# CS724: Topics in Algorithms
# Matrix Reducibility

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## Definition

A *permutation* of the set $\{1, \ldots, n\}$ is a bijection $\phi : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$.

A permutation can be represented as

$$\phi : \begin{pmatrix} 1 & 2 & \cdots & n \\ \phi(1) & \phi(2) & \cdots & \phi(n) \end{pmatrix}$$

## Example

The permutation of $\{1, 2, 3, 4\}$ defined by

$$\phi(1) = 3, \phi(2) = 1, \phi(3) = 4, \phi(4) = 2$$

can be written as

$$\phi : \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}.$$

An alternative representation of permutations is by *permutation matrices*.

## Definition

The permutation matrix of the permutation

$$\phi : \begin{pmatrix} 1 & 2 & \cdots & n \\ \phi(1) & \phi(2) & \cdots & \phi(n) \end{pmatrix}$$

is the matrix $P_\phi$ given by

$$P_\phi = [\mathbf{e}_{\phi(1)} \ \mathbf{e}_{\phi(2)} \ \cdots \ \mathbf{e}_{\phi(n)}]$$

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## Example

The matrix of the permutation

$$\phi : \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}.$$

is

$$\begin{aligned} P_\phi &= [\mathbf{e}_3\ \mathbf{e}_1\ \mathbf{e}_4\ \mathbf{e}_2] \\ &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \end{aligned}$$

A permutation matrix has exactly a single 1 in every row and column.

If $A \in \mathbb{R}^{n \times n}$ is a matrix and $\phi : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$ is a permutation, then $P_\phi A P'_\phi$ is the matrix obtained from $A$ by renumbering the rows and columns of $A$ according to permutation $\phi$.

## Example

By using the permutation

$$\phi : \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}.$$

defined above row 1 and column 1 become row 3 and column 3, etc.

The result of the matrix product $P_\phi A P'_\phi$ can be observed below:

```
> P <- matrix(c(0,0,1,0,1,0,0,0,0,0,0,1,0,1,0,0),ncol=4)
> P
     [,1] [,2] [,3] [,4]
[1,]    0    1    0    0
[2,]    0    0    0    1
[3,]    1    0    0    0
[4,]    0    0    1    0
> A <-matrix(c(1:16),4,4)
> A
     [,1] [,2] [,3] [,4]
[1,]    1    5    9   13
[2,]    2    6   10   14
[3,]    3    7   11   15
[4,]    4    8   12   16
> P \%*\% A \%*\% t(P)
     [,1] [,2] [,3] [,4]
[1,]    6   14    2   10
[2,]    8   16    4   12
[3,]    5   13    1    9
[4,]    7   15    3   11
```

## Definition

A matrix $A \in \mathbb{R}^{n \times n}$ is *reducible* if there exists a permutation matrix $P_\phi$ such that

$$A = P_\phi \begin{pmatrix} U & V \\ O_{p,q} & W \end{pmatrix} P'_\phi,$$

where $U \in \mathbb{R}^{p \times p}$ and $V \in \mathbb{R}^{q \times q}$ are square matrices. Otherwise, $A$ is said to be *irreducible*.

$A \in \mathbb{R}^{n \times n}$ is irreducible if and only if there is no partition $\{I, J\}$ of the set $\{1, \ldots, n\}$ such that $a_{ij} = 0$ when $i \in I$ and $j \in J$.
Note that if $A$ is a reducible matrix,

$$A = \begin{pmatrix} U & V \\ O_{p,q} & W \end{pmatrix},$$

where $U \in \mathbb{R}^{p \times p}$ and $V \in \mathbb{R}^{q \times q}$, and we seek to solve the linear system $A\mathbf{x} = \mathbf{b}$, by partitioning $\mathbf{x}$ and $\mathbf{b}$ as

$$\mathbf{x} = \begin{pmatrix} \mathbf{y} \\ \mathbf{z} \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} \mathbf{c} \\ \mathbf{d} \end{pmatrix},$$

where $\mathbf{y}, \mathbf{c} \in \mathbb{R}^p$ and $\mathbf{z}, \mathbf{d} \in \mathbb{R}^q$, the original system can be replaced by the equivalent system

$$\begin{aligned} U\mathbf{y} + V\mathbf{z} &= \mathbf{c}, \\ W\mathbf{z} &= \mathbf{d}. \end{aligned}$$

The solution of the initial system has been reduced of solving this new system. It is clear that $\mathbf{z} = W^{-1}\mathbf{d}$ and $\mathbf{y} = U^{-1}(\mathbf{c} - VW^{-1}\mathbf{d})$.

