CS724: Topics in Algorithms
Graph Laplacians

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Definition

The *Laplacian matrix of the weighted graph* $G = (V, E, c)$ is the symmetric matrix $L_G = D_G - A_G$, where D_G is the diagonal matrix of the degrees of vertices and A_G is the adjacency matrix.
The spectrum of the Laplacian matrix is referred to as the *Laplacian spectrum* of the weighted graph.

If G is clear from context, we omit the subscript G. Also, we will consider other variants of the Laplacian; to refer to current version, we will designate it as the *ordinary Laplacian* or just the Laplacian.
Note that the off-diagonal elements of L_G are non-positive numbers.

Example

Let $G = (\{v_1, v_2\}, \{(v_1, v_2)\}, c)$ be the weighted graph shown in Figure 1, where $c(v_1, v_2) = k$.

Figure: Two-vertex weighted graph

We have

$$D_G = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \text{ and } A_G = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}.$$

Thus, the Laplacian matrix is

$$L_G = \begin{pmatrix} k & -k \\ -k & k \end{pmatrix}.$$

Lemma

Let $G = (V, E, c)$ *be a weighted graph, where* $V = \{v_1, \ldots, v_m\}$. *For* $\mathbf{x} \in \mathbb{R}^m$ *we have* $\mathbf{x}' L_G \mathbf{x} = \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} c_{ij}(x_i - x_j)^2$.

Proof

We have

$$\begin{aligned}\mathbf{x}' L_G \mathbf{x} &= \mathbf{x}'(D_G - A_G)\mathbf{x} = \mathbf{x}' D_G \mathbf{x} - \mathbf{x}' A_G \mathbf{x} \\ &= \sum_{i=1}^{m} \mathrm{d}(v_i) x_i^2 - \sum_{i=1}^{m} \sum_{j=1}^{m} c_{ij} x_i x_j.\end{aligned}$$

The right hand side of the equality to be proven can be rewritten as shown on the next slide.

$$
\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij}(x_i - x_j)^2 \\
&= \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij}(x_i^2 - 2x_ix_j + x_j^2) \\
&= \frac{1}{2}\left(\sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij}x_i^2 - 2\sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij}x_ix_j + \sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij}x_j^2\right) \\
&= \sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij}x_i^2 - \sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij}x_ix_j \\
&\quad \text{(because } A_G \text{ is a symmetric matrix)} \\
&= \sum_{i=1}^{m} \mathrm{d}(v_i)x_i^2 - \sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij}x_ix_j \\
&\quad \text{(because } \textstyle\sum_{j=1}^{m} c_{ij} = \mathrm{d}(v_i)\text{)}.
\end{aligned}
$$

which concludes the argument.

Theorem

The Laplacian of a weighted graph $G = (V, E, c)$ is a symmetric and positive semi-definite matrix that has 0 as its smallest eigenvalue (and $\mathbf{1}$ as a corresponding eigenvector).

Proof

- Let $L_G = D_G - A_G$ be the Laplacian of G. Since both D_G and A_G are symmetric matrices, so is L_G.
  The positive definiteness of L_G follows immediately.
- Since the sum of elements of each row of L_G is 0 we have $L_G\mathbf{1} = \mathbf{0}$, which shows that 0 is an eigenvalue of L_G and $\mathbf{1}$ is an eigenvector of this eigenvalue.
- Also, all eigenvalues of L_G are real and non-negative, and L_G has a full set of n real and orthogonal eigenvectors. Thus, 0 is the smallest eigenvalue of L_G.

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Theorem

Let $G = (V, E, c)$ be a weighted graph, where $|V| = m$. The number of connected components of G equals the algebraic multiplicity of the eigenvalue 0, and the the characteristic vector of each connected component is an eigenvector of A_G that corresponds to the eigenvalue 0.

Proof

Let k the number of connected components of G.
When $k = 1$ the graph is connected and this is the case that we examine initially. If $\mathbf{x}$ is an eigenvector that corresponds to the eigenvalue 0 we have $\mathbf{x}' L_G \mathbf{x} = 0$, hence $\sum_{i=1}^{m} \sum_{j=1}^{m} c_{ij}(x_i - x_j)^2 = 0$. Thus, $c_{ij}(x_i - x_j)^2 = 0$ for all i, j, $1 \leqslant i, j \leqslant m$. If the vertices v_i and v_j are connected we have $c_{ij} > 0$, so $x_i = x_j$. Thus, the values of the components of $\mathbf{x}$ must be the same because the graph is connected. Consequently, the invariant subspace of the eigenvalue 0 is generated by the vector $\mathbf{1}$, which is also the characteristic vector of the connected component.

