6

Norms and Inner Products

6.1 Introduction

The notion of norm is introduced for evaluating the magnitude of vectors and, in turn, allows the definition of certain metrics on linear spaces equipped with norms.

After presenting some useful inequalities on linear spaces, we introduce norms and the topologies they induce on linear spaces. Then, we discuss inner products, angles between vectors, and the orthogonality of vectors.

We study unitary, orthogonal, positive definite and positive semidefinite matrices that describe important classes of linear transformations. The notion of orthogonality leads to the study of projection on subspaces and the Gram-Schmidt orthogonalization algorithm.

6.2 Inequalities on Linear Spaces

We begin with a technical result.

Lemma 6.1. *Let $p, q \in \mathbb{R} - \{0, 1\}$ be two numbers such that $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$. Then, for every $a, b \in \mathbb{R}_{\geqslant 0}$, we have*

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q},$$

where the equality holds if and only if $a = b^{-\frac{1}{1-p}}$.

Proof. Let $f : [0, \infty) \longrightarrow \mathbb{R}$ be the function defined by $f(x) = x^p - px + p - 1$. Note that $f(1) = 0$ and that $f'(x) = p(x^p - 1)$. This implies that f has a minimum in $x = 1$ and, therefore, $x^p - px + p - 1 \geqslant 0$ for $x \in [0, \infty)$. Substituting $ab^{-\frac{1}{p-1}}$ for x yields the desired inequality.

Theorem 6.2 (The Hölder Inequality). *Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be $2n$ nonnegative numbers, and let p and q be two numbers such that $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$. We have*

$$\sum_{i=1}^{n} a_i b_i \leqslant \left(\sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{n} b_i^q \right)^{\frac{1}{q}}.$$

Proof. Define the numbers

$$x_i = \frac{a_i}{\left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}}} \text{ and } y_i = \frac{b_i}{\left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}}}$$

for $1 \leqslant i \leqslant n$. Lemma 6.1 applied to x_i, y_i yields

$$\frac{a_i b_i}{\left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}}} \leqslant \frac{1}{p} \frac{a_i^p}{\sum_{i=1}^{n} a_i^p} + \frac{1}{q} \frac{b_i^p}{\sum_{i=1}^{n} b_i^p}.$$

Adding these inequalities, we obtain

$$\sum_{i=1}^{n} a_i b_i \leqslant \left(\sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i^q \right)^{\frac{1}{q}}.$$

Theorem 6.3. *Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be $2n$ real numbers and let p and q be two numbers such that $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$. We have*

$$\left| \sum_{i=1}^{n} a_i b_i \right| \leqslant \left(\sum_{i=1}^{n} |a_i|^p \right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{n} |b_i|^q \right)^{\frac{1}{q}}.$$

Proof. By Theorem 6.2, we have

$$\sum_{i=1}^{n} |a_i||b_i| \leqslant \left(\sum_{i=1}^{n} |a_i|^p \right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{n} |b_i|^q \right)^{\frac{1}{q}}.$$

The needed equality follows from the fact that

$$\left| \sum_{i=1}^{n} a_i b_i \right| \leqslant \sum_{i=1}^{n} |a_i||b_i|.$$

Corollary 6.4 (The Cauchy Inequality). *Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be $2n$ real numbers. We have*

$$\left| \sum_{i=1}^{n} a_i b_i \right| \leqslant \sqrt{\sum_{i=1}^{n} |a_i|^2} \cdot \sqrt{\sum_{i=1}^{n} |b_i|^2}.$$

Proof. The inequality follows immediately from Theorem 6.3 by taking $p = q = 2$.

Theorem 6.5. (Minkowski's Inequality) *Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be $2n$ nonnegative numbers. If $p \geqslant 1$, we have*

$$\left(\sum_{i=1}^{n}(a_i+b_i)^p\right)^{\frac{1}{p}} \leqslant \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i^p\right)^{\frac{1}{p}}.$$

If $p < 1$, the inequality sign is reversed.

Proof. For $p = 1$, the inequality is immediate. Therefore, we can assume that $p > 1$. Note that

$$\sum_{i=1}^{n}(a_i+b_i)^p = \sum_{i=1}^{n} a_i(a_i+b_i)^{p-1} + \sum_{i=1}^{n} b_i(a_i+b_i)^{p-1}.$$

By Hölder's inequality for p, q such that $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\begin{aligned}\sum_{i=1}^{n} a_i(a_i+b_i)^{p-1} &\leqslant \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n}(a_i+b_i)^{(p-1)q}\right)^{\frac{1}{q}} \\ &= \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n}(a_i+b_i)^{p}\right)^{\frac{1}{q}}.\end{aligned}$$

Similarly, we can write

$$\sum_{i=1}^{n} b_i(a_i+b_i)^{p-1} \leqslant \left(\sum_{i=1}^{n} b_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n}(a_i+b_i)^{p}\right)^{\frac{1}{q}}.$$

Adding the last two inequalities yields

$$\sum_{i=1}^{n}(a_i+b_i)^p \leqslant \left(\left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i^p\right)^{\frac{1}{p}}\right) \left(\sum_{i=1}^{n}(a_i+b_i)^{p}\right)^{\frac{1}{q}},$$

which is equivalent to the desired inequality

$$\left(\sum_{i=1}^{n}(a_i+b_i)^p\right)^{\frac{1}{p}} \leqslant \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i^p\right)^{\frac{1}{p}}.$$

6.3 Norms on Linear Spaces

Definition 6.6. *Let L be a linear space (real or complex). A* seminorm *on L is a mapping $\nu : L \longrightarrow \mathbb{R}$ that satisfies the following conditions:*
(i) $\nu(\boldsymbol{x}+\boldsymbol{y}) \leqslant \nu(\boldsymbol{x}) + \nu(\boldsymbol{y})$ *(subadditivity), and*
(ii) $\nu(a\boldsymbol{x}) = |a|\nu(\boldsymbol{x})$ *(positive homogeneity),*
for $\boldsymbol{x}, \boldsymbol{y} \in L$ and every scalar a.

By taking $a = 0$ in the second condition of the definition we have $\nu(\mathbf{0}) = 0$ for every seminorm on a real or complex space.

A seminorm can be defined on every linear space L. Indeed, if B is a basis of L, $B = \{\mathbf{v}_i \mid i \in I\}$, J is a finite subset of I, and $\mathbf{x} = \sum_{i\in I} x_i \mathbf{v}_i$, define $\nu_J(\mathbf{x})$ as

$$\nu_J(\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x} = \mathbf{0}, \\ \sum_{j\in J} |a_j| & \text{otherwise} \end{cases}$$

for $\mathbf{x} \in L$. We leave to the reader the verification of the fact that ν_J is indeed a seminorm.

Theorem 6.7. *If L is a real or complex linear space and $\nu : L \longrightarrow \mathbb{R}$ is a seminorm on L, then $\nu(\boldsymbol{x} - \boldsymbol{y}) \geqslant |\nu(\boldsymbol{x}) - \nu(\boldsymbol{y})|$ for $\boldsymbol{x}, \boldsymbol{y} \in L$.*

Proof. We have $\nu(\mathbf{x}) \leqslant \nu(\mathbf{x} - \mathbf{y}) + \nu(\mathbf{y})$, so $\nu(\mathbf{x}) - \nu(\mathbf{y}) \leqslant \nu(\mathbf{x} - \mathbf{y})$. Since $\nu(\mathbf{x}-\mathbf{y}) = |-1|\nu(\mathbf{y}-\mathbf{x}) \geqslant \nu(\mathbf{y})-\nu(\mathbf{x})$, we have $-(\nu(\mathbf{x})-\nu(\mathbf{y})) \leqslant \nu(\mathbf{x})-\nu(\mathbf{y})$.

Corollary 6.8. *If $\nu : L \longrightarrow \mathbb{R}$ is a seminorm on the linear space L, then $\nu(\boldsymbol{x}) \geqslant 0$ for $\boldsymbol{x} \in L$.*

Proof. By choosing $\mathbf{y} = \mathbf{0}$ in the inequality of Theorem 6.7 we have $\nu(\mathbf{x}) \geqslant |\nu(\mathbf{x})| \geqslant 0$.

Definition 6.9. *Let L be a real or complex linear space. A* norm *on L is a seminorm $\nu : L \longrightarrow \mathbb{R}$ such that $\nu(\boldsymbol{x}) = 0$ implies $\boldsymbol{x} = \boldsymbol{0}$ for $\boldsymbol{x} \in L$.*

The pair (L, ν) is referred to as a normed linear space*.*

Example 6.10. The set of real-valued continuous functions defined on the interval $[-1, 1]$ is a real linear space. The addition of two such functions f, g, is defined by $(f + g)(x) = f(x) + g(x)$ for $x \in [-1, 1]$; the multiplication of f by a scalar $a \in \mathbb{R}$ is $(af)(x) = af(x)$ for $x \in [-1, 1]$.

Define $\nu(f) = \sup\{|f(x)| \mid x \in [-1, 1]\}$. Since $|f(x)| \leqslant \nu(f)$ and $|g(x)| \leqslant \nu(g)$ for $x \in [-1, 1]\}$, it follows that $|(f+g)(x)| \leqslant |f(x)|+|g(x)| \leqslant \nu(f)+\nu(g)$. Thus, $\nu(f + g) \leqslant \nu(f) + \nu(g)$. We leave to the reader the verification of the remaining properties of Definition 6.6.

We denote $\nu(f)$ by $\| f \|$.

Corollary 6.11. *For $p \geqslant 1$, the function $\nu_p : \mathbb{C}^n \longrightarrow \mathbb{R}_{\geqslant 0}$ defined by*

$$\nu_p(\boldsymbol{x}) = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}},$$

is a norm on the linear space $(\mathbb{C}^n, +, \cdot)$.

Proof. Let $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$. Minkowski's inequality applied to the nonnegative numbers $a_i = |x_i|$ and $b_i = |y_i|$ amounts to

$$\left(\sum_{i=1}^{n} (|x_i| + |y_i|)^p \right)^{\frac{1}{p}} \leqslant \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |y_i|^p \right)^{\frac{1}{p}}.$$

Since $|x_i + y_i| \leqslant |x_i| + |y_i|$ for every i, we have

$$\left(\sum_{i=1}^{n} (|x_i + y_i|)^p \right)^{\frac{1}{p}} \leqslant \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |y_i|^p \right)^{\frac{1}{p}},$$

that is, $\nu_p(\mathbf{x} + \mathbf{y}) \leqslant \nu_p(\mathbf{x}) + \nu_p(\mathbf{y})$. Thus, ν_p is a norm on $\mathbb{C}^n$.

We refer to ν_p as a *Minkowski norm* on $\mathbb{C}^n$.

The normed linear space $(\mathbb{C}^n, \nu_p)$ is denoted by ℓ_p^n.

Example 6.12. Consider the mappings $\nu_1, \nu_\infty : \mathbb{C}^n \longrightarrow \mathbb{R}$ given by

$$\nu_1(\mathbf{x}) = |x_1| + |x_2| + \cdots + |x_n| \text{ and } \nu_\infty(\mathbf{x}) = \max\{|x_1|, |x_2|, \ldots, |x_n|\},$$

for every $\mathbf{x} \in \mathbb{C}^n$. Both ν_1 and ν_∞ are norms on $\mathbb{C}^n$; the corresponding linear spaces are denoted by ℓ_1^n and ℓ_∞^n.

To verify that ν_∞ is a norm we start from the inequality $|x_i + y_i| \leqslant |x_i| + |y_i| \leqslant \nu_\infty(\mathbf{x}) + \nu_\infty(\mathbf{y})$ for $1 \leqslant i \leqslant n$. This in turn implies

$$\nu_\infty(\mathbf{x} + \mathbf{y}) = \max\{|x_i + y_i| \mid 1 \leqslant i \leqslant n\} \leqslant \nu_\infty(\mathbf{x}) + \nu_\infty(\mathbf{y}),$$

which gives the desired inequality.

This norm can be regarded as a limit case of the norms ν_p. Indeed, let $\mathbf{x} \in \mathbb{C}^n$ and let $M = \max\{|x_i| \mid 1 \leqslant i \leqslant n\} = |x_{l_1}| = \cdots = |x_{l_k}|$ for some $l_1, \ldots, l_k$, where $1 \leqslant l_1, \ldots, l_k \leqslant n$. Here $x_{l_1}, \ldots, x_{l_k}$ are the components of $\mathbf{x}$ that have the maximal absolute value and $k \geqslant 1$. We can write

$$\lim_{p \to \infty} \nu_p(\mathbf{x}) = \lim_{p \to \infty} M \left(\sum_{i=1}^{n} \left(\frac{|x_i|}{M} \right)^p \right)^{\frac{1}{p}} = \lim_{p \to \infty} M(k)^{\frac{1}{p}} = M,$$

which justifies the notation ν_∞.

We will frequently use the alternative notation $\| \mathbf{x} \|_p$ for $\nu_p(\mathbf{x})$. We refer to the norm ν_2 as the *Euclidean norm*.

