{j} j! S(n, j) \\ &= \sum_{j=1}^m m(m-1) \cdots (m-j+1) S(n, j) = \sum_{j=1}^m (m)_n S(n, j) \end{aligned}$$

for every  $m \geq 1$ .



The *Bell number*  $B_n$  is the total number of partitions of a set of  $n$  objects, that is,

$$B_n = \sum_{k=1}^n S(n, k).$$

### Example

For  $n = 4$ , we have shown that there exist 7 partitions having two blocks, one partition with one block and one partition with 4 blocks. It is easy to see that there are 6 partitions with 3 blocks, so  $B_4 = 1 + 7 + 6 + 1 = 15$ .

The first 10 values of the Bell numbers are given below.

$n$	1	2	3	4	5	6	7	8	9	10
$B_n$	1	2	5	15	52	203	877	4140	21147	115975



## Definition

A relation  $\rho$  is a *partial order* on a set  $S$  if  $\rho$  is reflexive, antisymmetric and transitive.

A *partially ordered set* (or, a *poset*) is a pair  $(S, \rho)$ , where  $\rho$  is a partial order on  $S$ .

In general, we denote partial orders using the symbol " $\leq$ " or similar symbols; furthermore, instead of writing  $(x, y) \in \leq$ , we write  $x \leq y$ .



## Example

Let  $T$  be a set. The set of subsets of  $T$ ,  $\mathcal{P}(T)$  equipped with the set inclusion " $\subseteq$ " yields the poset  $(\mathcal{P}(T), \subseteq)$ .





## Example

The pair  $(\mathbb{P}, |)$ , where “ $|$ ” is the divisibility relation is a poset defined by  $p|q$  if there exists  $k \in \mathbb{P}$  such that  $q = pk$ . Indeed, we have  $p|p$  for every  $p \in \mathbb{P}$ , so “ $|$ ” is reflexive. If  $p|q$  and  $q|p$ , we have  $q = pk$  and  $p = qh$ , hence  $hk = 1$  which implies  $h = k = 1$ . Thus,  $p = q$ , which shows that “ $|$ ” is antisymmetric. Finally, if  $p|q$  and  $q|r$  we have  $q = pk$  and  $r = qh$  for some  $k, h \in \mathbb{P}$ . Thus,  $r = pkh$ , so  $p|r$ .



## Example

Let  $(\mathbb{P}, |)$  be the poset of positive numbers equipped with the divisibility relation.

We have  $p \prec y$  if  $x$  is none of the largest divisors of  $y$ . For example, we have  $6 \prec 12$  because there is no number  $z$  distinct from 6 and 12 such that  $6|z$  and  $z|12$ . Note that  $3|12$  but  $3 \not\prec 12$ .

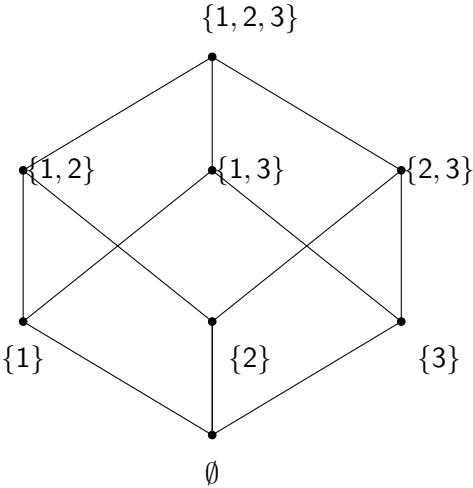


Finite posets can be represented graphically using *Hasse diagrams*.

- Each element is represented by a dot.
- If  $x, y$  are elements of a poset  $(P, \leq)$  and  $x \prec y$ , then the dot representing  $y$  is placed at a greater height than  $x$  and a link between the dots is drawn.