Proof cont'd

Suppose now that we have k connected components. Without loss of generality we can assume that the vertices of the graph are numbered such that the numbers attributed to the vertices that belong to a connected component are consecutive. In this case, the Laplacian L_G has a block-diagonal form

$$L_G = \begin{pmatrix} L_1 & O & \cdots & O \\ O & L_2 & \cdots & O \\ \vdots & \vdots & \cdots & \vdots \\ O & O & \cdots & L_k \end{pmatrix}.$$

Note that 0 is an eigenvalue of L and of each of the matrices L_j, where $1 \leqslant j \leqslant k$. Furthermore, each L_i is the Laplacian of a connected component of G, so it has 0 as an eigenvalue of multiplicity 1. Thus, L has 0 as an eigenvalue of multiplicity k.

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The characteristic vector $\mathbf{v}_{C_p} \in \mathbb{R}^m$ of a connected component C_p, where $1 \leqslant p \leqslant k$, is given by

$$(\mathbf{v}_{C_p})_i = \begin{cases} 1 & \text{if } i \text{ is the number of a row that corresponds to } L_i \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that each such vector is an eigenvector of L_G.

Corollary

Let $G = (V, E, c)$ be a weighted graph. If G is a connected graph, $rank(L_G) = |V| - 1$.

Proof.

We saw that if G is connected, 0 has algebraic multiplicity 1. Thus, L_G has rank $|V| - 1$. □

A graph $G = (V, E)$ can be regarded as a weighted graph (V, E, c) such that $c(e) = 1$ for every $e \in E$.

Definition

Let $G = (V, E)$ be a graph. The *algebraic connectivity* of G is the number $\alpha(G)$ which is the second smallest eigenvalue of the Laplacian L_G.

Recall that the smallest eigenvalue of L_G is 0.
If $|V| = n$ let $S_n = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x}'\mathbf{1}_n = 0 \text{ and } \parallel \mathbf{x} \parallel = 1\}$. Note that $\mathbf{0}_n \notin S_n$.
The Courant-Fisher Theorem implies

$$\begin{aligned}\alpha(G) &= \min_{\mathbf{x}}\{\mathbf{x}^{\mathsf{H}} L_G \mathbf{x} \mid \mathbf{x} \in S_n\} \\ &= \min_{\mathbf{x}} \sum \{(x_i - x_j)^2 \mid \mathbf{x} \in S_n, i < j \text{ and } \{v_i, v_j\} \in E\}.\end{aligned}$$

Example

Let $P(a) \in \mathbb{C}^{n\times n}$ be the matrix

$$P(a) = \begin{pmatrix} a & 1 & \cdots & 1 \\ 1 & a & \cdots & 1 \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & \cdots & a \end{pmatrix}.$$

To find the eigenvalues of $P(a)$ we need to solve the equation

$$\begin{vmatrix} \lambda - a & -1 & \cdots & -1 \\ -1 & \lambda - a & \cdots & -1 \\ \vdots & \vdots & \cdots & \vdots \\ -1 & -1 & \cdots & \lambda - a \end{vmatrix} = 0.$$

By adding the first $n-1$ columns to the last and factoring out $\lambda - (a+n-1)$, we obtain the equivalent equation

$$(\lambda - (a+n-1)) \begin{vmatrix} \lambda - a & -1 & \cdots & 1 \\ -1 & \lambda - a & \cdots & 1 \\ \vdots & \vdots & \cdots & \vdots \\ -1 & -1 & \cdots & 1 \end{vmatrix} = 0.$$

Adding the last column from the first $n-1$ columns and expanding the determinant yields the equation

$$(\lambda - (a+n-1))(\lambda - a + 1)^{n-1} = 0,$$

which allows us to conclude that $P(a)$ has the eigenvalue $a+n-1$ with $\text{algm}(P(a), a+n-1) = 1$ and the eigenvalue $a-1$ with $\text{algm}(P(a), a-1) = n-1$.

In the special case when $a = 1$ we have $P(1) = J_{n,n}$. Thus, $J_{n,n}$ has the eigenvalue $\lambda_1 = n$ with algebraic multiplicity 1 and the eigenvalue 0 with algebraic multiplicity $n - 1$.