Example 6.13. Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{C}^2$ be a unit vector in the sense of the Euclidean norm. We have $|x_1|^2 + |x_2|^2 = 1$. Since x_1 and x_2 are complex numbers we can write $x_1 = r_1 e^{i\alpha_1}$ and $x_2 = r_2 e^{i\alpha_2}$, where $r_1^2 + r_2^2 = 1$. Thus, there exists $\theta \in (0, \pi/2)$ such that $r_1 = \cos\theta$ and $r_2 = \sin\theta$, which allows us to write

$$\mathbf{x} = \begin{pmatrix} e^{i\alpha_1}\cos\theta \\ e^{i\alpha_2}\sin\theta \end{pmatrix}.$$

Theorem 6.14. *Each norm $\nu : L \longrightarrow \mathbb{R}_{\geqslant 0}$ on a real or complex linear space $(L, +, \cdot)$ generates a metric on the set L defined by $d_\nu(\boldsymbol{x}, \boldsymbol{y}) = \| \boldsymbol{x} - \boldsymbol{y} \|$ for $\boldsymbol{x}, \boldsymbol{y} \in L$.*

Proof. Note that if $d_\nu(\mathbf{x}, \mathbf{y}) = \| \mathbf{x} - \mathbf{y} \| = 0$, it follows that $\mathbf{x} - \mathbf{y} = \mathbf{0}$, so $\mathbf{x} = \mathbf{y}$.

The symmetry of d_ν is obvious and so we need to verify only the triangular axiom. Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in L$. We have

$$\nu(\mathbf{x} - \mathbf{z}) = \nu(\mathbf{x} - \mathbf{y} + \mathbf{y} - \mathbf{z}) \leqslant \nu(\mathbf{x} - \mathbf{y}) + \nu(\mathbf{y} - \mathbf{z})$$

or, equivalently, $d_\nu(\mathbf{x}, \mathbf{z}) \leqslant d_\nu(\mathbf{x}, \mathbf{y}) + d_\nu(\mathbf{y}, \mathbf{z})$, for every $\mathbf{x}, \mathbf{y}, \mathbf{z} \in L$, which concludes the argument.

We refer to d_ν as the *metric induced by the norm ν on the linear space* $(L, +, \cdot)$.

For $p \geqslant 1$, then d_p denotes the metric d_{ν_p} induced by the norm ν_p on the linear space $(\mathbb{C}^n, +, \cdot)$ known as the *Minkowski metric* on $\mathbb{R}^n$.

The metrics d_1, d_2 and d_∞ defined on $\mathbb{R}^n$ are given by

$$d_1(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} |x_i - y_i|, \tag{6.1}$$

$$d_2(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2}, \tag{6.2}$$

$$d_\infty(\mathbf{x}, \mathbf{y}) = \max\{|x_i - y_i| \mid 1 \leqslant i \leqslant n\}, \tag{6.3}$$

for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

These metrics are visualized in Figure 6.1 for the special case of $\mathbb{R}^2$. If $\mathbf{x} = (x_0, x_1)$ and $\mathbf{y} = (y_0, y_1)$, then $d_1(\mathbf{x}, \mathbf{y})$ is the sum of the lengths of the two legs of the triangle, $d_2(\mathbf{x}, \mathbf{y})$ is the length of the hypotenuse of the right triangle and $d_\infty(\mathbf{x}, \mathbf{y})$ is the largest of the lengths of the legs.

Theorem 6.16 to follow allows us to compare the norms ν_p (and the metrics of the form d_p) that were introduced on $\mathbb{R}^n$. We begin with a preliminary result.

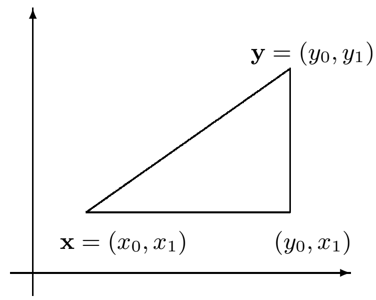

Fig. 6.1. The distances $d_1(\mathbf{x}, \mathbf{y})$ and $d_2(\mathbf{x}, \mathbf{y})$.

Lemma 6.15. *Let $a_1, \ldots, a_n$ be n positive numbers. If p and q are two positive numbers such that $p \leqslant q$, then $(a_1^p + \cdots + a_n^p)^{\frac{1}{p}} \geqslant (a_1^q + \cdots + a_n^q)^{\frac{1}{q}}$.*

Proof. Let $f : \mathbb{R}^{>0} \longrightarrow \mathbb{R}$ be the function defined by $f(r) = (a_1^r + \cdots + a_n^r)^{\frac{1}{r}}$. Since

$$\ln f(r) = \frac{\ln (a_1^r + \cdots + a_n^r)}{r},$$

it follows that

$$\frac{f'(r)}{f(r)} = -\frac{1}{r^2}(a_1^r + \cdots + a_n^r) + \frac{1}{r} \cdot \frac{a_1^r \ln a_1 + \cdots + a_n^r \ln a_r}{a_1^r + \cdots + a_n^r}.$$

To prove that $f'(r) < 0$, it suffices to show that

$$\frac{a_1^r \ln a_1 + \cdots + a_n^r \ln a_r}{a_1^r + \cdots + a_n^r} \leqslant \frac{\ln (a_1^r + \cdots + a_n^r)}{r}.$$

This last inequality is easily seen to be equivalent to

$$\sum_{i=1}^{n} \frac{a_i^r}{a_1^r + \cdots + a_n^r} \ln \frac{a_i^r}{a_1^r + \cdots + a_n^r} \leqslant 0,$$

which holds because

$$\frac{a_i^r}{a_1^r + \cdots + a_n^r} \leqslant 1$$

for $1 \leqslant i \leqslant n$.

Theorem 6.16. *Let p and q be two positive numbers such that $p \leqslant q$. We have $\| \boldsymbol{u} \|_p \geqslant \| \boldsymbol{u} \|_q$ for $\boldsymbol{u} \in \mathbb{C}^n$.*

Proof. This statement follows immediately from Lemma 6.15.

Corollary 6.17. *Let p, q be two positive numbers such that $p \leqslant q$. For every $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^n$, we have $d_p(\boldsymbol{x}, \boldsymbol{y}) \geqslant d_q(\boldsymbol{x}, \boldsymbol{y})$.*

Proof. This statement follows immediately from Theorem 6.16.

Theorem 6.18. *Let $p \geqslant 1$. We have $\| \boldsymbol{x} \|_\infty \leqslant \| \boldsymbol{x} \|_p \leqslant n \| \boldsymbol{x} \|_\infty$ for $\boldsymbol{x} \in \mathbb{C}^n$.*

Proof. The first inequality is an immediate consequence of Theorem 6.16. The second inequality follows by observing that

$$\| \mathbf{x} \|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \leqslant n \max_{1 \leqslant i \leqslant n} |x_i| = n \| \mathbf{x} \|_\infty .$$

Corollary 6.19. *Let p and q be two numbers such that $p, q \geqslant 1$. For $\boldsymbol{x} \in \mathbb{C}^n$ we have:*

$$\frac{1}{n} \| \boldsymbol{x} \|_q \leqslant \| \boldsymbol{x} \|_p \leqslant n \| \boldsymbol{x} \|_q .$$

Proof. Since $\| \mathbf{x} \|_\infty \leqslant \| \mathbf{x} \|_p$ and $\| x \|_q \leqslant n \| \mathbf{x} \|_\infty$, it follows that $\| x \|_q \leqslant n \| \mathbf{x} \|_p$. Exchanging the roles of p and q, we have $\| x \|_p \leqslant n \| \mathbf{x} \|_q$, so

$$\frac{1}{n} \| \mathbf{x} \|_q \leqslant \| \mathbf{x} \|_p \leqslant n \| \mathbf{x} \|_q$$

for every $\mathbf{x} \in \mathbb{C}^n$.

For $p = 1$ and $q = 2$ and $\mathbf{x} \in \mathbb{R}^n$ we have the inequalities

$$\frac{1}{n} \sqrt{\sum_{i=1}^n x_i^2} \leqslant \sum_{i=1}^n |x_i| \leqslant n \sqrt{\sum_{i=1}^n x_i^2}. \tag{6.4}$$

Corollary 6.20. *For every $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^n$ and $p \geqslant 1$, we have $d_\infty(\boldsymbol{x}, \boldsymbol{y}) \leqslant d_p(\boldsymbol{x}, \boldsymbol{y}) \leqslant n d_\infty(\boldsymbol{x}, \boldsymbol{y})$. Further, for $p, q > 1$, there exist $c, d \in \mathbb{R}_{>0}$ such that*

$$c\, d_q(\boldsymbol{x}, \boldsymbol{y}) \leqslant d_p(\boldsymbol{x}, \boldsymbol{y}) \leqslant c\, d_q(\boldsymbol{x}, \boldsymbol{y})$$

for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^n$.

Proof. This follows from Theorem 6.18 and from Corollary 6.20.

Corollary 6.17 implies that if $p \leqslant q$, then the closed sphere $B_{d_p}(\mathbf{x}, r)$ is included in the closed sphere $B_{d_q}(\mathbf{x}, r)$. For example, we have

$$B_{d_1}(\mathbf{0}, 1) \subseteq B_{d_2}(\mathbf{0}, 1) \subseteq B_{d_\infty}(\mathbf{0}, 1).$$

In Figures 6.2 (a) - (c) we represent the closed spheres $B_{d_1}(\mathbf{0}, 1)$, $B_{d_2}(\mathbf{0}, 1)$, and $B_{d_\infty}(\mathbf{0}, 1)$.

An useful consequence of Theorem 6.2 is the following statement:

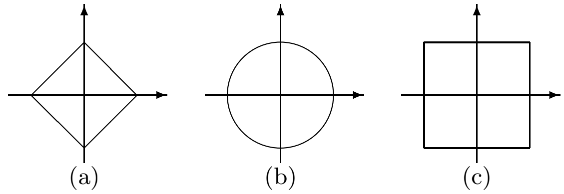

Fig. 6.2. Spheres $B_{d_p}(\mathbf{0}, 1)$ for $p = 1, 2, \infty$.

Theorem 6.21. *Let $x_1, \ldots, x_m$ and $y_1, \ldots, y_m$ be $2m$ nonnegative numbers such that $\sum_{i=1}^m x_i = \sum_{i=1}^m y_i = 1$ and let p and q be two positive numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. We have*

$$\sum_{j=1}^{m} x_j^{\frac{1}{p}} y_j^{\frac{1}{q}} \leqslant 1.$$

Proof. The Hölder inequality applied to $x_1^{\frac{1}{p}}, \ldots, x_m^{\frac{1}{p}}$ and $y_1^{\frac{1}{q}}, \ldots, y_m^{\frac{1}{q}}$ yields the needed inequality

$$\sum_{j=1}^{m} x_j^{\frac{1}{p}} y_j^{\frac{1}{q}} \leqslant \sum_{j=1}^{m} x_j \sum_{j=1}^{m} y_j = 1$$

The linear space $\mathbf{Seq}_\infty(\mathbb{C})$ discussed in Example 5.3 can be equipped with norms similar to the $\{\nu_p \mid p > 1\}$ family.

For $\mathbf{x} = (x_0, x_1, \ldots)$ is a sequence of complex numbers define

$$\nu_p(\mathbf{x}) = \left(\sum_{i=0}^{\infty} |x_i|^p \right)^{\frac{1}{p}}.$$

Theorem 6.22. *The set of sequences $\boldsymbol{x} \in \mathbf{Seq}_\infty(\mathbb{C})$ such that $\nu_p(\boldsymbol{x})$ is finite is a normed linear space.*

Proof. In Example 5.3 we saw that $\mathbf{Seq}_\infty(\mathbb{C})$ can be organized as a linear space. Let $\mathbf{x}, \mathbf{y} \in \mathbf{Seq}_\infty(\mathbb{C})$ be two sequences such that $\nu_p(\mathbf{x})$ and $\nu_p(\mathbf{y})$ are finite. By Minkowski's inequality, if $p \geqslant 1$ we have

$$\left(\sum_{i=1}^{n} |x_i + y_i|^p \right)^{\frac{1}{p}} \leqslant \left(\sum_{i=1}^{n} (|x_i| + |y_i|)^p \right)^{\frac{1}{p}} \leqslant \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |y_i|^p \right)^{\frac{1}{p}}.$$

When n tends to ∞ we have $\nu_p(\mathbf{x} + \mathbf{y}) \leqslant \nu_p(\mathbf{x}) + \nu_p(y)$, so ν_p is a norm.

The normed linear space $(\mathbf{Seq}_\infty(\mathbb{C}), \nu_p)$ is denoted by ℓ_p.

6.4 Inner Products

Inner product spaces are linear spaces equipped with an additional operation that associates to a pair of vectors a scalar called their *inner product*.

Definition 6.23. *Let L be a $\mathbb{C}$-linear space. An* inner product *on L is a function $f : L \times L \longrightarrow \mathbb{C}$ that has the following properties:*
(i) $f(a\boldsymbol{x} + b\boldsymbol{y}, \boldsymbol{z}) = af(\boldsymbol{x}, \boldsymbol{z}) + bf(\boldsymbol{y}, \boldsymbol{z})$ *(linearity in the first argument);*
(ii) $f(\boldsymbol{x}, \boldsymbol{y}) = \overline{f(\boldsymbol{y}, \boldsymbol{x})}$ *for $\boldsymbol{y}, \boldsymbol{x} \in L$ (conjugate symmetry);*
(iii) *if $\boldsymbol{x} \neq \boldsymbol{0}$, then $f(\boldsymbol{x}, \boldsymbol{x})$ is a positive real number (positivity),*
(iv) $f(\boldsymbol{x}, \boldsymbol{x}) = 0$ *if and only if $\boldsymbol{x} = \boldsymbol{0}$ (definiteness),*
for every $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in L$ and $a, b \in \mathbb{C}$.

The pair (L, f) is called an inner product space.