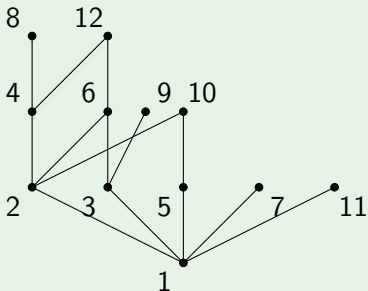


The Hasse diagram of the poset  $(\mathcal{P}(\{1, 2, 3\}), \subseteq)$  equipped with the inclusion relation is given below:



## Example

The Hasse diagram of the poset  $(\{1, \dots, 12\}, |)$  equipped with the divisibility relation is given below:



If  $(S, \rho)$  is a poset and  $T \subseteq S$ , it is easy to see that the relation  $\rho_T = \rho \cap (T \times T)$  is itself a partial order; we will refer to it as the *trace of  $(S, \rho)$  on  $T$* . Often, we will use the same symbol  $\rho$  instead of  $\rho_T$  to denote the partial order on  $T$ .

### Example

Let  $S \subseteq \mathbb{P}$  be the set  $\{1, 2, 3, 4, 5, 6\}$ . The trace of  $\mathbb{P}$  on  $S$  consists of the pairs:

$$\begin{aligned} &(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), \\ &(2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), \\ &(5, 5), (6, 6). \end{aligned}$$



## Definition

A *totally ordered set* is a pair  $(S, \rho)$ , where  $\rho$  is a partial order with the additional property that for all  $x, y \in S$  we have either  $(x, y) \in \rho$ , or  $(y, x) \in \rho$ . The relation  $\rho$  is referred to as a *total order*.

## Example

The real numbers  $\mathbb{R}$  equipped with the standard less-than-or-equal relation  $\leq$  is a totally ordered set.



## Definition

A sequence  $\mathbf{x} \in \mathbf{Seq}(S)$  is a *subsequence of a sequence*  $\mathbf{y} \in \mathbf{Seq}(S)$ , if  $\mathbf{y} = \mathbf{uxv}$  for some sequences  $\mathbf{u}, \mathbf{v} \in \mathbf{Seq}(S)$ . This is denoted by  $\mathbf{x} \sqsubseteq \mathbf{y}$ .

## Example

Let  $S = \{0, 1\}$ . The sequence  $\mathbf{y} = 1011$  is a subsequence of  $\mathbf{x} = 010110110101100$ .

The relation “ $\sqsubseteq$ ” is a partial order on  $\mathbf{Seq}(S)$ .





## Definition

Let  $(P, \leq)$  be a poset. An element  $y$  *covers* an element  $x$  of  $P$  if  $x \leq y$  and there is no  $z \in P$ ,  $z \neq x$  and  $z \neq y$  such that  $x \leq z \leq y$ .

We denote the fact that  $y$  covers  $x$  by  $x \prec y$ .



## Definition

Let  $(S, \leq)$  be a poset and let  $T$  be a subset of  $S$ . The *set of upper bounds* of  $T$  is the set

$$T^s = \{y \in S \mid \text{for all } x \in T \text{ we have } x \leq y\}.$$

The *set of lower bounds* of  $T$  is the set

$$T^i = \{y \in S \mid \text{for all } x \in T \text{ we have } y \leq x\}.$$



If  $T_1, T_2$  are two subsets of  $S$ ,  $T_1 \subseteq T_2$  implies  $T_2^s \subseteq T_1^s$  and  $T_2^i \subseteq T_1^i$ .

## Theorem

Let  $(S, \leq)$  be a poset and let  $T$  be a subset of  $S$ . The sets  $T \cap T^s$  and  $T \cap T^i$  contain at most one element of  $S$ .

## Proof.

Suppose that  $x, y \in T \cap T^s$ . Since  $x \in T$  and  $y \in T^s$ , it follows that  $x \leq y$ . On other hand, since  $x \in T^s$  and  $y \in T$  we have  $y \leq x$ . Therefore,  $x = y$ , which implies that the set  $T \cap T^s$  contains at most one element. The argument for  $T \cap T^i$  is similar.  $\square$



## Definition

Let  $(S, \leq)$  be a poset and let  $T$  be a subset of  $S$ . If  $T \cap T^s = \{u\}$ , then  $u$  is the *largest element* of set  $T$ .