Let K_m be the complete graph having m vertices. Its Laplacian is

$$L_{K_m} = \begin{pmatrix} m-1 & -1 & \cdots & -1 \\ -1 & m-1 & \cdots & -1 \\ \vdots & \vdots & \cdots & \vdots \\ -1 & -1 & \cdots & m-1 \end{pmatrix}$$

Note that $L_{K_m} = -P(1 - m)$. Its characteristic equation is $\det(\lambda I_m - L_{K_m}) = 0$, or $\det(\lambda I_m + P(1 - m)) = 0$. Thus, the eigenvalues of L_{K_m} are the opposites of the eigenvalues of $P(1 - m)$. In other words, L_{K_m} has 0 and m as eigenvalues with multiplicities 1 and $m - 1$, respectively. So $\alpha(K_m) = m$.

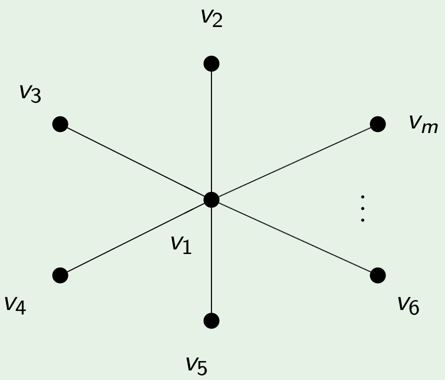
Note that if $G = (V, E)$ is a graph and $\overline{G}$ is its complement, then

$$L_G + L_{\overline{G}} = nI_n - J_{n,n}.$$

Example

The Laplacian of the star graph G is

$$L_G = \begin{pmatrix} m-1 & -1 & \cdots & -1 \\ -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ -1 & 0 & \cdots & 1 \end{pmatrix}.$$



Let $Q_n(a, b)$ be the determinant:

$$Q_n(a,b) = \begin{vmatrix} a & 1 & 1 & 1 & \cdots & 1 \\ 1 & b & 1 & 0 & \cdots & 0 \\ 1 & 0 & b & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & b \end{vmatrix}$$

For $n = 2$ we have $Q_2(a, b) = \begin{vmatrix} a & 1 \\ 1 & b \end{vmatrix} = ab - 1$.

To compute the value of $Q_n(a,b)$ note that, by expanding this determinant by its last column we have:

$$Q_n(a,b) = (-1)^{n+1} \begin{vmatrix} 1 & b & 0 & 0 & \cdots 0 & \\ 1 & 0 & b & 0 & \cdots 0 & \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ 1 & 0 & 0 & 0 & \cdots b & \\ 1 & 0 & 0 & 0 & \cdots 0 & \end{vmatrix} + bQ_{n-1}(a,b) = -b^{n-2} + bQ_{n-1}(a,b).$$

It is easy to verify that $Q_n(a,b) = b^{n-1}a - (n-1)b^{n-2}$ for $n \geqslant 2$. Thus, the eigenvalues of L_G are the solutions of the equation $Q_m(m-1-\lambda, 1-\lambda) = (1-\lambda)^{m-2}\lambda(\lambda - m) = 0$, that is, $0, 1$ and m, so $\alpha(G) = 1$.

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Theorem

Let $G = (V, E)$ be a k-regular graph with $|V| = m$. If the ordinary spectrum of G is $k = \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_m$, then its Laplacian spectrum consists of the numbers $0 = k - \lambda_1 \leqslant k - \lambda_2 \leqslant \cdots \leqslant k - \lambda_m$.

Proof.

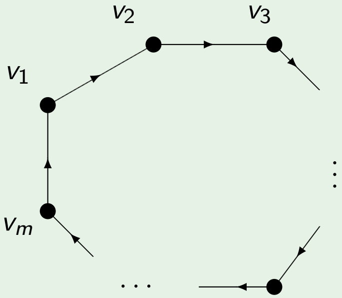
Since G is a k-regular graph its Laplacian has the form $L_G = kI_n - A_G$. Therefore, L_G has the eigenvalues
$0 = k - \lambda_1 \leqslant k - \lambda_2 \leqslant \cdots \leqslant k - \lambda_m$. □

For directed graphs the definition of the ordinary spectrum is exactly the same as for undirected graphs. Note however, that unlike undirected graphs whose adjacency matrices are symmetric and, therefore, have real eigenvalues, in the case of directed graphs their spectra may consist of complex numbers.

Example

Let $\mathcal{D}_m$ be a directed simple cycle that has m vertices:

$$\mathcal{D}_m = (\{v_1, \ldots, v_m\}, \{(v_i, v_{i+1}) \mid 1 \leq i \leq m-1\} \cup \{(v_m, v_1)\})$$



Its adjacency matrix is

$$A_{\mathcal{D}_m} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

The characteristic polynomial of this matrix is

$$p_{A_{\mathcal{D}_m}} = \begin{vmatrix} \lambda & -1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & -1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ -1 & 0 & 0 & 0 & \cdots & \lambda \end{vmatrix} = \lambda^m - 1.$$

Thus, the ordinary spectrum of $\mathcal{D}_m$ consists of the m-ary complex roots $z_0 = 1, z_1, \ldots, z_{m-1}$ of 1, where

$$z_k = \cos \frac{2k\pi}{m} + i \sin \frac{2k\pi}{m} = e^{\frac{2ki\pi}{m}},$$

for $0 \leq k \leq m-1$.

