Homework 2

Posted: February 20, 2019
Due: March 6, 2019

1. Let $B = \{x_1, \ldots, x_n\}$ be a finite subset of a metric space (S, d). Prove that
$$(n-1)\sum_{i=1}^{n} d(x, x_i) \geqslant \sum\{d(x_i, x_j) \mid 1 \leqslant i < j \leqslant n\}$$
for every $x \in S$.
Explain why this inequality can be seen as a generalization of the triangular inequality.
2. Let (S, d) be a metric space and let $u \in S$ be a fixed element of S. Define the function $d_u : S^2 \longrightarrow \mathbb{R}_{\geqslant 0}$ by
$$d_u(x,y) = \begin{cases} 0 & \text{if } x = y, \\ d(x,u) + d(u,y) & \text{otherwise,} \end{cases}$$
for $x, y \in S$. Prove that d_u is a metric on S.
3. Let (S, d) be a metric space. Prove that $\sqrt{d}$ and $\frac{d}{1+d}$ are also metrics on S. What can be said about d^2?
4. Let (S, d) be an ultrametric space. Prove that if $a \geqslant 0$, then the mapping $d_a : S \times S \longrightarrow \mathbb{R}_{\geqslant 0}$ defined by $d_a(x,y) = (d(x,y))^a$ is also an ultrametric metric on S.
5. Let (S, d) be a metric space. Prove that d is an ultrametric on S if and only if for every $a > 0$ the mapping $d_a : S \times S \longrightarrow \mathbb{R}_{\geqslant 0}$ defined by $d_a(x,y) = (d(x,y))^a$ for $x, y \in S$ is a metric on S
6. Let (S, d) be a dissimilarity space. Prove that d is an quasi-ultrametric if and only if for every $u, v \in S$ we have $B[u, d(u,v)] = B[v, d(u,v)]$.