For the second argument of a scalar product we have the property of *conjugate linearity*, that is,

$$f(\mathbf{z}, a\mathbf{x} + b\mathbf{y}) = \bar{a}f(\mathbf{z}, \mathbf{x}) + \bar{b}f(\mathbf{z}, \mathbf{y})$$

for every $\mathbf{x}, \mathbf{y}, \mathbf{z} \in L$ and $a, b \in \mathbb{C}$. Indeed, by the conjugate symmetry property we can write

$$\begin{aligned} f(\mathbf{z}, a\mathbf{x} + b\mathbf{y}) &= \overline{f(a\mathbf{x} + b\mathbf{y}, \mathbf{z})} = \overline{af(\mathbf{x}, \mathbf{z}) + bf(\mathbf{y}, \mathbf{z})} \\ &= \bar{a}\overline{f(\mathbf{x}, \mathbf{z})} + \bar{b}\overline{f(\mathbf{y}, \mathbf{z})} = \bar{a}f(\mathbf{z}, \mathbf{x}) + \bar{b}f(\mathbf{z}, \mathbf{y}). \end{aligned}$$

Observe that conjugate symmetry property on inner products implies that for $\mathbf{x} \in L$, $f(\mathbf{x}, \mathbf{x})$ is a real number because $f(\mathbf{x}, \mathbf{x}) = \overline{f(\mathbf{x}, \mathbf{x})}$.

When L is a real linear space the definition of the inner product becomes simpler because the conjugate of a real number a is a itself. Thus, for real linear spaces, the conjugate symmetry is replaced by the plain symmetry property, $f(\mathbf{x}, \mathbf{y}) = f(\mathbf{y}, \mathbf{x})$, for $\mathbf{x}, \mathbf{y} \in L$ and f is linear in both arguments.

Example 6.24. Let $\mathbb{C}^n$ be the linear space of n-tuples of complex numbers. If $a_1, \ldots, a_n$ are n real, positive numbers, then the function $f : \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}$ defined by $f(\mathbf{x}, \mathbf{y}) = a_1 x_1 \bar{y}_1 + a_2 x_2 \bar{y}_2 + \cdots + a_n x_n \bar{y}_n$ is an inner product on $\mathbb{C}^n$, as the reader can easily verify.

If $a_1 = \cdots = a_n = 1$, we have the *Euclidean inner product*:

$$f(\mathbf{x}, \mathbf{y}) = x_1 \bar{y}_1 + \cdots + x_n \bar{y}_n = \mathbf{y}^{\mathsf{H}}\mathbf{x}.$$

For the linear space $\mathbb{R}^n$, the *Euclidean inner product* is

$$f(\mathbf{x}, \mathbf{y}) = x_1 y_1 + \cdots + x_n y_n = \mathbf{y}'\mathbf{x} = \mathbf{x}'\mathbf{y},$$

where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

To simplify notations we denote an inner product $f(\mathbf{x}, \mathbf{y})$ by $(\mathbf{x}, \mathbf{y})$ when there is no risk of confusion.

A fundamental property of the inner product defined on $\mathbb{C}^n$ in Example 6.24 is the equality

$$(A\mathbf{x}, \mathbf{y}) = (\mathbf{x}, A^{\mathsf{H}}\mathbf{y}), \tag{6.5}$$

which holds for every $A \in \mathbb{C}^{n\times n}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$. Indeed, we have

$$\begin{aligned}(A\mathbf{x}, \mathbf{y}) &= \sum_{i=1}^{n} (A\mathbf{x})_i \bar{y}_i = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_j \bar{y}_i = \sum_{j=1}^{n} x_j \sum_{i=1}^{n} a_{ij}\bar{y}_i \\ &= \sum_{j=1}^{n} x_j \sum_{i=1}^{n} \overline{\bar{a}_{ij} y_i} = (\mathbf{x}, A^{\mathsf{H}}\mathbf{y}).\end{aligned}$$

More generally we have the following definition.

Definition 6.25. *A matrix $B \in \mathbb{C}^{n\times n}$ is the* adjoint *of a matrix $A \in \mathbb{C}^{n\times n}$ relative to the inner product $(\cdot,\cdot)$ if $(A\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{x}, B\boldsymbol{y})$ for every $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^n$.*

A matrix is *self-adjoint* if it equals its own adjoint, that is if $(A\mathbf{x}, \mathbf{y}) = (\mathbf{x}, A\mathbf{y})$ for every $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$. Thus, a Hermitian matrix is self-adjoint relative to the inner product $(\mathbf{x}, \mathbf{y}) = \mathbf{x}^{\mathsf{H}}\mathbf{y}$ for $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$. If we use the Euclidean inner product we omit the reference to this product and refer to the adjoint of A relative to this product simply as the adjoint of A.

Example 6.26. An inner product on $\mathbb{C}^{n\times n}$, the linear space of matrices of format $n \times n$, can be defined as $(X, Y) = \mathit{trace}(XY^{\mathsf{H}})$ for $X, Y \in \mathbb{C}^{n\times n}$.

A linear form on $\mathbb{R}^n$ can be expressed using the Euclidean inner product. Let f be a linear form defined on $\mathbb{R}^n$ and let $\mathbf{x} \in \mathbb{R}^n$. If $\mathbf{u}_1, \ldots, \mathbf{u}_n$ is a basis in $\mathbb{R}^n$ and $\mathbf{x} = x_1\mathbf{u}_1 + \cdots + x_n\mathbf{u}_n$, then $f(\mathbf{x}) = x_1 f(\mathbf{u}_1) + \cdots + x_n f(\mathbf{u}_n)$ and $f(\mathbf{x})$ can be written as $f(\mathbf{x}) = (\mathbf{x}, \mathbf{a})$, where $a_i = f(\mathbf{u}_i)$ for $1 \leqslant i \leqslant n$. The vector $\mathbf{a}$ is uniquely determined for a linear form. Indeed, suppose that there exists $\mathbf{b} \in \mathbb{R}^n$ such that $f(\mathbf{x}) = (\mathbf{x}, \mathbf{b})$ for $\mathbf{x} \in \mathbb{R}^n$. Since $(\mathbf{x}, \mathbf{a}) = (\mathbf{x}, \mathbf{b})$ for every $\mathbf{x} \in \mathbb{R}^n$, it follows that $(\mathbf{x}, \mathbf{a} - \mathbf{b}) = 0$. Choosing $\mathbf{x} = \mathbf{a} - \mathbf{b}$, it follows that $\| \mathbf{a} - \mathbf{b} \|_2 = 0$, so $\mathbf{a} = \mathbf{b}$.

Theorem 6.27. *Any inner product on a linear space L generates a norm on that space defined by $\| \boldsymbol{x} \| = \sqrt{(\boldsymbol{x}, \boldsymbol{x})}$ for $\boldsymbol{x} \in L$.*

Proof. We need to verify that the norm satisfies the conditions of Definition 6.6. Applying the properties of the inner product we have

$$\begin{aligned}\| \mathbf{x} + \mathbf{y} \|^2 &= (\mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y}) = (\mathbf{x}, \mathbf{x}) + 2(\mathbf{x}, \mathbf{y}) + (\mathbf{y}, \mathbf{y}) \\ &= \| \mathbf{x} \|^2 + 2(\mathbf{x}, \mathbf{y}) + \| \mathbf{y} \|^2 \leqslant \| \mathbf{x} \|^2 + 2 \| \mathbf{x} \| \| \mathbf{y} \| + \| \mathbf{y} \|^2 \\ &= (\| \mathbf{x} \| + \| \mathbf{y} \|)^2.\end{aligned}$$

Because $\parallel \mathbf{x} \parallel \geqslant 0$ it follows that $\parallel \mathbf{x}+\mathbf{y} \parallel \leqslant \parallel \mathbf{x} \parallel + \parallel \mathbf{y} \parallel$, which is the subadditivity property.

If $a \in \mathbb{C}$, then $\parallel a\mathbf{x} \parallel = \sqrt{(a\mathbf{x}, a\mathbf{x})} = \sqrt{a\bar{a}(\mathbf{x},\mathbf{x})} = \sqrt{|a|^2(\mathbf{x},\mathbf{x})} = |a|\sqrt{(\mathbf{x},\mathbf{x})} = |a| \parallel \mathbf{x} \parallel$.

Finally, from the definiteness property of the inner product it follows that $\parallel \mathbf{x} \parallel = 0$ if and only if $\mathbf{x} = \mathbf{0}$, which allows us to conclude that $\parallel \cdot \parallel$ is indeed a norm.

Observe that if the inner product $(\mathbf{x}, \mathbf{y})$ of the vectors $\mathbf{x} \in \mathbb{C}^n$ and $y \in \mathbb{C}^n$ is defined as in Example 6.24 with $a_1 = \cdots = a_n = 1$, then

$$\parallel \mathbf{x} \parallel^2 = (\mathbf{x},\mathbf{x}) = x_1\bar{x}_1 + x_2\bar{x}_2 + \cdots + x_n\bar{x}_n = \sum_{i=1}^{2} |x_i|^2,$$

which shows that the norm induced by the inner product is precisely $\parallel \mathbf{x} \parallel_2$.

Not every norm can be induced by an inner product. A characterization of this type of norms in linear spaces is presented next.

This equality shown in the next theorem is known as the *parallelogram equality.*

Theorem 6.28. *Let L be a real linear space. A norm $\parallel \cdot \parallel$ is induced by an inner product if and only if*

$$\parallel \boldsymbol{x}+\boldsymbol{y} \parallel^2 + \parallel \boldsymbol{x}-\boldsymbol{y} \parallel^2 = 2(\parallel \boldsymbol{x} \parallel^2 + \parallel \boldsymbol{y} \parallel^2),$$

for every $\boldsymbol{x}, \boldsymbol{y} \in L$.

Proof. Suppose that the norm is induced by an inner product. In this case we can write for every $\mathbf{x}$ and $\mathbf{y}$:

$$\begin{aligned}(\mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}) &= (\mathbf{x},\mathbf{x}) + 2(\mathbf{x},\mathbf{y}) + (\mathbf{y},\mathbf{y}),\\ (\mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{y}) &= (\mathbf{x},\mathbf{x}) - 2(\mathbf{x},\mathbf{y}) + (\mathbf{y},\mathbf{y}).\end{aligned}$$

Thus,

$$(\mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}) + (\mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{y}) = 2(\mathbf{x},\mathbf{x}) + 2(\mathbf{y},\mathbf{y}),$$

which can be written in terms of the norm generated as the inner product as

$$\parallel \mathbf{x}+\mathbf{y} \parallel^2 + \parallel \mathbf{x}-\mathbf{y} \parallel^2 = 2(\parallel \mathbf{x} \parallel^2 + \parallel \mathbf{y} \parallel^2).$$

Conversely, suppose that the condition of the theorem is satisfied by the norm $\parallel \cdot \parallel$. Consider the function $f : V \times V \longrightarrow \mathbb{R}$ defined by

$$f(\mathbf{x},\mathbf{y}) = \frac{1}{4}\left(\parallel \mathbf{x}+\mathbf{y} \parallel^2 - \parallel \mathbf{x}-\mathbf{y} \parallel^2\right), \tag{6.6}$$

for $\mathbf{x}, \mathbf{y} \in L$. The symmetry of f is immediate, that is, $f(\mathbf{x},\mathbf{y}) = f(\mathbf{y},\mathbf{x})$ for $\mathbf{x}, \mathbf{y} \in L$.

The definition of f implies

$$f(\mathbf{0},\mathbf{y}) = \frac{1}{4}\left(\| \mathbf{y} \|^2 - \| -\mathbf{y} \|^2\right) = 0. \tag{6.7}$$

We prove that f is a bilinear form that satisfies the conditions of Definition 6.23.

Starting from the parallelogram equality we can write

$$\begin{aligned}
&\| \mathbf{u}+\mathbf{v}+\mathbf{y} \|^2 + \| \mathbf{u}+\mathbf{v}-\mathbf{y} \|^2 = 2(\| \mathbf{u}+\mathbf{v} \|^2 + \| \mathbf{y} \|^2),\\
&\| \mathbf{u}-\mathbf{v}+\mathbf{y} \|^2 + \| \mathbf{u}-\mathbf{v}-\mathbf{y} \|^2 = 2(\| \mathbf{u}-\mathbf{v} \|^2 + \| \mathbf{y} \|^2).
\end{aligned}$$

Subtracting these equality yields:

$$\begin{aligned}
&\| \mathbf{u}+\mathbf{v}+\mathbf{y} \|^2 + \| \mathbf{u}+\mathbf{v}-\mathbf{y} \|^2 - \| \mathbf{u}-\mathbf{v}+\mathbf{y} \|^2 - \| \mathbf{u}-\mathbf{v}-\mathbf{y} \|^2\\
&= 2(\| \mathbf{u}+\mathbf{v} \|^2 - \| \mathbf{u}-\mathbf{v} \|^2).
\end{aligned}$$

This equality can be written as

$$f(\mathbf{u}+\mathbf{y},\mathbf{v}) + f(\mathbf{u}-\mathbf{y},\mathbf{v}) = 2f(\mathbf{u},\mathbf{v}).$$

Choosing $\mathbf{y}=\mathbf{u}$ implies

$$f(2\mathbf{u},\mathbf{v}) = 2f(\mathbf{u},\mathbf{v}), \tag{6.8}$$

due to Equality (6.7).

Let $\mathbf{t}=\mathbf{u}+\mathbf{y}$ and $\mathbf{s}=\mathbf{u}-\mathbf{y}$. Since $\mathbf{u}=\frac{1}{2}(\mathbf{t}+\mathbf{s})$ and $\mathbf{y}=\frac{1}{2}(\mathbf{t}-\mathbf{s})$ we have

$$f(\mathbf{t},\mathbf{v}) + f(\mathbf{s},\mathbf{v}) = 2f\left(\frac{1}{2}(\mathbf{t}+\mathbf{s}),\mathbf{v}\right) = f(\mathbf{t}+\mathbf{s},\mathbf{v}),$$

by Equality (6.8).