If  $T \cap T^i = \{v\}$ , then  $v$  is the *least element* of set  $T$ .

## Example

Not every subset of a poset has a least or a greatest element. The subset  $\{1, 2, 3, 6\}$  of the poset  $(\{1, \dots, 12\}, |)$  considered before has 1 as its least element and 6 as the largest element. In contrast, the set  $\{4, 5, 6\}$  has neither a least nor a largest element.



If  $T$  is a subset of a poset  $(S, \leq)$  we will consider the sets  $(T^s)^i$  and  $(T^i)^s$  denoted by  $T^{si}$  and  $T^{is}$ , respectively.

Observe that the set  $T^s \cap T^{si} = T^s \cap (T^s)^i$  may contain at most one element, by a previous observation applied to the set  $T^s$ . Similarly, the set  $T^i \cap T^{is}$  may contain at most one element.



## Definition

Let  $(S, \leq)$  be a poset and let  $T$  be a subset of  $S$ . If  $T^s \cap T^{si} = \{u\}$ ,  $u$  is the *supremum* of the set  $T$ .

If  $T^i \cap T^{is} = \{v\}$ ,  $v$  is the *infimum* of  $T$ .

The supremum and infimum of a set  $T$  (if they exist) are unique and are denoted by  $\sup T$  and  $\inf T$ , respectively.



## Example

In the poset  $(\mathcal{P}(T), \subseteq)$  introduced in before, for every  $\mathcal{C} \in \mathcal{P}(X)$  we have

$$\inf \mathcal{C} = \bigcap \mathcal{C} \text{ and } \sup \mathcal{C} = \bigcup \mathcal{C}.$$



## Example

In the poset  $(\mathbb{P}, |)$ , we have

$$\inf\{p, q\} = \gcd(p, q) \text{ and } \sup\{p, q\} = \text{lcm}(p, q),$$

where  $\gcd(p, q)$  is the greatest common divisor of  $p$  and  $q$ , and  $\text{lcm}(p, q)$  is the least common multiple of  $p$  and  $q$ .





## Definition

A poset  $(S, \leq)$  is a *lattice* if for every two elements  $x, y \in S$  there exist  $\inf\{x, y\}$  and  $\sup\{x, y\}$ . If  $(S, \leq)$  is a lattice we use the notations

$$x \wedge y = \inf\{x, y\} \text{ and } x \vee y = \sup\{x, y\}.$$

The element  $x \wedge y$  is referred to as the *meet* of  $x$  and  $y$ ;  $x \vee y$  is the *join* of  $x$  and  $y$ .

A poset  $(S, \leq)$  is a *complete lattice* if for every  $X \in \mathcal{P}(S)$  there exist  $\inf X$  and  $\sup X$ .



## Example

- $(\mathbb{P}, |)$  is a lattice;
- $(\mathcal{P}(T), \subseteq)$  is a complete lattice.



Note that if  $(S, \leq)$  is a complete lattice and  $S \neq \emptyset$ , then this poset has a least element  $0_S = \inf S$ , and a greatest element  $1_S = \sup S$ .

## Theorem

*Let  $(S, \leq)$  be a complete lattice and let  $W$  be a subset of  $S$  such that  $1_S \in W$  and  $T \subseteq W$  implies that  $\inf T$  in  $S$  belongs to  $W$ . Then  $W$  is a complete lattice.*



# Proof

For every nonvoid subset  $T$  of  $W$ ,  $\inf T \in W$  and is the infimum of  $T$  in  $S$ . Let  $U$  be a subset of  $W$  defined as  $U = T^s$ . We have  $U \neq \emptyset$  because  $1_S \in W$ . Then,  $\inf U \in W$  is also an upper bound of  $T$ , and is actually the least upper bound of  $U$ . Thus,  $(W, \leq)$  is a complete lattice.