Observe that

$$A_{\mathcal{D}_m} \begin{pmatrix} 1 \\ z_k \\ \vdots \\ z_k^{m-1} \end{pmatrix} = \begin{pmatrix} z_k \\ \vdots \\ z_k^{m-1} \\ 1 \end{pmatrix} = z_k \begin{pmatrix} 1 \\ \vdots \\ z_k^{m-2} \\ z_k^{m-1} \end{pmatrix},$$

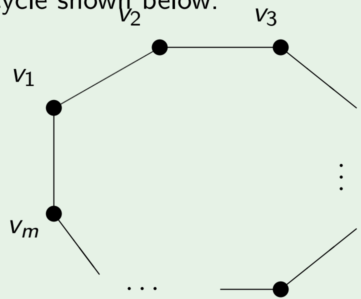
which shows that each ordinary eigenvalue z_k of $\mathcal{D}_m$ has

$$= \begin{pmatrix} 1 \\ z_k \\ \vdots \\ z_k^{m-1} \end{pmatrix}$$

as an eigenvector.

Example

The graph $C_m = (\{v_1, \ldots, v_m\}, \{(v_i, v_{i+1}) | 1 \leqslant i \leqslant m\} \cup \{(v_m, v_1)\})$ is a simple cycle shown below.



The adjacency matrix of C_m is $A_{C_m} = A_{D_m} + A'_{D_m}$.

The Laplacian of C_m is

$$L_{C_m} = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & -1 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & 0 \\ -1 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}.$$

C_m has the Laplacian spectrum

$$\left\{ 2 - 2\cos\frac{2k\pi}{m} \middle| 0 \leqslant k \leqslant m-1 \right\}$$

because C_m is a 2-regular graph.

Example

Let $\mathcal{P}_m = (\{v_1, \ldots, v_m\}, \{(v_i, v_{i+1}) | 1 \leqslant i \leqslant m-1\}$ be the path graph.

The Laplacian matrix of P_m is the tridiagonal matrix

$$L_{P_m} = \begin{pmatrix} 1 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ 0 & 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

To compute the Laplacian spectrum of P_m let z be a complex number such that $z^{2m} = 1$ and let

$$\mathbf{u} = \begin{pmatrix} 1 + z^{2m-1} \\ z + z^{2m-2} \\ \vdots \\ z^j + z^{2m-1-j} \\ \vdots \\ z^{m-2} + z^{m+1} \\ z^{m-1} + z^m \end{pmatrix}.$$

The equalities

$$1 + z^{2m-1} - (z + z^{2m-2}) = (2 - z - z^{-1})(1 + z^{2m-1}),$$

$$-(z^{j-1} + z^{2m-j}) + 2(z^j + z^{2m-1-j}) - (z^{j+1} + z^{2m-2-j}) = (2 - z - z^{-1})(z^j + z^{2m-1-j}),$$

and

$$-(z^{m-2} + z^{m+1}) + z^{m-1} + z^m = (2 - z - z^{-1})(z^{m-1} + z^m)$$

can be directly verified and they show that $\mathbf{u}$ is an eigenvector of L_{P_m} that corresponds to the eigenvalue $2 - z - z^{-1}$.

If $z_k = \cos \frac{2k\pi}{2m} + i \sin \frac{2k\pi}{2m}$, then the Laplacian spectrum of P_m consists of numbers of the form

$$2 - 2\cos \frac{2k\pi}{2m} = 4\sin^2 \frac{k\pi}{2m},$$

where $0 \leqslant k \leqslant m - 1$.

Theorem

For a graph $G = (V, E)$ *we have* $\alpha(G) \geqslant 0$; $\alpha(G) = 0$ *if and only if* G *is not connected.*

Proof

It is obvious that $\alpha(G) \geqslant 0$.
Suppose that G is not connected, and let $G_1 = (V_1, E_1)$ be one of its components. Let $G_1' = (V - V_1, E_1')$ be the subgraph of G determined by the set $V - V_1$. If $|V| = n$ define $\mathbf{y} \in \mathbb{R}^n$ as

$$y_i = \begin{cases} \frac{1}{\sqrt{n}}\sqrt{\frac{|V-V_1|}{|V_1|}} & \text{if } v_i \in V_1 \\ -\frac{1}{\sqrt{n}}\sqrt{\frac{|V_1|}{|V-V_1|}} & \text{if } v_i \in V_2 \end{cases}$$

It is immediate that $\mathbf{y} \in S_n$ and $\mathbf{y}'L(G)\mathbf{y} = 0$. Thus, $\alpha(G) = 0$.