Next, we show that $f(a\mathbf{x},\mathbf{y}) = af(\mathbf{x},\mathbf{y})$ for $a \in \mathbb{R}$ and $\mathbf{x},\mathbf{y} \in L$. Consider the function $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $\phi(a) = f(a\mathbf{x}+\mathbf{y})$.

The basic properties of norms imply that

$$\Big| \| a\mathbf{x}+\mathbf{y} \| - \| b\mathbf{x}+\mathbf{y} \| \Big| \leqslant \| (a-b)\mathbf{x} \|$$

for every $a,b \in \mathbb{R}$ and $\mathbf{x},\mathbf{y} \in L$. Therefore, the function $\phi,\psi : \mathbb{R} \longrightarrow \mathbb{R}$ given by $\phi(a) = \| a\mathbf{x}+\mathbf{y} \|$ and $\psi(a) = \| a\mathbf{x}-\mathbf{y} \|$ for $a \in \mathbb{R}$ are continuous. The continuity of these functions implies that the function f defined by Equality (6.6) is continuous relative to a.

Define the set:

$$S = \{a \in \mathbb{R} \mid f(a\mathbf{x},\mathbf{y}) = af(\mathbf{x},\mathbf{y})\}.$$

Clearly, we have $1 \in S$. Further, if $a,b \in S$, then $a+b \in S$ and $a-b \in S$, which implies $\mathbb{Z} \subseteq S$.

If $b \neq 0$ and $b \in S$, then, by substituting $\mathbf{x}$ by $\frac{1}{b}\mathbf{x}$ in the equality $f(b\mathbf{x},\mathbf{y}) = bf(\mathbf{x},\mathbf{y})$ we have $f(\mathbf{x},\mathbf{y}) = bf(\frac{1}{b}\mathbf{x},\mathbf{y})$, so $\frac{1}{b}f(\mathbf{x},\mathbf{y}) = f(\frac{1}{b}\mathbf{x},\mathbf{y})$. Thus, if $a,b \in S$

and $b \neq 0$, we have $f(\frac{a}{b}\mathbf{x}, \mathbf{y}) = \frac{a}{b} f(\mathbf{x}, \mathbf{y})$, so $\mathbb{Q} \subseteq S$. Consequently, $S = \mathbb{R}$. This allows us to conclude that f is linear in its first argument. The symmetry of f implies the linearity in its second argument, so f is bilinear.

Observe that $f(\mathbf{x}, \mathbf{x}) = \parallel \mathbf{x} \parallel^2$. The definition of norms implies that $f(\mathbf{x}, \mathbf{x}) = 0$ if and only if $\mathbf{x} = \mathbf{0}$ and if $\mathbf{x} \neq \mathbf{0}$, then $f(\mathbf{x}, \mathbf{x}) > 0$. Thus, f is indeed an inner product and $\parallel \mathbf{x} \parallel = \sqrt{f(\mathbf{x}, \mathbf{x})}$.

Theorem 6.29. *Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ be two vectors such that $x_1 \geqslant x_1 \geqslant \cdots \geqslant x_n$, $y_1 \geqslant y_2 \geqslant \cdots \geqslant y_n$. For every permutation matrix P we have $\boldsymbol{x}'\boldsymbol{y} \geqslant \boldsymbol{x}'(P\boldsymbol{y})$.*

If $x_1 \geqslant x_1 \geqslant \cdots \geqslant x_n$ and $y_1 \leqslant y_2 \leqslant \cdots \leqslant y_n$, then for every permutation matrix P we have $\boldsymbol{x}'\boldsymbol{y} \leqslant \boldsymbol{x}'(P\boldsymbol{y})$.

Proof. Let ϕ be the permutation that corresponds to the permutation matrix P and suppose that $\phi = \psi_p \ldots \psi_1$, where $p = inv(\phi)$ and $\psi_1, \ldots, \psi_p$ are standard transpositions that correspond to all standard inversions of ϕ.

Let ψ be a standard transposition of $\{1, \ldots, n\}$,

$$\psi : \begin{pmatrix} 1 & \cdots & i & i+1 & \cdots & n \\ 1 & \cdots & i+1 & i & \cdots & n \end{pmatrix}.$$

We have

$$\mathbf{x}'(P\mathbf{y}) = x_1y_1 + \cdots + x_{i-1}y_{i-1} + x_iy_{i+1} + x_{i+1}y_i + \cdots + x_ny_n,$$

so the inequality $\mathbf{x}'\mathbf{y} \geqslant \mathbf{x}'(P\mathbf{y})$ is equivalent to

$$x_iy_i + x_{i+1}y_{i+1} \geqslant x_iy_{i+1} + x_{i+1}y_i.$$

This, in turn is equivalent to $(x_{i+1} - x_i)(y_{i+1} - y_i) \geqslant 0$, which obviously holds in view of the hypothesis.

As we observed previously, $P_\phi = P_{\psi_1} \cdots P_{\psi_p}$, so

$$\mathbf{x}'\mathbf{y} \geqslant \mathbf{x}'(P_{\psi_p}\mathbf{y}) \geqslant \mathbf{x}'(P_{\psi_{p-1}}P_{\psi_p}\mathbf{y}) \geqslant \cdots \geqslant \mathbf{x}'(P_{\psi_1} \cdots P_{\psi_p}\mathbf{y}) = \mathbf{x}(P\mathbf{y}),$$

which concludes the proof of the first part of the theorem.

To prove the second part of the theorem apply the first part to the vectors $\mathbf{x}$ and $-\mathbf{y}$.

Corollary 6.30. *Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ be two vectors such that $x_1 \geqslant x_1 \geqslant \cdots \geqslant x_n$, $y_1 \geqslant y_2 \geqslant \cdots \geqslant y_n$. For every permutation matrix P we have*

$$\parallel \boldsymbol{x} - \boldsymbol{y} \parallel_F \leqslant \parallel \boldsymbol{x} - P\boldsymbol{y} \parallel_F .$$

If $x_1 \geqslant x_1 \geqslant \cdots \geqslant x_n$ and $y_1 \leqslant y_2 \leqslant \cdots \leqslant y_n$, then for every permutation matrix P we have

$$\parallel \boldsymbol{x} - \boldsymbol{y} \parallel_F \geqslant \parallel \boldsymbol{x} - P\boldsymbol{y} \parallel_F .$$

Proof. Note that

$$\begin{aligned} \| \mathbf{x} - \mathbf{y} \|_F^2 &= \| \mathbf{x} \|_F^2 + \| \mathbf{y} \|_F^2 - 2\mathbf{x}'\mathbf{y}, \\ \| \mathbf{x} - P\mathbf{y} \|_F^2 &= \| \mathbf{x} \|_F^2 + \| P\mathbf{y} \|_F^2 - 2\mathbf{x}'(P\mathbf{y}) \\ &= \| \mathbf{x} \|_F^2 + \| \mathbf{y} \|_F^2 - 2\mathbf{x}'(P\mathbf{y}) \end{aligned}$$

because $\| P\mathbf{y} \|_F^2 = \| \mathbf{y} \|_F^2$. Then, by Theorem 6.29, $\| \mathbf{x} - \mathbf{y} \|_F \leqslant \| \mathbf{x} - P\mathbf{y} \|_F$.

The argument for the second part of the corollary is similar.

6.5 Orthogonality

The Cauchy-Schwarz Inequality implies that $|(\mathbf{x}, \mathbf{y})| \leqslant \| \mathbf{x} \|_2 \| \mathbf{y} \|_2$. Equivalently, this means that

$$-1 \leqslant \frac{(\mathbf{x}, \mathbf{y})}{\| \mathbf{x} \|_2 \| \mathbf{y} \|_2} \leqslant 1.$$

This double inequality allows us to introduce the notion of angle between two vectors $\mathbf{x}, \mathbf{y}$ of a real linear space L.

Definition 6.31. *The* angle *between the vectors* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *is the number* $\alpha \in [0, \pi]$ *defined by*

$$\cos \alpha = \frac{(\boldsymbol{x}, \boldsymbol{y})}{\| \boldsymbol{x} \|_2 \| \boldsymbol{y} \|_2}.$$

This angle will be denoted by $\angle(\boldsymbol{x}, \boldsymbol{y})$.

Example 6.32. Let $\mathbf{u} = (u_1, u_2) \in \mathbb{R}^2$ be a unit vector. Since $u_1^2 + u_2^2 = 1$, there exists $\alpha \in [0, 2\pi]$ such that $u_1 = \cos \alpha$ and $u_2 = \sin \alpha$. Thus, for any two unit vectors in $\mathbb{R}^2$, $\mathbf{u} = (\cos \alpha, \sin \alpha)$ and $\mathbf{v} = (\cos \beta, \sin \beta)$ we have $(\mathbf{u}, \mathbf{v}) = \cos \alpha \cos \beta + \sin \alpha \sin \beta = \cos(\alpha - \beta)$, where $\alpha, \beta \in [0, 2\pi]$. Consequently, $\angle(\mathbf{u}, \mathbf{v})$ is the angle in the interval $[0, \pi]$ that has the same cosine as $\alpha - \beta$.

Theorem 6.33. (The Cosine Theorem) *Let* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *be two vectors in* $\mathbb{R}^n$ *equipped with the Euclidean inner product. We have:*

$$\| \boldsymbol{x} - \boldsymbol{y} \|^2 = \| \boldsymbol{x} \|^2 + \| \boldsymbol{y} \|^2 - 2 \| \boldsymbol{x} \| \| \boldsymbol{y} \| \cos \alpha,$$

where $\alpha = \angle(\boldsymbol{x}, \boldsymbol{y})$.

Proof. Since the norm is induced by the inner product we have

$$\begin{aligned} \| \mathbf{x} - \mathbf{y} \|^2 &= (\mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{y}) \\ &= (\mathbf{x}, \mathbf{x}) - 2(\mathbf{x}, \mathbf{y}) + (\mathbf{y}, \mathbf{y}) \\ &= \| \mathbf{x} \|^2 - 2 \| \mathbf{x} \| \| \mathbf{y} \| \cos \alpha + \| \mathbf{y} \|^2, \end{aligned}$$

which is the desired equality.

The notion of angle between two vectors allows the introduction of the notion of *orthogonality*.

Definition 6.34. *Let L be an inner product space. Two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ of L are* orthogonal *if $(\boldsymbol{x}, \boldsymbol{y}) = 0$.*

A pair of orthogonal vectors $(\mathbf{x}, \mathbf{y})$ is denoted by $\mathbf{x} \perp \mathbf{y}$. If $T \subseteq V$, then the set $T^{\perp}$ is defined by

$$T^{\perp} = \{\mathbf{v} \in L \mid \mathbf{v} \perp \mathbf{t} \text{ for every } \mathbf{t} \in T\}$$

Note that $T \subseteq U$ implies $U^{\perp} \subseteq T^{\perp}$.

If S, T are two subspaces of an inner product space, then S and T are *orthogonal* if $\mathbf{s} \perp \mathbf{t}$ for every $\mathbf{s} \in S$ and every $\mathbf{t} \in T$. This is denoted as $S \perp T$.

Theorem 6.35. *Let L be an inner product space and let T be a subset of L. The set $T^{\perp}$ is a subspace of L. Furthermore, $\langle T \rangle^{\perp} = T^{\perp}$.*

Proof. Let $\mathbf{x}$ and $\mathbf{y}$ be two members of T. We have $(\mathbf{x}, \mathbf{t}) = (\mathbf{y}, \mathbf{t}) = 0$ for every $\mathbf{t} \in T$. Therefore, for every $a, b \in F$, by the linearity of the inner product we have $(a\mathbf{x} + b\mathbf{y}, \mathbf{t}) = a(\mathbf{x}, \mathbf{t}) + b(\mathbf{y}, \mathbf{t}) = 0$, for $\mathbf{t} \in T$, so $a\mathbf{x} + b\mathbf{t} \in T^{\perp}$. Thus, $T^{\perp}$ is a subspace of L.

By a previous observation, since $T \subseteq \langle T \rangle$, we have $\langle T \rangle^{\perp} \subseteq T^{\perp}$. To prove the converse inclusion, let $\mathbf{z} \in T^{\perp}$.

If $\mathbf{y} \in \langle T \rangle$, $\mathbf{y}$ is a linear combination of vectors of T, $\mathbf{y} = a_1\mathbf{t}_1 + \cdots + a_m\mathbf{t}_m$, so $(\mathbf{y}, \mathbf{z}) = a_1(\mathbf{t}_1, \mathbf{z}) + \cdots + a_m(\mathbf{t}_m, \mathbf{z}) = 0$. Therefore, $\mathbf{z} \perp \mathbf{y}$, which implies $\mathbf{z} \in \langle T \rangle^{\perp}$. This allows us to conclude that $\langle T \rangle^{\perp} = T^{\perp}$.

We refer to $T^{\perp}$ as the *orthogonal complement* of T.

Note that $T \cap T^{\perp} \subseteq \{\mathbf{0}\}$. If T is a subspace, then this inclusion becomes an equality, that is, $T \cap T^{\perp} = \{\mathbf{0}\}$.

Theorem 6.36. *Let T be a subspace of the finite-dimensional linear space L. We have* $\dim(T) + \dim(T^{\perp}) = \dim(L)$.

Proof. This statement follows directly from Theorem 5.19.