Let $A$ be a symmetric matrix. The *degree of reducibility* of $A$ is $k$, where $0 \leq k \leq n-1$ if there exists a partition $\{I_1, \ldots, I_{k+1}\}$ of $\{1, \ldots, n\}$ such that:

- any submatrix $A\begin{bmatrix} I_i \\ I_i \end{bmatrix}$ is irreducible for $1 \leq i \leq k+1$;
- $a_{pq} = 0$ whenever $p \in I_i$, $q \in I_j$, and $i \neq j$.

The degree of reducibility of $A$ is denoted by $\text{red}(A)$.

The adjacency matrix is the usual matrix representation of a directed graph (or a digraph). In this section we are concerned with the reverse process that associates a graph to a matrix.

## Definition

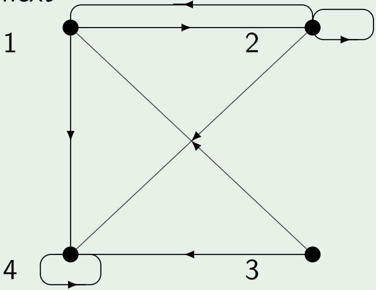
Let $A \in \mathbb{C}^{n \times n}$ be a square matrix. The *directed graph* of $A$ is the graph $G_A = (\{1, \ldots, n\}, E)$, where $(i, j) \in E$ if and only if $a_{ij} \neq 0$.

# Example

The directed graph of the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 4 \\ -1 & 2 & 0 & 5 \\ -5 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

is shown next



Observe that if the vertices of the directed graph $G_A$ are renumbered by replacing each number $j$ by $\phi(j)$, where $\phi$ is a permutation of the set $\{1, \ldots, n\}$, the new resulting graph $G'$ corresponds to the matrix $B = P_\phi A \Phi_\phi^{-1} = P_\phi A \Phi'_\phi$.

### Definition

A matrix $A \in \mathbb{R}^{n \times n}$ is *reducible* if there exists a permutation matrix $P_\phi$ such that

$$A = P_\phi \begin{pmatrix} U & V \\ O_{p,q} & W \end{pmatrix} P'_\phi,$$

where $U \in \mathbb{R}^{p \times p}$ and $V \in \mathbb{R}^{q \times q}$ are square matrices. Otherwise, $A$ is said to be *irreducible*.

$A \in \mathbb{R}^{n \times n}$ is irreducible if and only if there is no partition $\{I, J\}$ of the set $\{1, \ldots, n\}$ such that $a_{ij} = 0$ when $i \in I$ and $j \in J$.

### Theorem

*Let $A \in \mathbb{R}^{n \times n}$ be a matrix. If $G_A$ is a strongly connected digraph, then $A$ is irreducible.*

### Proof.

Let $A \in \mathbb{R}^{n \times n}$ be a reducible matrix. There exists a partition $\{I, J\}$ of the set $\{1, \ldots, n\}$ such that $a_{ij} = 0$ when $i \in I$ and $j \in J$. Therefore, there is no edge from a vertex in $J$ to a vertex in $I$, which implies that there exists a vertex $i \in I$ and a vertex $j \in J$ such that there is no path leading from $j$ to $i$. Thus, $G_A$ is not strongly connected. This shows that if $G_A$ is strongly connected, then $A$ is irreducible. □

**Theorem**

*The directed graph $G_A$ of an irreducible matrix $A$ is strongly connected.*

Suppose that $G_A$ is not strongly connected and let $V_1, \ldots, V_k$ be the strong connected components of $G_A$, where $k > 1$. Since the condensed digraph $\mathrm{C}(G_A)$ is acyclic we may assume without loss of generality that its vertices $V_1, \ldots, V_k$ are numbered in topological order. In other words, the existence of an edge $(V_i, V_j)$ in $\mathrm{C}(G)$ implies $i < j$.