Proof cont'd

Conversely, let $\alpha(G) = 0$. There exists $\mathbf{y} \in S_n$ such that $\mathbf{y}'L(G)\mathbf{y} = 0$. Let $k = \min\{i \mid 1 \leqslant i \leqslant n, y_i \neq 0\}$. Since $\mathbf{y} \in S_n$ we have $\mathbf{y} \neq \mathbf{0}_n$, so k exists. Let $V_1 = \{v_i \mid 1 \leqslant i \leqslant n, y_i = y_k\}$. Clearly, $\emptyset \neq V_1 \neq V$ and there is no edge between a vertex in V_1 and a vertex in $V - V_1$ because $\mathbf{y}'L(G)\mathbf{y} = 0$. Thus, G is not connected.

Theorem

Let $G_i = (V, E_i)$, $i = 1, 2$ *be two graphs having the same set of vertices such that* $E_1 \cap E_2 = \emptyset$. *Define the graph* $G_1 \cup G_2 = (V, E_1 \cup E_2)$, *then* $\alpha(G_1 \cup G_2) \geqslant \alpha(G_1) + \alpha(G_2)$.

Proof

It is easy to see that $L_{G_1 \cup G_2} = L_{G_1} + L_{G_2}$, so

$$\begin{aligned}
\alpha(G_1 \cup G_2) &= \min_{\mathbf{x}} \{ \mathbf{x}^{\mathsf{H}} L_{G_1} \mathbf{x} + bfx^{\mathsf{H}} L_{G_2} \mathbf{x} \mid \parallel \mathbf{x} \parallel_2 = 1 \text{ and } \mathbf{x}'\mathbf{1} = 0 \} \\
&\geqslant \min_{\mathbf{x}} \{ \mathbf{x}^{\mathsf{H}} L_{G_1} \mathbf{x} \mid \parallel \mathbf{x} \parallel_2 = 1 \text{ and } \mathbf{x}'\mathbf{1} = 0 \} \\
&\quad + \min_{\mathbf{x}} \{ \mathbf{x}^{\mathsf{H}} L_{G_2} \mathbf{x} \mid \parallel \mathbf{x} \parallel_2 = 1 \text{ and } \mathbf{x}'\mathbf{1} = 0 \} \\
&= \alpha(G_1) + \alpha(G_2).
\end{aligned}$$

If $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ are two graphs having the same set of vertices we write $G_1 \subseteq G_2$ if $E_1 \subseteq E_2$.

Corollary

If G_1 and G_2 are two graphs on the same set of vertices, then $G_1 \subseteq G_2$ implies $\alpha(G_1) \leqslant \alpha(G_2)$.

Theorem

Let $G' = (V', E')$ be a graph obtained from the graph $G = (V, E)$ by removing k vertices and all edges incident to these vertices. Then, $\alpha(G') \geqslant \alpha(G) - k$.

Proof

The proof is by induction on $k \geqslant 1$ and the base case $k = 1$ is the only non-trivial part.

For $k = 1$ suppose that $V = \{v_1, \ldots, v_m\}$ and $V' = V - \{v_m\}$. Define the graph G_1 as

$$G_1 = (V' \cup \{v_m\}, E' \cup \{(v_i, v_m) \mid 1 \leqslant i \leqslant m-1\}).$$

Clearly $G \subseteq G_1$, so $\alpha(G) \leqslant \alpha(G_1)$. The Laplacian of G_1 has the form

$$L_{G_1} = \begin{pmatrix} L_{G'} + I_{m-1} & -\mathbf{1}_{m-1} \\ -\mathbf{1}'_{m-1} & m-1 \end{pmatrix}.$$

Proof cont'd

If $\mathbf{t}$ is an eigenvector of $L_{G'}$, then

$$\mathbf{z} = \begin{pmatrix} \mathbf{t} \\ 0 \end{pmatrix}$$

is an eigenvector of L_{G_1} and we have

$$L_{G_1} \begin{pmatrix} \mathbf{t} \\ 0 \end{pmatrix} = (\alpha(G') + 1) \begin{pmatrix} \mathbf{t} \\ 0 \end{pmatrix},$$

which proves that $\alpha(G') + 1$ is a non-zero eigenvalue of L_{G_1}. In other words, we have $\alpha(G_1) \leqslant \alpha(G') + 1$, which implies $\alpha(G') \geqslant \alpha(G) - 1$. The induction step is immediate.