If $\mathbf{x}$ and $\mathbf{y}$ are orthogonal, by Theorem 6.33 we have

$$\| \mathbf{x} - \mathbf{y} \|^2 = \| \mathbf{x} \|^2 + \| \mathbf{y} \|^2,$$

which is the well-known *Pythagora's Theorem.*

Theorem 6.37. *Let T be a subspace of $\mathbb{C}^n$. We have $(T^{\perp})^{\perp} = T$.*

Proof. Observe that $T \subseteq (T^\perp)^\perp$. Indeed, if $\mathbf{t} \in T$, then $(\mathbf{t}, \mathbf{z}) = 0$ for every $\mathbf{z} \in T^\perp$, so $\mathbf{t} \in (T^\perp)^\perp$.

To prove the reverse inclusion, let $\mathbf{x} \in (T^\perp)^\perp$. Theorem 6.39 implies that we can write $\mathbf{x} = \mathbf{u} + \mathbf{v}$, where $\mathbf{u} \in T$ and $\mathbf{v} \in T^\perp$, so $\mathbf{x} - \mathbf{u} = \mathbf{v} \in T^\perp$.

Since $T \subseteq (T^\perp)^\perp$, we have $\mathbf{u} \in (T^\perp)^\perp$, so $\mathbf{x} - \mathbf{u} \in (T^\perp)^\perp$. Consequently, $\mathbf{x} - \mathbf{u} \in T^\perp \cap (T^\perp)^\perp = \{\mathbf{0}\}$, so $\mathbf{x} = \mathbf{u} \in T$. Thus, $(T^\perp)^\perp \subseteq T$, which concludes the argument.

Corollary 6.38. *Let Z be a subset of $\mathbb{C}^n$. We have $(Z^\perp)^\perp = \langle Z \rangle$.*

Proof. Let Z be a subset of $\mathbb{C}^n$. Since $Z \subseteq \langle Z \rangle$ it follows that $\langle Z \rangle^\perp \subseteq Z^\perp$. Let now $\mathbf{y} \in Z^\perp$ and let $\mathbf{z} = a_1\mathbf{z}_1 + \cdots + a_p\mathbf{z}_p \in \langle Z \rangle$, where $\mathbf{z}_1, \ldots, \mathbf{z}_p \in Z$. Since

$$(\mathbf{y}, \mathbf{z}) = a_1(\mathbf{y}, \mathbf{z}_1) + \cdots + a_p(\mathbf{y}, \mathbf{z}_p) = 0,$$

it follows that $\mathbf{y} \in \langle Z \rangle^\perp$. Thus, we have $Z^\perp = \langle Z \rangle^\perp$.

This allows us to write $(Z^\perp)^\perp = (\langle Z \rangle^\perp)^\perp$. Since $\langle Z \rangle$ is a subspace of $\mathbb{C}^n$, by Theorem 6.37, we have $(\langle Z \rangle^\perp)^\perp = \langle Z \rangle$, so $(Z^\perp)^\perp = \langle Z \rangle$.

Theorem 6.39. *Let U be a subspace of $\mathbb{C}^n$. Then, $\mathbb{C}^n = U \boxplus U^\perp$.*

Proof. If $U = \{\mathbf{0}\}$, then $U^\perp = \mathbb{C}^n$ and the statement is immediate. Therefore, we can assume that $U \neq \{\mathbf{0}\}$.

In Theorem 6.35 we saw that $U^\perp$ is a subspace of $\mathbb{C}^n$. Thus, we need to show that $\mathbb{C}^n$ is the direct sum of the subspaces U and $U^\perp$. We need to verify only that every $\mathbf{x} \in \mathbb{C}^n$ can be uniquely written as a sum $\mathbf{x} = \mathbf{u} + \mathbf{v}$, where $\mathbf{u} \in U$ and $\mathbf{v} \in U^\perp$.

Let $\mathbf{u}_1, \ldots, \mathbf{u}_m$ be an orthonormal basis of U, that is, a basis such that

$$(\mathbf{u}_i, \mathbf{u}_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$$

for $1 \leqslant i, j \geqslant m$. Define $\mathbf{u} = (\mathbf{x}, \mathbf{u}_1)\mathbf{u}_1 + \cdots + (\mathbf{x}, \mathbf{u}_m)\mathbf{u}_m$ and $\mathbf{v} = \mathbf{x} - \mathbf{u}$.

The vector $\mathbf{v}$ is orthogonal to every vector $\mathbf{u}_i$ because

$$(\mathbf{v}, \mathbf{u}_i) = (\mathbf{x} - \mathbf{u}, \mathbf{u}_i) = (\mathbf{x}, \mathbf{u}_i) - (\mathbf{u}, \mathbf{u}_i) = 0.$$

Therefore $\mathbf{v} \in U^\perp$ and $\mathbf{x}$ has the necessary decomposition. To prove that the decomposition is unique suppose that $\mathbf{x} = \mathbf{s} + \mathbf{t}$, where $\mathbf{s} \in U$ and $\mathbf{t} \in U_\perp$. Since $\mathbf{s} + \mathbf{t} = \mathbf{u} + \mathbf{v}$ we have $\mathbf{s} - \mathbf{u} = \mathbf{v} - \mathbf{t} \in U \cap U^\perp = \{\mathbf{0}\}$, which implies $\mathbf{s} = \mathbf{u}$ and $\mathbf{t} = \mathbf{v}$.

Let $W = \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ be a basis in the real n-dimensional inner product space L. If $\mathbf{x} = x_1\mathbf{w}_1 + \cdots + x_n\mathbf{w}_n$ and $\mathbf{y} = y_1\mathbf{w}_1 + \cdots + y_n\mathbf{w}_n$, then

$$(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i y_j (\mathbf{w}_i, \mathbf{w}_j),$$

due to the bilinearity of the inner product.

Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ be the matrix defined by $a_{ij} = (\mathbf{w}_i, \mathbf{w}_j)$ for $1 \leqslant i, j \leqslant n$. The symmetry of the inner product implies that the matrix A itself is symmetric. Now, the inner product can be expressed as

$$(\mathbf{x}, \mathbf{y}) = (x_1, \ldots, x_n) A \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}.$$

We refer to A as the *matrix associated* with W.

Definition 6.40. *An* orthogonal set of vectors *in an inner product space* $(V, (\cdot, \cdot))$ *is a subset* W *of* L *such that for every* $\boldsymbol{u}, \boldsymbol{v} \in W$ *we have* $\boldsymbol{u} \perp \boldsymbol{v}$.

If, in addition, $\| \boldsymbol{u} \| = 1$ *for every* $\boldsymbol{u} \in W$*, then we say that* W *is* orthonormal.

Theorem 6.41. *If* W *is a set of non-zero orthogonal vectors, then* W *is linearly independent.*

Proof. Let $a_1 \mathbf{w}_1 + \cdots + a_n \mathbf{w}_n = \mathbf{0}$ for a linear combination of elements of W. This implies $a_i \| \mathbf{w}_i \|^2 = 0$, so $a_i = 0$ because $\| \mathbf{w}_i \|^2 \neq 0$, and this holds for every i, where $1 \leqslant i \leqslant n$. Thus, W is linearly independent.

Corollary 6.42. *Let* L *be an* n*-dimensional linear space. If* W *is an orthogonal (orthonormal) set and* $|W| = n$*, then* W *is an orthogonal (orthonormal) basis of* L.

Proof. This statement is an immediate consequence of Theorem 6.41.

Theorem 6.43. *Let* S *be a subspace of* $\mathbb{C}^n$ *such that* $\dim(S) = k$*. There exists a matrix* $A \in \mathbb{C}^{n \times k}$ *having orthonormal columns such that* $S = \mathsf{Ran}(A)$.

Proof. Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be an orthonormal basis of S. Define the matrix A as $A = (\mathbf{v}_1, \ldots, \mathbf{v}_k)$. We have $\mathbf{x} \in S$, if and only if $\mathbf{x} = a_1 \mathbf{v}_1 + \cdots + a_k \mathbf{v}_k$, which is equivalent to $\mathbf{x} = A\mathbf{a}$. This amounts to $\mathbf{x} \in \mathsf{Ran}(A)$, so $S = \mathsf{Ran}(A)$.

For an orthonormal basis in an n-dimensional space, the associated matrix is the diagonal matrix I_n. In this case, we have

$$(\mathbf{x}, \mathbf{y}) = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

for $\mathbf{x}, \mathbf{y} \in L$.

Observe that if $W = \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ is an orthonormal set and $\mathbf{x} \in \langle W \rangle$, which means that $\mathbf{x} = a_1 \mathbf{w}_1 + \cdots + a_n \mathbf{w}_n$, then $a_i = (\mathbf{x}, \mathbf{w}_i)$ for $1 \leqslant i \leqslant n$.

Definition 6.44. *Let* $W = \{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n\}$ *be an orthonormal set and let* $\boldsymbol{x} \in \langle W \rangle$*. The equality*

$$\boldsymbol{x} = (\boldsymbol{x}, \boldsymbol{w}_1)\boldsymbol{w}_1 + \cdots + (\boldsymbol{x}, \boldsymbol{w}_n)\boldsymbol{w}_n \tag{6.9}$$

is the Fourier expansion of $\boldsymbol{x}$ *with respect to the orthonormal set* W*.*

Furthermore, we have *Parseval's equality*:

$$\| \mathbf{x} \|^2 = (\mathbf{x}, \mathbf{x}) = \sum_{i=1}^{n} (\mathbf{x}, \mathbf{w}_i)^2. \tag{6.10}$$

Thus, if $1 \leqslant q \leqslant n$ we have

$$\sum_{i=1}^{q} (\mathbf{x}, \mathbf{w}_i)^2 \leqslant \| \mathbf{x} \|^2 . \tag{6.11}$$

It is easy to see that a square matrix $C \in \mathbb{C}^{n \times n}$ is unitary if and only if its set of columns is an orthonormal set in $\mathbb{C}^n$.

Example 6.45. Let

$$C = \begin{pmatrix} x_1 + ix_2 & y_1 + iy_2 \\ u_1 + iu_2 & v_1 + iv_2 \end{pmatrix} \in \mathbb{C}^2$$

be a unitary matrix, where $x_i, y_i, u_i, v_i \in \mathbb{R}$ for $i \in \{1, 2\}$. We have

$$C^{\mathsf{H}} C = \begin{pmatrix} x_1 - ix_2 & u_1 - iu_2 \\ y_1 - iy_2 & v_1 - iv_2 \end{pmatrix} \begin{pmatrix} x_1 + ix_2 & y_1 + iy_2 \\ u_1 + iu_2 & v_1 + iv_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This equality implies the equalities

$$\begin{aligned} x_1^2 + x_2^2 + u_1^2 + u_2^2 &= 1, \\ y_1^2 + y_2^2 + v_1^2 + v_2^2 &= 1, \\ x_1 y_1 + x_2 y_2 + u_1 v_1 + u_2 v_2 &= 0, \\ x_1 y_2 - x_2 y_1 + u_1 v_2 - u_2 v_1 &= 0. \end{aligned}$$

It is easy to verify that the matrices

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

known as the *Pauli matrices* are both Hermitian and unitary.

Definition 6.46. *Let* $\boldsymbol{w} \in \mathbb{R}^n - \{\boldsymbol{0}\}$ *and let* $a \in \mathbb{R}$*. The* hyperplane *determined by* $\boldsymbol{w}$ *and* a *is the set* $H_{\boldsymbol{w},a} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{w}'\boldsymbol{x} = a\}$*.*

If $\mathbf{x}_0 \in H_{\mathbf{w},a}$, then $\mathbf{w}'\mathbf{x}_0 = a$, so $H_{\mathbf{w},a}$ is also described by the equality

$$H_{\mathbf{w},a} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{w}'(\mathbf{x} - \mathbf{x}_0) = 0\}.$$

Any hyperplane $H_{\mathbf{w},a}$ partitions $\mathbb{R}^n$ into three sets:

$$\begin{aligned} H^{>}_{\mathbf{w},a} &= \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{w}'\mathbf{x} > a\}, \\ H^{0}_{\mathbf{w},a} &= H_{\mathbf{w},a}, \\ H^{<}_{\mathbf{w},a} &= \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{w}'\mathbf{x} < a\}. \end{aligned}$$

The sets $H^{>}_{\mathbf{w},a}$ and $H^{<}_{\mathbf{w},a}$ are the *positive* and *negative open* half-spaces determined by $H_{\mathbf{w},a}$, respectively. The sets

$$\begin{aligned} H^{\geqslant}_{\mathbf{w},a} &= \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{w}'\mathbf{x} \geqslant a\}, \\ H^{\leqslant}_{\mathbf{w},a} &= \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{w}'\mathbf{x} \leqslant a\}. \end{aligned}$$

are the *positive* and *negative closed* half-spaces determined by $H_{\mathbf{w},a}$, respectively.

If $\mathbf{x}_1, \mathbf{x}_2 \in H_{\mathbf{w},a}$, then $\mathbf{w} \perp \mathbf{x}_1 - \mathbf{x}_2$. This justifies referring to $\mathbf{w}$ as the *normal to the hyperplane* $H_{\boldsymbol{w},a}$. Observe that a hyperplane is fully determined by a vector $\mathbf{x}_0 \in H_{\mathbf{w},a}$ and by $\mathbf{w}$.