## Corollary

Let  $\mathcal{K}$  be a closure system on a set  $S$ . The subsets of  $S$  in  $\mathcal{K}$  form a complete lattice in which  $\inf \mathcal{C} = \bigcap \mathcal{C}$  and  $\sup \mathcal{C} = \bigcap \{T \in \mathcal{P} \mid C \subseteq T \text{ for every } C \in \mathcal{K}\}$ .



Let  $\pi, \sigma$  be two partitions of  $S$ . We write  $\pi \leq \sigma$  if each block  $B$  of  $\pi$  is included in a block  $C$  of  $\sigma$ .

### Theorem

*The pair  $(PART(S), \leq)$  is a partially ordered set.*



# Proof

The relation " $\leq$ " is obviously reflexive.

Suppose that we have both  $\pi \leq \sigma$  and  $\sigma \leq \pi$ . Then, a block  $B$  of  $\pi$  is included in a block  $C$  of  $\sigma$ , and  $C$ , in turn, is included in a block  $B'$  of  $\pi$ . Thus,  $B \subseteq C \subseteq B'$ , which implies  $B = C = B'$  because no block of  $\pi$  can be included into another block. Thus,  $\pi \subseteq \sigma$ . In the same manner, starting from a block  $C$  of  $\sigma$  we can show that  $\sigma \subseteq \pi$ , so  $\pi = \sigma$ . This shows that the relation " $\leq$ " is antisymmetric. It is immediate that " $\leq$ " is transitive



## Theorem

Let  $\pi, \sigma \in \text{PART}(S)$  be two partitions,  $\pi = \{B_i \mid i \in I\}$  and  $\sigma = \{C_j \mid j \in J\}$ . We have  $\pi \leq \sigma$  if and only if for each  $j \in J$  there exists a subset  $I_j$  of  $I$  such that  $C_j = \bigcup\{B_i \mid i \in I_j\}$ .

Suppose that  $\pi \leq \sigma$  and let  $C \in \sigma$ . Suppose that  $B \cap C \neq \emptyset$ . Since each block  $B$  of  $\pi$  is included in a block  $C'$  of  $\sigma$  we must have  $C' = C$  because, otherwise  $C'$  and  $C$  would have a non-empty intersection. Thus, if a block  $B$  of  $\pi$  has a non-empty intersection with a block  $C$  of  $\sigma$  we must have  $B \subseteq C$ . This implies that a block  $C$  of  $\sigma$  is a union of block of  $\pi$ . The converse implication is immediate.





## Example

If  $\pi \in PART(S)$  we have  $\alpha_S \leq \pi \leq \omega_S$ . Thus,  $\alpha_S$  is the smallest element of  $(PART(S), \leq)$  and  $\omega_S$  is its largest element.



## Definition

Let  $\pi, \sigma$  be two partitions of a set  $S$ . The partition  $\sigma$  covers  $\pi$  if  $\pi < \sigma$  and there is no partition  $\tau \in PART(S)$  such that  $\pi < \tau < \sigma$ .



## Theorem

*Let  $\pi, \sigma$  be two partitions of a set  $S$ . The partition  $\sigma$  covers  $\pi$  if and only if there exists a block  $C$  of  $\sigma$  that is the union of two blocks  $B$  and  $B'$  of  $\pi$ , and every other block of  $\sigma$  that is distinct of  $C$  is a block of  $\pi$ .*



# Proof

Suppose that  $\sigma$  is a partition that covers the partition  $\pi$ . Since  $\pi \leq \sigma$ , every block of  $\sigma$  is a union of blocks of  $\pi$ . Suppose that there exists a block  $E$  of  $\sigma$  that is the union of more than two blocks of  $\pi$ ; that is,  $E = \bigcup\{B_i \mid i \in I\}$ , where  $|I| \geq 3$ , and let  $B_{i_1}, B_{i_2}, B_{i_3}$  be three blocks of  $\pi$  included in  $E$ . Consider the partitions

$$\begin{aligned}\sigma_1 &= \{C \in \sigma \mid C \neq E\} \cup \{B_{i_1}, B_{i_2}, B_{i_3}\}, \\ \sigma_2 &= \{C \in \sigma \mid C \neq E\} \cup \{B_{i_1} \cup B_{i_2}, B_{i_3}\}.\end{aligned}$$

It is easy to see that  $\pi \leq \sigma_1 < \sigma_2 < \sigma$ , which contradicts the fact that  $\sigma$  covers  $\pi$ . Thus, each block of  $\sigma$  is the union of at most two blocks of  $\pi$ .