Theorem

Let $G_1 = (V_1, E_1)$ *and* $G_2 = (V_2, E_2)$ *be two graphs, where* $V_1 = \{v_1, \ldots, v_p\}$ *and* $V_2 = \{u_1, \ldots, u_q\}$. *We have* $L_{G_1 \times G_2} = L_{G_1} \oplus L_{G_2}$.

Proof

If the vertex (v_i, u_j) of $G_1 \times G_2$ occupies the $\ell^{\rm th}$ place in the list of vertices we have

$$(D_{G_1 \times G_2})_{\ell\ell} = \mathrm{d}_{G_1}(v_i) + \mathrm{d}_{G_2}(u_j),$$

where $i = \left\lceil \frac{\ell}{q} \right\rceil$ and $j = \ell - q\left(\left\lceil \frac{\ell}{q} \right\rceil - 1\right)$. This shows that

$$D_{G_1 \times G_2} = D_{G_1} \oplus D_{G_2}.$$

Thus,

$$\begin{aligned} L_{G_1 \times G_2} = D_{G_1 \times G_2} - A_{G_1 \times G_2} & \\ &= (D_{G_1} \oplus D_{G_2}) - (A_{G_1} \oplus A_{G_2}) \\ &= (D_{G_1} \otimes I_q + I_p D_{G_2}) - (A_{G_1} \otimes I_q + I_p A_{G_2}) \\ &= (D_{G_1} - A_{G_1}) \otimes I_q + I_p \otimes (D_{G_2} - A_{G_2}) \\ &= L_{G_1} \oplus L_{G_2}. \end{aligned}$$

Theorem

Let $G_1 = (V_1, E_1)$ *and* $G_2 = (V_2, E_2)$ *be two graphs. We have* $\alpha(G_1 \times G_2) = \min\{\alpha(G_1), \alpha(G_2)\}$.

Proof

The eigenvalues of the Laplacian $L_{G_1 \times G_2}$ have the form $\lambda + \mu$, where λ is an eigenvalue of L_{G_1} and μ is an eigenvalue of L_{G_2}. Therefore, the second smallest eigenvalue of $L_{G_1 \times G_2}$ is either $\alpha(G_1) + 0$ or $0 + \alpha(G_2)$, which implies the desired statement.

The notion of connectivity can be extended to weighted graphs.

Definition

Let $G = (V, E, c)$ be a weighted graph. Its *connectivity* $\alpha(G)$ is the second smallest eigenvalue of its Laplacian L_G.

Theorem

Let $G = (V, E, c)$ be a connected weighted graph such that $|V| = m$ and $c(v_i, v_j) > 0$ for every $(v_i, v_j) \in E$.
The algebraic connectivity $\alpha(G)$ is positive and is equal to the minimum of the function $\phi : \mathbb{R}^m \longrightarrow \mathbb{R}$ defined by

$$\phi(\mathbf{x}) = m \frac{\sum\{c_{ij}(x_i - x_j)^2 \mid (v_i, v_j) \in E\}}{\sum\{(x_i - x_j)^2 \mid i < j\}},$$

over all vectors $\mathbf{x} \in \mathbb{R}^m$ having distinct components. The corresponding eigenvectors to $\alpha(G)$ are those vectors $\mathbf{y}$ having distinct components for which the minimum of ϕ is attained and for which $\sum_{i=1}^m y_i = 0$.

Proof

Since $L_G \mathbf{1}_m = 0$ and L_G is positive semidefinite, the smallest eigenvalue of L_G is 0. Since G is connected it follows that $\mathbf{1}_m$ is the only linearly independent solution of $(L_G\mathbf{x}, \mathbf{x}) = 0$, so 0 is a simple eigenvalue. All other eigenvalues are positive, and all eigenvectors that correspond to these values are orthogonal to $\mathbf{1}$. Thus, the second smallest eigenvalue $\alpha(G)$ is given by

$$\alpha(G) = \min \left\{ \frac{\sum\{c_{ij}(x_i - x_j)^2 \mid (v_i, v_j) \in E\}}{\sum_{i=1}^m x_i^2} \middle| \mathbf{x} \neq \mathbf{0}, \mathbf{x}'\mathbf{1} = 0 \right\}$$

and the minimum is attained for any eigenvector corresponding to $\alpha(G)$.