Let $\mathbf{x}_0 \in \mathbb{R}^n$ and let $H_{\mathbf{w},a}$ be a hyperplane. We seek $\mathbf{x} \in H_{\mathbf{w},a}$ such that $\| \mathbf{x} - \mathbf{x}_0 \|_2$ is minimal. Finding $\mathbf{x}$ amounts to minimizing the function $f(\mathbf{x}) = \| \mathbf{x} - \mathbf{x}_0 \|_2^2 = \sum_{i=1}^{n} (x_i - x_{0i})^2$ subjected to the constraint $\mathbf{w}_1\mathbf{x}_1 + \cdots + w_n\mathbf{x}_n - a = 0$. Using the Lagrangean $\Lambda(\mathbf{x}) = f(\mathbf{x}) + \lambda(\mathbf{w}'\mathbf{x} - a)$ and the multiplier λ we impose the conditions

$$\frac{\partial \Lambda}{\partial x_i} = 0 \text{ for } 1 \leqslant i \leqslant n$$

which amount to

$$\frac{\partial f}{\partial x_i} + \lambda w_i = 0$$

for $1 \leqslant i \leqslant n$. These equalities yield $2(x_i - x_{0i}) + \lambda \mathbf{w}_i = 0$, so we have $x_i = x_{0i} - \frac{1}{2}\lambda \mathbf{w}_i$. Consequently, we have $\mathbf{x} = \mathbf{x}_0 - \frac{1}{2}\lambda \mathbf{w}$. Since $\mathbf{x} \in H_{\mathbf{w},a}$ this implies

$$\mathbf{w}'\mathbf{x} = \mathbf{w}'\mathbf{x}_0 - \frac{1}{2}\lambda \mathbf{w}'\mathbf{w} = a.$$

Thus,

$$\lambda = 2\frac{\mathbf{w}'\mathbf{x}_0 - a}{\mathbf{w}'\mathbf{w}} = 2\frac{\mathbf{w}'\mathbf{x}_0 - a}{\| \mathbf{w} \|_2^2}.$$

We conclude that the closest point in $H_{\mathbf{w},a}$ to $\mathbf{x}_0$ is

$$\mathbf{x} = \mathbf{x}_0 - \frac{\mathbf{w}'\mathbf{x}_0 - a}{\| \mathbf{w} \|_2^2}\mathbf{w}.$$

The smallest distance between $\mathbf{x}_0$ and a point in the hyperplane $H_{\mathbf{w},a}$ is given by

$$\parallel \mathbf{x}_0 - \mathbf{x} \parallel = \frac{\mathbf{w}'\mathbf{x}_0 - a}{\parallel \mathbf{w} \parallel_2}.$$

If we define the distance $d(H_{\mathbf{w},a}, \mathbf{x}_0)$ between $\mathbf{x}_0$ and $H_{\mathbf{w},a}$ as this smallest distance we have

$$d(H_{\mathbf{w},a}, \mathbf{x}_0) = \frac{\mathbf{w}'\mathbf{x}_0 - a}{\parallel \mathbf{w} \parallel_2}. \tag{6.12}$$

6.6 Unitary and Orthogonal Matrices

Lemma 6.47. *Let $A \in \mathbb{C}^{n\times n}$. If $\boldsymbol{x}^{\mathsf{H}} A \boldsymbol{x} = 0$ for every $\boldsymbol{x} \in \mathbb{C}^n$, then $A = O_{n,n}$.*

Proof. If $\mathbf{x} = \mathbf{e}_k$, then $\mathbf{x}^{\mathsf{H}} A\mathbf{x} = a_{kk}$ for every k, $1 \leqslant k \leqslant n$, so all diagonal entries of A equal 0. Choose now $\mathbf{x} = \mathbf{e}_k + \mathbf{e}_j$. Then,

$$\begin{aligned}(\mathbf{e}_k + \mathbf{e}_j)^{\mathsf{H}} A(\mathbf{e}_k + \mathbf{e}_j) &= \mathbf{e}_k^{\mathsf{H}} A\mathbf{e}_k + \mathbf{e}_k^{\mathsf{H}} A\mathbf{e}_j + \mathbf{e}_j^{\mathsf{H}} A\mathbf{e}_k + \mathbf{e}_j^{\mathsf{H}} A\mathbf{e}_j \\ &= \mathbf{e}_k^{\mathsf{H}} A\mathbf{e}_j + \mathbf{e}_j^{\mathsf{H}} A\mathbf{e}_k = a_{kj} + a_{jk} = 0.\end{aligned}$$

Similarly, if we choose $\mathbf{x} = \mathbf{e}_k + i\mathbf{e}_j$ we obtain:

$$\begin{aligned}(\mathbf{e}_k + i\mathbf{e}_j)^{\mathsf{H}} A(\mathbf{e}_k + i\mathbf{e}_j) &= (\mathbf{e}_k^{\mathsf{H}} - i\mathbf{e}_j^{\mathsf{H}}) A(\mathbf{e}_k + i\mathbf{e}_j) \\ &= \mathbf{e}_k^{\mathsf{H}} A\mathbf{e}_k - i\mathbf{e}_j^{\mathsf{H}} A\mathbf{e}_k + i\mathbf{e}_k^{\mathsf{H}} A\mathbf{e}_j + \mathbf{e}_j^{\mathsf{H}} A\mathbf{e}_j \\ &= -ia_{jk} + ia_{kj} = 0.\end{aligned}$$

The equalities $a_{kj} + a_{jk} = 0$ and $-a_{jk} + a_{kj} = 0$ imply $a_{kj} = a_{jk} = 0$. Thus, all off-diagonal elements of A are also 0, hence $A = O_{n,n}$.

Theorem 6.48. *A matrix $U \in \mathbb{C}^{n\times n}$ is unitary if and only if $\parallel U\boldsymbol{x} \parallel_2 = \parallel \boldsymbol{x} \parallel_2$ for every $\boldsymbol{x} \in \mathbb{C}^n$.*

Proof. If U is unitary we have $\parallel U\mathbf{x} \parallel_2^2 = (U\mathbf{x})^{\mathsf{H}} U\mathbf{x} = \mathbf{x}^{\mathsf{H}} U^{\mathsf{H}} U\mathbf{x} = \parallel \mathbf{x} \parallel_2^2$ because $U^{\mathsf{H}} U = I_n$. Thus, $\parallel U\mathbf{x} \parallel_2 = \parallel \mathbf{x} \parallel_2$.

Conversely, let U be a matrix such that $\parallel U\mathbf{x} \parallel_2 = \parallel \mathbf{x} \parallel_2$ for every $\mathbf{x} \in \mathbb{C}^n$. This implies $\mathbf{x}^{\mathsf{H}} U^{\mathsf{H}} U\mathbf{x} = \mathbf{x}^{\mathsf{H}}\mathbf{x}$, hence $\mathbf{x}^{\mathsf{H}}(U^{\mathsf{H}} U - I_n)\mathbf{x} = 0$ for $\mathbf{x} \in \mathbb{C}^n$. By Lemma 6.47 this implies $U^{\mathsf{H}} U = I_n$, so U is a unitary matrix.

Corollary 6.49. *The following statements that concern a matrix $U \in \mathbb{C}^{n\times n}$ are equivalent:*
(i) *U is unitary;*
(ii) *$\parallel U\boldsymbol{x} - U\boldsymbol{y} \parallel_2 = \parallel \boldsymbol{x} - \boldsymbol{y} \parallel_2$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^n$;*
(iii) *$(U\boldsymbol{x}, U\boldsymbol{y}) = (\boldsymbol{x}, \boldsymbol{y})$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^n$.*

Proof. This statement is a direct consequence of Theorem 6.48.

The counterpart of unitary matrices in the set of real matrices are introduced next.

Definition 6.50. *A matrix* $A \in \mathbb{R}^{n \times n}$ *is* orthogonal *if it is unitary.*

In other words, $A \in \mathbb{R}^{n \times n}$ is orthogonal if and only if $A'A = AA' = I_n$. Clearly, A is orthogonal if and only if A' is orthogonal.

Theorem 6.51. *If* $A \in \mathbb{R}^{n \times n}$ *is an orthogonal matrix, then* $\det(A) \in \{-1, 1\}$.

Proof. By Corollary 5.131, $|\det(A)| = 1$. Since $\det(A)$ is a real number, it follows that $\det(A) \in \{-1, 1\}$.

Corollary 6.52. *Let A be a matrix in $\mathbb{R}^{n \times n}$. The following statements are equivalent:*
(i) *A is orthogonal;*
(ii) *A is invertible and $A^{-1} = A'$;*
(iii) *A' is invertible and $(A')^{-1} = A$;*
(iv) *A' is orthogonal.*

Proof. The equivalence between these statements is an immediate consequence of definitions.

Corollary 6.52 implies that the columns of a square matrix form an orthonormal set of vectors if and only if the set of rows of the matrix is an orthonormal set.

Theorem 6.48 specialized to orthogonal matrices shows that a matrix A is orthogonal if and only if it preserves the length of vectors.

Theorem 6.53. *Let S be an r-dimensional subspace of $\mathbb{R}^n$ and let $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r\}$, $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r\}$ be two orthonormal bases of the space S. The orthogonal matrices $B = (\boldsymbol{u}_1 \ \cdots \ \boldsymbol{u}_r) \in \mathbb{R}^{n \times r}$ and $C = (\boldsymbol{v}_1 \ \cdots \ \boldsymbol{v}_r) \in \mathbb{R}^{n \times r}$ of the any two such bases are related by the equality $B = CT$, where $T = C'B \in \mathbb{C}^{r \times r}$ is an orthogonal matrix.*

Proof. Since the columns of B form a basis for S, each vector $\mathbf{v}_i$ can be written as

$$\mathbf{v}_i = \mathbf{v}_1 t_{1i} + \cdots + \mathbf{v}_r t_{ri}$$

for $1 \leqslant i \leqslant r$. Thus, $B = CT$. Since B and C are orthogonal, we have

$$B^{\mathsf{H}} B = T^{\mathsf{H}} C^{\mathsf{H}} C T = T^{\mathsf{H}} T = I_r,$$

so T is an orthogonal matrix and because it is a square matrix, it is also a unitary matrix. Furthermore, we have $C^{\mathsf{H}} B = C^{\mathsf{H}} C T = T$, which concludes the argument.

Definition 6.54. *A* rotation matrix *is an orthogonal matrix* $R \in \mathbb{R}^{n\times n}$ *such that* $\det(R) = 1$. *A* reflexion matrix *is an orthogonal matrix* $R \in \mathbb{R}^{n\times n}$ *such that* $\det(R) = -1$.

Example 6.55. In the two dimensional case, $n = 2$, a rotation is a matrix $R \in \mathbb{R}^{2\times 2}$,

$$R = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}$$

such that

$$r_{11}^2 + r_{21}^2 = 1, r_{12}^2 + r_{22}^2 = 1,$$
$$r_{11}r_{12} + r_{21}r_{22} = 0, r_{11}r_{22} - r_{12}r_{21} = 1.$$

These equalities implies

$$r_{22}(r_{11}r_{12} + r_{21}r_{22}) - r_{12}(r_{11}r_{22} - r_{12}r_{21}) = -r_{12},$$

or

$$r_{21}(r_{22}^2 + r_{12}^2) = -r_{12},$$

so $r_{21} = -r_{12}$.

If $r_{21} = -r_{21} = 0$, the above equalities imply that either $r_{11} = r_{22} = 1$ or $r_{11} = r_{22} = -1$. Otherwise, the equality $r_{11}r_{12} + r_{21}r_{22} = 0$ implies $r_{11} = r_{22}$.

Since $r_{11}^2 \leqslant 1$ it follows that there exits θ such that $r_{11} = \cos\theta$. This shows that R has the form

$$R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

The vector $\mathbf{y} = R\mathbf{x}$, where

$$\mathbf{y} = \begin{pmatrix} x_1\cos\theta + x_2\sin\theta \\ -x_1\sin\theta + x_2\cos\theta \end{pmatrix},$$

is obtained from $\mathbf{x}$ by a clockwise rotation through an angle θ. It is easy to see that $\det(R) = 1$, so the term "rotation matrix" is clearly justified for R. To mark the dependency of R on θ we will use the notation

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

A extension of this example is the *Givens matrix* $G(p, q, \theta) \in \mathbb{R}^{n\times n}$ defined as

$$\begin{array}{c} \begin{pmatrix} 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & \cos\theta & \cdots & \sin\theta & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & -\sin\theta & \cdots & \cos\theta & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \end{pmatrix} \begin{array}{c} \\ \\ p \\ \\ q \\ \\ \\ \end{array} \\ \quad\quad p \quad\quad\quad q \end{array}$$

We can write

$$G(p, q, \theta) = (\mathbf{e}_1 \ \cdots \ \cos\theta \mathbf{e}_p - \sin\theta \mathbf{e}_q \ \cdots \ \sin\theta \mathbf{e}_p + \cos\theta \ \mathbf{e}_q \ \cdots \ \mathbf{e}_n).$$

It is easy to verify that $G(p, q, \theta)$ is a rotation matrix since it is orthogonal and $\det(G(p, q, \theta)) = 1$.

Since

$$G(p, q, \theta) \begin{pmatrix} v_1 \\ \vdots \\ v_p \\ \vdots \\ v_q \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} v_1 \\ \vdots \\ \cos\theta v_p + \sin\theta v_q \\ \vdots \\ -\sin\theta v_p + \cos\theta v_q \\ \vdots \\ v_n \end{pmatrix},$$

the the multiplication of a vector $\mathbf{v}$ be a Givens matrix amounts to a clockwise rotation by θ in the plane of the coordinates (v_p, v_q).

If $v_p \neq 0$, then the rotation described by the Givens matrix can be used to zero the q^{th} component of the resulting vector by taking θ such that $\tan\theta = \frac{v_q}{v_p}$.

It is easy to see that $R(\theta)^{-1} = R(-\theta)$ and that $R(\theta_2)R(\theta_1) = R(\theta_1 + \theta_2)$.