Assume initially that the vertices of the strong component $V_i$ are

$$v_p, v_{p+1}, \ldots, v_{p+|V_i|-1},$$

where $p = 1 + \sum_{j=1}^{i-1} |V_j|$ for $1 \leq i \leq j \leq k$. Under this assumption we have $a_{pq} = 0$ if $v_p \in V_i$, $v_q \in V_\ell$ and $i > \ell$. In other words the matrix $A$ has the form

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ O & A_{22} & \cdots & A_{2k} \\ \vdots & \vdots & \ldots & \vdots \\ O & O & \cdots & A_{kk} \end{pmatrix},$$

where $A_{ii}$ is the incidence matrix of the subgraph induced by $V_i$. Thus, $A$ is not irreducible.

If the vertices of $G_A$ are not numbered according to the previous assumptions, let $\phi$ be a permutation that rearranges the vertices in the needed order. Then $P_\phi A P'_\phi$ has the necessary form and, again, $A$ is not irreducible.

## Corollary

*A matrix $A$ is irreducible if and only if its directed graph $G_A$ is strongly connected. Moreover, $\text{red}(A) = k - 1$ if and only if $G_A$ has $k$ strong components. In this case, there is a permutation matrix $P_\phi$ such that $P_\phi A P'_\phi$ is an upper block triangular matrix*

$$P_\phi A P'_\phi = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ O & A_{22} & \cdots & A_{2k} \\ \vdots & \vdots & \cdots & \vdots \\ O & O & \cdots & A_{kk} \end{pmatrix},$$

*having irreducible diagonal blocks.*

**Corollary**

*A matrix $A$ is irreducible if and only if its transpose is irreducible.*

**Proof.**

This statement follows immediately by observing that the digraph $G_{A'}$ is obtained from $G_A$ by reversing the direction of all edges. □

## Theorem

*Let $A \in \mathbb{R}^{n \times n}$ be a non-negative matrix. For $m \geq 1$ we have $(A^m)_{ij} > 0$ if and only if there exists a path of length $m$ in the graph $G_A$ from $i$ to $j$.*

# Proof

The argument is by induction on $m \geq 1$. The base case, $m = 1$, is immediate.

Suppose that the theorem holds for numbers less than $m$. Then, $(A^m)_{ij} = \sum_{k=1}^{n} (A^{m-1})_{ik} A_{kj}$. $(A^m)_{ij} > 0$ if and only if there is a positive term $(A^{m-1})_{ik} A_{kj}$ in the right-hand sum because all terms are non-negative. By the inductive hypothesis this is the case if and only if there exists a path of length $m-1$ joining $i$ to $k$ and an edge joining $k$ to $j$, that is, a path of length $m$ joining $i$ to $j$.

## Theorem

*Let* $A \in \mathbb{R}^{n\times n}$ *be an irreducible matrix such that* $A \geq O$*. If* $k_i > 0$ *for* $1 \leq i \leq n-1$*, then* $\sum_{i=0}^{n-1} k_i A^i > 0$*.*

# Proof

Since $A$ is an irreducible matrix, the graph $G_A$ is strongly connected. Thus, there exists a path of length no larger than $n-1$ that joins any two distinct vertices $i$ and $j$ of the graph $G_A$. This implies that for some $m \leq n-1$ we have $(A^m)_{ij} > 0$. Since

$$\left(\sum_{i=0}^{n-1} k_i A^i\right)_{ij} = \sum_{i=0}^{n-1} k_i (A^m)_{ij},$$

and all numbers that occur in this equality are non-negative, it follows that for $i \neq j$ we have $\left(\sum_{i=0}^{n-1} k_i A^i\right)_{ij} > 0$. If $i = j$, the same inequality follows from the fact that $k_0 I > O$.

## Corollary

*Let $A \in \mathbb{R}^{n \times n}$ be an irreducible matrix such that $A \geq O$, then $(I + A)^n > 0$.*

If we choose $k_i = \binom{n}{i}$ for $0 \leq i \leq n - 1$ the desired inequality follows immediately.