By the elementary identity

$$m\sum_{i=1}^{m} x_i^2 - \left(\sum_{i=1}^{m} x_i\right)^2 = \sum_{i<j}(x_i - x_j)^2,$$

taking into account that $\mathbf{x}'\mathbf{1}_m = \sum_{i=1}^{m} x_i = 0$, we have

$$m\sum_{i=1}^{m} x_i^2 = \sum_{i<j}(x_i - x_j)^2,$$

which yields the desired equality. Observe that the value of $\phi(\mathbf{x})$ is invariant with respect to adding a multiple of $\mathbf{1}$ to $\mathbf{x}$.

Corollary

Let $G = (V, E, c)$ be a connected weighted graph such that $c(v_i, v_j) > 0$ for every $(v_i, v_j) \in E$. If $|V| = m$ we have the inequality

$$m \sum \{c_{ij}(x_i - x_j)^2 \mid (v_i, v_j) \in E\} \geq \alpha(G) \sum \{(x_i - x_j)^2 \mid i < j\}$$

for every $\mathbf{x} \in \mathbb{R}^n$.

Definition

Let $G = (V, E, c)$ be a weighted graph. A *Fiedler vector* of the weighted graph is an eigenvector that corresponds to the second smallest eigenvalue $\alpha(G)$.

A Fiedler vector is determined up to a non-zero factor. If $\mathbf{y}$ is a Fiedler vector of the weighted graph $((\{v_1, \ldots, v_m\}, E), c)$, its component y_i corresponds to the vertex v_i is the $\mathbf{y}$-*valuation of the vertex* v_i.

Theorem

(Fiedler's Graph Theorem) *Let* $G = (V, E)$ *be a connected graph such that* $|V| = m$ *and let* $\mathbf{y}$ *be a Fiedler vector. For* $r \geqslant 0$ *let* $V_r = \{v_i \in V \mid y_i \geq r\}$. *Then, the subgraph defined by* V_r *is connected.*

Proof

Let $B \in \mathbb{R}^{m \times m}$ be the symmetric matrix defined by

$$b_{ij} = \begin{cases} w_{ij} & \text{if } i \neq j \text{ and } (v_i, v_j) \in E, \\ 0 & \text{if } i \neq j \text{ and } (v_i, v_j) \notin E, \\ -\sum\{b_{ij} \mid 1 \leqslant j \leq m, j \neq i\} & \text{if } i = j. \end{cases}$$

Since $-(B\mathbf{x}, \mathbf{x}) = \sum\{w_{ij}(x_i - x_j)^2 \mid (v_i, v_j) \in E\}$ we have $B = -L_G$.

Proof cont'd

Since the off-diagonal entries of B are non-negative, $B + \sigma I_m$ is non-negative for sufficiently large σ. The eigenvectors of L_G are identical to those of $B + \sigma I_m$ and the second smallest eigenvalue of L_G corresponds to the second largest eigenvalue of $B + \sigma I_m$. If $\mathbf{y}$ is a Fiedler vector, $\mathbf{y} + e\mathbf{1}$ has the property that the submatrix of $B + \sigma I_m$ that corresponds to the set V_r is irreducible. Thus, the subgraph defined by V_r is connected.

Definition

The *symmetric Laplacian* of a graph $G = (V, E, c)$ is the matrix $L_{G,\text{sym}}$ given by

$$L_{G,\text{sym}} = D_G^{-\frac{1}{2}} L_G D_G^{-\frac{1}{2}} = I - D_G^{-\frac{1}{2}} A_G D_G^{-\frac{1}{2}}.$$

The *random walk Laplacian* of G is the matrix $L_{G,\text{rw}}$ defined as

$$L_{G,\text{rw}} = D_G^{-1} L_G = I - D_G^{-1} A_G.$$

Theorem

The pair $(\lambda, \mathbf{t})$ *is an eigenpair of the symmetric Laplacian* $L_{G,sym}$*, if and only if* $(\lambda, D^{-\frac{1}{2}}\mathbf{t})$ *is an eigenpair of the random walk Laplacian* $L_{G,rw}$*.*

Proof

Let $(\lambda, \mathbf{t})$ be an eigenpair of the symmetric Laplacian $L_{G,\text{sym}}$. We have $L_{G,\text{sym}}\mathbf{t} = \lambda\mathbf{t}$, or $D_G^{-\frac{1}{2}} L_G D_G^{-\frac{1}{2}}\mathbf{t} = \lambda\mathbf{t}$, so $L_G D_G^{-\frac{1}{2}} = \lambda D_G^{\frac{1}{2}}\mathbf{t}$. By multiplying this equality to the left by D^{-1} we obtain finally
$D^{-1} L_{G,\text{sym}}(D_G^{\frac{1}{2}}\mathbf{t}) = \lambda(D_G^{\frac{1}{2}}\mathbf{t})$, or $L_{G,\text{rw}}(D_G^{\frac{1}{2}}\mathbf{t}) = \lambda(D_G^{\frac{1}{2}}\mathbf{t})$, which proves that $(\lambda, D_G^{\frac{1}{2}}\mathbf{t})$ is an eigenpair of $L_{G,\text{rw}}$.
The reverse implication follows by observing that all implications mentioned in the previous argument hold in reverse.