Example 6.56. Let $\mathbf{v} \in \mathbb{C}^n - \{\mathbf{0}_n\}$ be a unit vector. The *Householder matrix* $H_{\mathbf{v}} \in \mathbb{C}^{n \times n}$ is defined by $H_{\mathbf{v}} = I_n - 2\mathbf{v}\mathbf{v}^{\mathsf{H}}$.

The matrix $H_{\mathbf{v}}$ is clearly Hermitian. Moreover, we have

$$HH^{\mathsf{H}} = HH = (I_n - 2\mathbf{v}\mathbf{v}^{\mathsf{H}})^2 = I_n - 4\mathbf{v}\mathbf{v}^{\mathsf{H}} + 4\mathbf{v}(\mathbf{v}^{\mathsf{H}}\mathbf{v})\mathbf{v}^{\mathsf{H}} = I_n,$$

so $H_{\mathbf{v}}$ is unitary and involutive. Since $\det(H_{\mathbf{v}}) = -1$, $H_{\mathbf{v}}$ is a reflexion. For a unit vector $\mathbf{v} \in \mathbb{R}^n$, $H_{\mathbf{v}}$ is an orthogonal and involutive matrix.

The vector $H_{\mathbf{v}}\mathbf{w}$ is a reflexion of the vector $\mathbf{w}$ relative to the hyperplane $H_{\mathbf{v},0}$ defined by $\mathbf{v}^{\mathsf{H}}\mathbf{x} = 0$, because the vector

$$\mathbf{w} - H_{\mathbf{v}}\mathbf{w} = (I_n - H_{\mathbf{v}})\mathbf{w} = 2\mathbf{v}(\mathbf{v}^{\mathsf{H}}\mathbf{w})$$

is orthogonal to the hyperplane $\mathbf{v}'\mathbf{x} = 0$. Furthermore, the vector $H_{\mathbf{v}}\mathbf{w}$ has the same norm as $\mathbf{w}$.

Theorem 6.57. *Let $A \in \mathbb{C}^{n \times n}$ and $B \in \mathbb{C}^{k \times k}$ be two matrices. If there exists $U \in \mathbb{C}^{n \times k}$ having an orthonormal set of columns such that $AU = UB$, then there exists $V \in \mathbb{C}^{n \times (n-k)}$ such that $(U \ V) \in \mathbb{C}^{n \times n}$ is a unitary matrix and*

$$(U \ V)^{\mathsf{H}} A (UV) = \begin{pmatrix} B & U^{\mathsf{H}}AV \\ O_{n-k,k} & V^{\mathsf{H}}AV \end{pmatrix}.$$

Proof. Since U has an orthonormal set of columns, there exists $V \in \mathbb{C}^{n\times(n-k)}$ such that $(U\ V)$ is a unitary matrix. We have

$$U^{\mathsf{H}}AU = U^{\mathsf{H}}UB = B \text{ and } V^{\mathsf{H}}AU = V^{\mathsf{H}}UB = O_{n-k,k}B = O_{n-k,k}.$$

The equality of the theorem follows immediately.

6.7 The Topology of Normed Linear Spaces

A normed space can be equipped with the topology of a metric space, using the metric defined by the norm. Since this topology is induced by a metric, any normed space is a Hausdorff space. Further, if $\mathbf{v} \in L$, then the collection of subsets $\{C_d(\mathbf{v}, r) \mid r > 0\}$ is a fundamental system of neighborhoods for $\mathbf{v}$.

By specializing the definition of local continuity of functions between metric spaces, a function $f : L \longrightarrow M$ between two normed spaces (L, ν) and (M, ν') is continuous in $\mathbf{x}_0 \in L$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\nu(\mathbf{x} - \mathbf{x}_0) < \delta$ implies $\nu'(f(\mathbf{x}) - f(\mathbf{x}_0)) < \epsilon$.

A sequence $(\mathbf{x}_0, \mathbf{x}_1, \ldots)$ of elements of L converges to $\mathbf{x}$ if for every $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ such that $n \geqslant n_\epsilon$ implies $\nu(\mathbf{x}_n - \mathbf{x}) < \epsilon$.

Theorem 6.58. *In a normed linear space (L, ν), the norm, the multiplication by scalars and the vector addition are continuous functions.*

Proof. Let $(\mathbf{x}_0, \mathbf{x}_1, \ldots)$ be a sequence in L such that $\lim_{n\to\infty} \mathbf{x}_n = \mathbf{x}$. By Theorem 6.7 we have $\nu(\mathbf{x}_n - \mathbf{x}) \geqslant |\nu(\mathbf{x}_n) - \nu(\mathbf{x})|$, which implies $\lim_{n\to\infty} \nu(\mathbf{x}_n) = \nu(\mathbf{x})$. Thus, the norm is continuous.

Suppose now that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} \mathbf{x}_n = \mathbf{x}$, where (a_n) is a sequence of scalars. Since the sequence $(\mathbf{x}_n)$ is bounded, we have

$$\begin{aligned}\nu(a\mathbf{x} - a_n\mathbf{x}_n) &\leqslant \nu(a\mathbf{x} - a_n\mathbf{x}) + \nu(a_n\mathbf{x} - a_n\mathbf{x}_n)\\ &\leqslant |a - a_n|\nu(\mathbf{x}) + a_n\nu(\mathbf{x} - \mathbf{x}_n),\end{aligned}$$

which implies that $\lim_{n\to\infty} a_n\mathbf{x}_n = a\mathbf{x}$. This shows that the multiplication by scalars is a continuous function.

To prove that the vector addition is continuous, let $(\mathbf{x}_n)$ and $(\mathbf{y}_n)$ be two sequences in L such that $\lim_{n\to\infty} \mathbf{x}_n = \mathbf{x}$ and $\lim_{n\to\infty} \mathbf{y}_n = \mathbf{y}$. Note that

$$\nu\left((\mathbf{x} + \mathbf{y}) - (\mathbf{x}_n + \mathbf{y}_n)\right) \leqslant \nu(\mathbf{x} - \mathbf{x}_n) + \nu(\mathbf{y} - \mathbf{y}_n),$$

which implies that $\lim_{n\to\infty}(\mathbf{x}_n + \mathbf{y}_n) = \mathbf{x} + \mathbf{y}$. Thus, the vector addition is continuous.

Definition 6.59. *Two norms ν and ν' on a linear space L are* equivalent *if they generate the same topology.*

Theorem 6.60. *Let L be a linear space and let $\nu : L \longrightarrow \mathbb{R}_{\geqslant 0}$ and $\nu' : L \longrightarrow \mathbb{R}_{\geqslant 0}$ be two norms on L that generate the topologies $\mathcal{O}$ and $\mathcal{O}'$ on L, respectively.*

The topology $\mathcal{O}'$ is finer than the topology $\mathcal{O}$ (that is, $\mathcal{O} \subseteq \mathcal{O}'$) if and only if there exists $c \in \mathbb{R}_{>0}$ such that $\nu(\boldsymbol{v}) \leqslant c\nu'(\boldsymbol{v})$ for every $\boldsymbol{v} \in L$.

Proof. Suppose that $\mathcal{O} \subseteq \mathcal{O}'$. Then, any open sphere $C_\nu(\mathbf{0}, r_0) = \{\mathbf{x} \in L \mid \nu(\mathbf{x}) < r_0\}$ (in $\mathcal{O}$) must be an open set in $\mathcal{O}'$. Therefore, there exists an open sphere $C_{\nu'}(\mathbf{0}, r_1)$ such that $C_{\nu'}(\mathbf{0}, r_1) \subseteq C_\nu(\mathbf{0}, r_0)$. This means that for $r_0 \in \mathbb{R}_{\geqslant 0}$ and $\mathbf{v} \in L$ there exists $r_1 \in \mathbb{R}_{\geqslant 0}$ such that $\nu'(\mathbf{v}) < r_1$ implies $\nu(\mathbf{v}) < r_0$ for every $\mathbf{u} \in L$. In particular, for $r_0 = 1$, there is $k > 0$ such that $\nu'(\mathbf{v}) < k$ implies $\nu(\mathbf{v}) < 1$, which is equivalent to $c\nu'(\mathbf{v}) < 1$ implies $\nu(\mathbf{v}) < 1$, for every $\mathbf{v} \in L$ and $c = \frac{1}{k}$.

For $\mathbf{w} = \frac{1}{c+\epsilon}\frac{\mathbf{v}}{\nu'(\mathbf{v})}$, where $\epsilon > 0$ it follows that

$$c\nu'(\mathbf{w}) = c\nu'\left(\frac{1}{c+\epsilon}\frac{\mathbf{v}}{\nu'(\mathbf{v})}\right) = \frac{c}{c+\epsilon} < 1,$$

so

$$\nu(\mathbf{w}) = \nu\left(\frac{1}{c+\epsilon}\frac{\mathbf{v}}{\nu'(\mathbf{v})}\right) = \frac{1}{c+\epsilon}\frac{\nu(\mathbf{v})}{\nu'(\mathbf{v})} < 1.$$

Since this inequality holds for every $\epsilon > 0$ it follows that $\nu(\mathbf{v}) \leqslant c\nu'(\mathbf{v})$.

Conversely, suppose that there exists $c \in \mathbb{R}_{>0}$ such that $\nu(\mathbf{v}) \leqslant c\nu'(\mathbf{v})$ for every $\mathbf{v} \in L$. Since

$$\left\{\mathbf{v} \mid \nu'(\mathbf{v}) \leqslant \frac{r}{c}\right\} \subseteq \{\mathbf{v} \mid \nu(\mathbf{v}) \leqslant r\},$$

for $\mathbf{v} \in L$ and $r > 0$ it follows that $\mathcal{O} \subseteq \mathcal{O}'$.

Corollary 6.61. *Let ν and ν' be two norms on a linear space L. Then, ν and ν' are equivalent if and only if there exist $a, b \in \mathbb{R}_{>0}$ such that $a\nu(\boldsymbol{v}) \leqslant \nu'(\boldsymbol{v}) \leqslant b\nu(\boldsymbol{v})$ for $\boldsymbol{v} \in V$.*

Proof. This statement follows directly from Theorem 6.60.

Example 6.62. By Corollary 6.19 any two norms ν_p and ν_q, on $\mathbb{R}^n$ (with $p, q \geqslant 1$) are equivalent.

Continuous linear operators between normed spaces have a simple characterization.

Theorem 6.63. *Let (L, ν) and (L', ν') be two normed F-linear spaces where F is either $\mathbb{R}$ or $\mathbb{C}$. A linear operator $f : L \longrightarrow L'$ is continuous if and only if there exists $M \in \mathbb{R}_{>0}$ such that $\nu'(f(\boldsymbol{x})) \leqslant M\nu(\boldsymbol{x})$ for every $\boldsymbol{x} \in L$.*

Proof. Suppose that $f : L \longrightarrow L'$ satisfies the condition of the theorem. Then,

$$f\left(C_\nu\left(\mathbf{0}, \frac{r}{M}\right)\right) \subseteq C_{\nu'}(\mathbf{0}, r),$$

for every $r > 0$, which means that f is continuous in $\mathbf{0}$ and, therefore, it is continuous everywhere (by Theorem 5.172).

Conversely, suppose that f is continuous. Then, there exists $\delta > 0$ such that $f(C_\nu(\mathbf{0}, \delta)) \subseteq C_{\nu'}(f(x), 1)$, which is equivalent to $\nu(\mathbf{x}) < \delta$ implies $\nu'(f(\mathbf{x})) < 1$. Let $\epsilon > 0$ and let $\mathbf{z} \in L$ be defined by

$$\mathbf{z} = \frac{\delta}{\nu(\mathbf{x}) + \epsilon}\mathbf{x}.$$

We have $\nu(\mathbf{z}) = \frac{\delta\nu(\mathbf{x})}{\nu(\mathbf{x})+\epsilon} < \delta$. This implies $\nu'(f(\mathbf{z})) < 1$, which is equivalent to

$$\frac{\delta}{\nu(\mathbf{x}) + \epsilon}\nu'(f(\mathbf{x})) < 1$$

because of the linearity of f. This means that

$$\nu'(f(\mathbf{x})) < \frac{\nu(\mathbf{x}) + \epsilon}{\delta}$$

for every $\epsilon > 0$, so $\nu'(f(\mathbf{x})) \leqslant \frac{1}{\delta}\nu(\mathbf{x})$.

Lemma 6.64. *Let (L, ν) and (L', ν') be two normed F-linear spaces where F is either $\mathbb{R}$ or $\mathbb{C}$. A linear function $f : L \longrightarrow L'$ is not injective if and only if there exists $\boldsymbol{u} \in L - \{\boldsymbol{0}\}$ such that $f(\boldsymbol{u}) = \boldsymbol{0}$.*

Proof. It is clear that the condition of the lemma is sufficient for failing injectivity. Conversely, suppose that f is not injective. There exist $\mathbf{t}, \mathbf{v} \in L$ such that $\mathbf{t} \neq \mathbf{v}$ and $f(\mathbf{t}) = f(\mathbf{v})$. The linearity of f implies $f(\mathbf{t} - \mathbf{v}) = 0$. By defining $\mathbf{u} = \mathbf{t} - \mathbf{v} \neq \mathbf{0}$, we have the desired element $\mathbf{u}$.