## Proof (cont'd)

Suppose that  $\sigma$  contains two blocks  $C'$  and  $C''$  that are unions of two blocks of  $\pi$ , namely  $C' = B_{i_0} \cup B_{i_1}$  and  $C'' = B_{i_2} \cup B_{i_3}$ . Define the partitions

$$\begin{aligned}\sigma' &= \{C \in \sigma \mid C \notin \{C', C''\}\} \cup \{C', B_{i_2}, B_{i_3}\}, \\ \sigma'' &= \{C \in \sigma \mid C \notin \{C', C''\}\} \cup \{B_{i_1}, B_{i_2}, C''\}.\end{aligned}$$

Since  $\pi < \sigma', \sigma'' < \sigma$ , this contradicts the fact that  $\sigma$  covers  $\pi$ . Thus, we obtain the conclusion of the theorem.

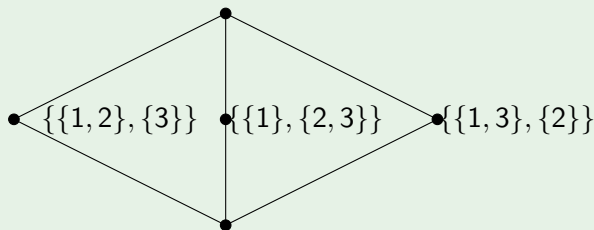


# Hasse diagrams

## Example

The Hasse diagram of  $(PART(\{1, 2, 3\}), \leq)$  is given below:

$\{\{1, 2, 3\}\}$



$\{\{1\}, \{2\}, \{3\}\}$



## Theorem

*The posets  $(EQS(S), \subseteq)$  and  $(PART(S), \leq)$  are isomorphic.*

Let  $f : EQS(S) \rightarrow PART(S)$  be the mapping defined by  $f(\rho) = S/\rho$ . We need to show that  $f$  is a monotonic bijective mapping and that its inverse mapping  $f^{-1}$  is also monotonic.

The bijectivity of  $f$  follows immediately from the remarks that precede the theorem. Let  $\rho_0, \rho_1$  be two equivalences such that  $\rho_0 \subseteq \rho_1$  and let  $S/\rho_0 = \{B_i \mid i \in I\}$ ,  $S/\rho_1 = \{C_j \mid j \in J\}$ . Let  $B_i$  be a block in  $S/\rho_0$  and assume that  $B_i = [x]_{\rho_0}$ . We have  $y \in B_i$  if and only if  $(x, y) \in \rho_0$ , so  $(x, y) \in \rho_1$ . Therefore,  $y \in [x]_{\rho_1}$ , which shows that every block  $B \in S/\rho_0$  is included in a block  $C \in \rho_1$ . This shows that  $f(\rho_0) \leq f(\rho_1)$ , so  $f$  is indeed monotonic.



Let  $\{\rho_i \mid i \in I\} \subseteq EQS(S)$  be a collection of equivalences. Then,  
 $\inf\{\rho_i \mid i \in I\} = \bigcap_{i \in I} \rho_i$ .

## Definition

Let  $S$  be a set and let  $\rho, \tau \in EQS(S)$ . A  $(\rho, \tau)$ -*alternating sequence that joins  $x$  to  $y$*  is a sequence  $(s_0, s_1, \dots, s_n)$  such that  $x = s_0$ ,  $y = s_n$ ,  $(s_i, s_{i+1}) \in \rho$  for every even  $i$  and  $(s_i, s_{i+1}) \in \tau$  for every odd  $i$ , where  $0 \leq i \leq n - 1$ .