Lemma

Let $G = (V, E, c)$ *be a weighted graph, where* $V = \{v_1, \ldots, v_m\}$. *For* $\mathbf{x} \in \mathbb{R}^m$ *we have* $\mathbf{x}' L_{G,sym} \mathbf{x} = \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} c_{ij} \left(\frac{x_i}{\sqrt{d(v_i)}} - \frac{x_j}{\sqrt{d(v_j)}} \right)^2$.

Proof

By the definition of the symmetric Laplacian we have $\mathbf{x}' L_{G,\mathrm{sym}} \mathbf{x} = \mathbf{x}' D_G^{-\frac{1}{2}} L_G D_G^{-\frac{1}{2}} \mathbf{x}$. Note that

$$D_G^{-\frac{1}{2}} \mathbf{x} = \mathrm{diag}\left(\frac{1}{\sqrt{\mathrm{d}(v_1)}}, \ldots, \frac{1}{\sqrt{\mathrm{d}(v_n)}} \right) \mathbf{x} = \begin{pmatrix} \frac{x_1}{\sqrt{\mathrm{d}(v_1)}} \\ \vdots \\ \frac{x_n}{\sqrt{\mathrm{d}(v_n)}} \end{pmatrix}.$$

An application of a previous result yields the desired conclusion.

Theorem

The symmetric Laplacian of a weighted graph $G = (V, E, c)$ *is a symmetric positive semi-definite matrix that has the eigenvalues* $0 = \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$. *The eigenvector that corresponds to the eigenvalue* 0 *is* $D^{\frac{1}{2}}\mathbf{1}_n$.

Proof

The symmetry and the positive semi-definiteness of $L_{G,\mathrm{sym}}$ are immediate. Therefore all eigenvalues of $L_{G,\mathrm{sym}}$ are real and non-negative. Finally, we have:

$$L_{G,\mathrm{sym}} D^{\frac{1}{2}} \mathbf{1}_n = D_G^{-\frac{1}{2}} L_G D_G^{-\frac{1}{2}} D^{\frac{1}{2}} \mathbf{1}_n = \mathbf{0}_n.$$

The Laplacian matrix of a weighted graph can be computed in **R** starting from the adjacency matrix A of the graph using the function `laplacian` defined below. Note that the function has a parameter `symm`. When this parameter is set to `TRUE` the function computes the symmetric Laplacian

```
laplacian <- function(A, symm = TRUE){
if(dim(A)[1] != dim(A)[2]){
          print(''Error: adjacency matrix must be square'')
        }

        n <- dim(A)[1]
        g <- colSums(A)
        if(symm) {
          D_half = diag(1/sqrt(g))
          return(diag(n) - D_half %*% A %*% D_half)
        }
        else
          return(diag(g) - A)
        }
}
```

Example

By applying the function `laplacian` to the adjacency matrix

```
> A
     [,1] [,2] [,3] [,4] [,5]
[1,]    0    3    0    0    2
[2,]    3    0    3    0    4
[3,]    0    3    0    5    0
[4,]    0    0    5    2    2
[5,]    2    4    0    2    0
```

we obtain:

```
> L <- laplacian(A)
> L
           [,1]       [,2]       [,3]       [,4]       [,5]
[1,]  1.0000000 -0.4242641  0.0000000  0.0000000 -0.3162278
[2,] -0.4242641  1.0000000 -0.3354102  0.0000000 -0.4472136
[3,]  0.0000000 -0.3354102  1.0000000 -0.5892557  0.0000000
[4,]  0.0000000  0.0000000 -0.5892557  0.7777778 -0.2357023
[5,] -0.3162278 -0.4472136  0.0000000 -0.2357023  1.0000000
```

A similar call

```
L <- laplacian(A,FALSE)
```

results in

```
> L
     [,1] [,2] [,3] [,4] [,5]
[1,]    5   -3    0    0   -2
[2,]   -3   10   -3    0   -4
[3,]    0   -3    8   -5    0
[4,]    0    0   -5    7   -2
[5,]   -2   -4    0   -2    8
```