## Lemma

Let  $S$  be a set and let  $\rho, \tau \in \text{EQS}(S)$ . If  $\mathbf{s}$  and  $\mathbf{z}$  are two  $(\rho, \tau)$ -alternating sequences joining  $x$  to  $y$  and  $y$  to  $z$ , respectively, then there exists a  $(\rho, \tau)$ -alternating sequence that joins  $x$  to  $z$ .



# Proof

Let  $(s_0, \dots, s_n)$  be a  $(\rho, \tau)$ -alternating sequences joining  $x$  to  $y$  and  $(w_0, \dots, w_m)$  a  $(\rho, \tau)$ -alternating sequences joining  $y$  to  $z$ , where  $x = s_0$ ,  $s_n = w_0 = y$  and  $w_m = z$ . If  $(s_{n-1}, s_n) \in \tau$ , then the sequence  $(s_0, \dots, s_n, w_1, \dots, w_m)$  is a  $(\rho, \tau)$ -alternating sequence joining  $x$  to  $z$ . Otherwise, that is, if  $(s_{n-1}, s_n) \in \rho$ , then taking into account the reflexivity of  $\tau$  we have  $(s_n, w_0) = (s_n, s_n) \in \tau$ . In this case,  $(s_0, \dots, s_n, s_n, w_1, \dots, w_m)$  is a  $(\rho, \tau)$ -alternating sequence joining  $x$  to  $z$ .



## Theorem

Let  $S$  be a set and let  $\rho, \tau \in \text{EQS}(S)$ . If  $\xi$  is the relation that consists of all pairs  $(x, y) \in S \times S$  that can be joined by a  $(\rho, \tau)$ -alternating sequence, then  $\xi = \sup\{\rho, \tau\}$ .



It is easy to verify that  $\xi$  is indeed an equivalence relation. Note that we have both  $\rho \subseteq \xi$  and  $\tau \subseteq \xi$ . Indeed, if  $(x, y) \in \rho$ , then  $(x, y, y)$  is a  $(\rho, \tau)$ -alternating sequence joining  $x$  to  $y$ . If  $(x, y) \in \tau$ , then  $(x, x, y)$  is the needed alternating sequence.

Let  $\zeta \in EQS(S)$  such that  $\rho \subseteq \zeta$  and  $\tau \subseteq \zeta$ . If  $(x, y) \in \xi$ , and  $(s_0, s_1, \dots, s_n)$  is a  $(\rho, \tau)$ -alternating sequence such that  $x = s_0$ ,  $y = s_n$ , then each pair  $(s_i, s_{i+1})$  belongs to  $\zeta$ . By the transitivity property,  $(x, y) \in \zeta$ , so  $\xi \subseteq \zeta$ . This implies that  $\xi = \sup\{\rho, \tau\}$ .



If  $\pi, \sigma \in PART(S)$  both the infimum and the supremum of the set  $\{\pi, \sigma\}$  exist and their description follows from the corresponding results that refer to the equivalence relations. Namely, if  $\pi, \sigma \in PART(S)$ , where  $\pi = \{B_i \mid i \in I\}$  and  $\sigma = \{C_j \mid j \in J\}$ , the partition  $\inf\{\pi, \sigma\}$  exists and is given by

$$\inf\{\pi, \sigma\} = \{B_i \cap C_j \mid i \in I, j \in J \text{ and } B_i \cap C_j \neq \emptyset\}.$$

The partition  $\inf\{\pi, \sigma\}$  will be denoted by  $\pi \wedge \sigma$ .



A block of the partition  $\sup\{\pi, \sigma\}$ , denoted by  $\pi \vee \sigma$ , is an equivalence class of the equivalence  $\theta = \sup\{\rho_\pi \wedge \rho_\sigma\}$ . We have  $y \in [x]_\theta$  if there exists a sequence  $(s_0, \dots, s_n) \in \mathbf{Seq}(S)$  such that  $x = s_0$ ,  $s_n = y$  and successive sets  $\{s_i, s_{i+1}\}$  are included, alternatively, in a block of  $\pi$  or in a block of  $\sigma$ .