Theorem 6.65. *Let (L, ν) and (L', ν') be two normed F-linear spaces where F is either $\mathbb{R}$ or $\mathbb{C}$. A linear function $f : L \longrightarrow L'$ is injective if and only if there exists $m \in \mathbb{R}_{>0}$ such that $\nu'(f(\boldsymbol{x})) \geqslant m\nu(\boldsymbol{x})$ for every $\boldsymbol{x} \in V_1$.*

Proof. Suppose that f is not injective. By Lemma 6.64, there exists $\mathbf{u} \in L - \{\mathbf{0}\}$ such that $f(\mathbf{u}) = \mathbf{0}$, so $\nu'(f(\mathbf{u})) < m\nu(\mathbf{u})$ for any $m > 0$. Thus, the condition of the theorem is sufficient for injectivity.

Suppose that f is injective, so the inverse function $f^{-1} : L' \longrightarrow L$ is a linear function. By Theorem 6.63, there exists $M > 0$ such that

$$\nu(f^{-1}(\mathbf{y})) \leqslant M\nu'(\mathbf{y})$$

for every $\mathbf{y} \in L'$. Choosing $\mathbf{y} = f(\mathbf{x})$ yields $\nu(\mathbf{x}) \leqslant M\nu'(f(\mathbf{x})$, so $\nu'(f(\mathbf{x})) \geqslant m\nu(\mathbf{x})$ for $m = \frac{1}{M}$, which concludes the argument.

Corollary 6.66. *Every linear function $f : \mathbb{C}^m \longrightarrow \mathbb{C}^n$ is continuous.*

Proof. Suppose that both $\mathbb{C}^m$ and $\mathbb{C}^n$ are equipped with the norm ν_1. If $\mathbf{x} \in \mathbb{C}^m$ we can write $\mathbf{x} = x_1\mathbf{e}_1 + \cdots + x_m\mathbf{x}_m$ and the linearity of f implies

$$\begin{aligned}\nu_1(f(\mathbf{x})) &= \nu_1\left(f\left(\sum_{i=1}^m x_i\mathbf{e}_i\right)\right) = \nu_1\left(\sum_{i=1}^m x_i f(\mathbf{e}_i)\right) \\ &\leqslant \sum_{i=1}^m |x_i|\nu_1(f(\mathbf{e}_i)) \leqslant M\sum_{i=1}^m |x_i| = M\nu_1(\mathbf{x}),\end{aligned}$$

where $M = \sum_{i=1}^m \nu_1(f(\mathbf{e}_i))$. By Theorem 6.63, the continuity of f follows.

Next, we introduce a norm on the linear space $\mathsf{Hom}(\mathbb{C}^m, \mathbb{C}^n)$ of linear functions from $\mathbb{C}^m$ to $\mathbb{C}^n$. Recall that if $f : \mathbb{C}^m \longrightarrow \mathbb{C}^n$ is a linear function and ν, ν' are norms on $\mathbb{C}^m$ and $\mathbb{C}^n$ respectively, then there exists a non-negative constant m such that $\nu'(f(\mathbf{x})) \leqslant M\nu(\mathbf{x})$ for every $\mathbf{x} \in \mathbb{C}^m$. Define the *norm of f*, $\mu(f)$, as

$$\mu(f) = \inf\{M \in \mathbb{R}_{\geqslant 0} \mid \nu'(f(\mathbf{x})) \leqslant M\nu(\mathbf{x}) \text{ for every } \mathbf{x} \in \mathbb{C}^m\}. \tag{6.13}$$

Theorem 6.67. *The mapping μ defined by Equality (6.13) is a norm on the linear space of linear functions $\mathsf{Hom}(\mathbb{C}^m, \mathbb{C}^n)$.*

Proof. Let f, g be two functions in $\mathsf{Hom}(\mathbb{C}^m, \mathbb{C}^n)$. There exist M_f and M_g in $\mathbb{R}_{\geqslant 0}$ such that $\nu'(f(\mathbf{x})) \leqslant M_f\nu(\mathbf{x})$ and $\nu'(g(\mathbf{x})) \leqslant M_g\nu(\mathbf{x})$ for every $\mathbf{x} \in V$. Thus,

$$\nu'((f+g)(\mathbf{x})) = \nu'(f(\mathbf{x}) + g(\mathbf{x})) \leqslant \nu'(f(\mathbf{x})) + \nu'(g(\mathbf{x})) \leqslant (M_f + M_g)\nu(\mathbf{x}),$$

so

$$M_f + M_g \in \{M \in \mathbb{R}_{\geqslant 0} \mid \nu'((f+g)(\mathbf{x})) \leqslant M\nu(\mathbf{x}) \text{ for every } \mathbf{x} \in V\}.$$

Therefore,

$$\mu(f+g) \leqslant \mu(f) + \mu(g).$$

We leave to the reader the verification of the remaining norm properties of μ.

Since the norm μ defined by Equality (6.13) depends on the norms ν and ν' we denote it by $N(\nu, \nu')$.

Theorem 6.68. *Let $f : \mathbb{C}^m \longrightarrow \mathbb{C}^n$ and $g : \mathbb{C}^n \longrightarrow \mathbb{C}^p$ and let $\mu = N(\nu, \nu')$, $\mu' = N(\nu', \nu'')$ and $\mu'' = N(\mu, \mu'')$, where ν, ν', ν'' are norms on $\mathbb{C}^m$, $\mathbb{C}^n$, and $\mathbb{C}^p$, respectively. We have $\mu''(gf) \leqslant \mu(f)\mu'(g)$.*

Proof. Let $\mathbf{x} \in \mathbb{C}^m$. We have $\nu'(f(\mathbf{x})) \leqslant (\mu(f) + \epsilon')\nu(\mathbf{x})$ for every $\epsilon' > 0$. Similarly, for $\mathbf{y} \in \mathbb{C}^n$, $\nu''(g(\mathbf{y})) \leqslant (\mu'(g) + \epsilon'')\nu'(\mathbf{y})$ for every $\epsilon'' > 0$. These inequalities imply

$$\nu''(g(f(\mathbf{x})) \leqslant (\mu'(g) + \epsilon'')\nu'(f(\mathbf{x})) \leqslant (\mu'(g) + \epsilon'')\mu(f) + \epsilon')\nu(\mathbf{x}).$$

Thus, we have $\mu''(gf) \leqslant (\mu'(g)+\epsilon'')\mu(f)+\epsilon')$, for every ϵ' and ϵ''. This allows us to conclude that $\mu''(fg) \leqslant \mu(f)\mu'(g)$.

Equivalent definitions of the norm $\mu = N(\nu, \nu')$ are given next.

Theorem 6.69. *Let $f : \mathbb{C}^m \longrightarrow \mathbb{C}^n$ and let ν and ν' be two norms defined on $\mathbb{C}^m$ and $\mathbb{C}^n$, respectively. If $\mu = N(\nu, \nu')$, we have*
(i) $\mu(f) = \inf\{M \in \mathbb{R}_{\geqslant 0} \mid \nu'(f(\boldsymbol{x})) \leqslant M\nu(\boldsymbol{x}) \text{ for every } \boldsymbol{x} \in \mathbb{C}^m\}$;
(ii) $\mu(f) = \sup\{\nu'(f(\boldsymbol{x})) \mid \nu(\boldsymbol{x}) \leqslant 1\}$;
(iii) $\mu(f) = \max\{\nu'(f(\boldsymbol{x})) \mid \nu(\boldsymbol{x}) \leqslant 1\}$;
(iv) $\mu(f) = \max\{\nu'(f(\boldsymbol{x})) \mid \nu(\boldsymbol{x}) = 1\}$;
(v) $\mu(f) = \sup\left\{\frac{\nu'(f(\boldsymbol{x}))}{\nu(\boldsymbol{x})} \mid \boldsymbol{x} \in \mathbb{C}^m - \{\boldsymbol{0}_m\}\right\}$.

Proof. The first equality is the definition of $\mu(f)$.

Let ϵ be a positive number. By the definition of the infimum, there exists M such that $\nu'(f(\mathbf{x})) \leqslant M\nu(\mathbf{x})$ for every $\mathbf{x} \in \mathbb{C}^m$ and $M \leqslant \mu(f) + \epsilon$. Thus, for any $\mathbf{x}$ such that $\nu(\mathbf{x}) \leqslant 1$, we have $\nu'(f(\mathbf{x})) \leqslant M \leqslant \mu(f) + \epsilon$. Since this inequality holds for every ϵ, it follows that $\nu'(f(\mathbf{x})) \leqslant \mu(f)$ for every $\mathbf{x} \in \mathbb{C}^m$ with $\nu(\mathbf{x}) \leqslant 1$.

Furthermore, if ϵ' is a positive number, we claim that there exists $\mathbf{x}_0 \in \mathbb{C}^m$ such that $\nu(\mathbf{x}_0) \leqslant 1$ and $\mu(f) - \epsilon' \leqslant \nu'(f(\mathbf{x}_0)) \leqslant \mu(f)$. Suppose that this is not the case. Then, for every $\mathbf{z} \in \mathbb{C}^n$ with $\nu(\mathbf{z}) \leqslant 1$ we have $\nu'(f(\mathbf{z})) \leqslant \mu(f) - \epsilon'$. If $\mathbf{x} \in \mathbb{C}^n$, then $\nu\left(\frac{1}{\nu(\mathbf{x})}\mathbf{x}\right) = 1$, so $\nu'(f(\mathbf{x})) \leqslant (\mu(f) - \epsilon')\nu(\mathbf{x})$, which contradicts the definition of $\mu(f)$. This allows us to conclude that $\mu(f) = \sup\{\nu'(f(\mathbf{x})) \mid \nu(\mathbf{x}) \leqslant 1\}$, which proves the second equality.

Observe that the third equality (where we replaced sup by max) holds because the closed sphere $B(\mathbf{0}, 1)$ is a compact set in $\mathbb{R}^n$. Thus, we have

$$\mu(A) = \max\{\nu'(f(\mathbf{x})) \mid \nu(\mathbf{x}) \leqslant 1\}. \tag{6.14}$$

For the fourth equality, since $\{\mathbf{x} \mid \nu(\mathbf{x}) = 1\} \subseteq \{\mathbf{x} \mid \nu(\mathbf{x}) \leqslant 1\}$, it follows that

$$\max\{\nu'(f(\mathbf{x})) \mid \nu(\mathbf{x}) = 1\} \leqslant \max\{\nu'(f(\mathbf{x})) \mid \nu(\mathbf{x}) \leqslant 1\} = \mu(f).$$

By the third equality there exists a vector $\mathbf{z} \in \mathbb{R}^n - \{\mathbf{0}\}$ such that $\nu(\mathbf{z}) \leqslant 1$ and $\nu'(f(\mathbf{z})) = \mu(f)$. Thus, we have

$$\mu(f) = \nu(\mathbf{z})\nu\left(f\left(\frac{\mathbf{z}}{\nu(\mathbf{z})}\right)\right) \leqslant \nu\left(f\left(\frac{\mathbf{z}}{\nu(\mathbf{z})}\right)\right).$$

Since $\nu\left(\frac{\mathbf{z}}{\nu(\mathbf{z})}\right) = 1$, it follows that $\mu(A) \leqslant \max\{\nu'(f(\mathbf{x})) \mid \nu(\mathbf{x}) = 1\}$. This yields the desired conclusion.

Finally, to prove the last equality observe that for every $\mathbf{x} \in \mathbb{C}^m - \{\mathbf{0}_m\}$, $\frac{1}{\nu(\mathbf{x})}\mathbf{x}$ is a unit vector. Thus, $\nu'(f(\frac{1}{\nu(\mathbf{x})}\mathbf{x}) \leqslant \mu(f)$, by the fourth equality. On the other hand, by the third equality, there exists $\mathbf{x}_0$ such that $\nu(\mathbf{x}_0) = 1$ and $\nu'(f(\mathbf{x}_0)) = \mu(f)$. This concludes the argument.

Definition 6.70. *A normed linear space is* complete *if it is complete as a metric space, that is, if every Cauchy sequence is convergent. A* Banach space *is a complete normed space.*

By Theorem 8.52, if T is a closed subspace of a a Banach space S, then T is complete; the reverse implication is immediate, so a subspace of a Banach space is closed if and only if it is complete.

Let $(\mathbf{x}_0, \mathbf{x}_1, \ldots)$ be a sequence in a normed linear space (L, ν). A *series* in (L, ν) is a sequence $(\mathbf{s}_0, \mathbf{s}_1, \ldots)$ such that $\mathbf{s}_n = \sum_{i=0}^{n} \mathbf{x}_i$ for $n \in \mathbb{N}$. We refer to the elements $\mathbf{x}_i$ as the terms of the series and to $\mathbf{s}_n$ as the n^{th} *partial sum* of the series. The series will be often be denoted by $\sum_{i=0}^{\infty} \mathbf{x}_i$.

If $\lim_{n\to\infty} \mathbf{s}_i = \mathbf{s}$ we say that $\mathbf{s}$ is the *sum* of the series $\sum_{i=0}^{\infty} \mathbf{x}_i$.

A series $\sum_{i=0}^{\infty} \mathbf{x}_i$ in a linear normed space (V, ν) is *absolutely convergent* if the numerical series $\sum_{i=0}^{\infty} \nu(\mathbf{x}_i)$ is convergent.

Theorem 6.71. *In a Banach space (L, ν) every absolutely convergent series is convergent.*

Proof. Let $\sum_{i=0}^{\infty} \mathbf{x}_i$ be an absolutely convergent series in (V, ν). We show that the sequence of its partial sums $(\mathbf{s}_0, \mathbf{s}_1, \ldots)$ is a Cauchy sequence. Let $\epsilon > 0$ and let n_0 be a number such that $\sum_{n=n_0}^{\infty} \nu(\mathbf{x}_i) < \epsilon$. Then, if $m > n \geqslant n_0$ we can write

$$\nu(\mathbf{s}_m - \mathbf{s}_n) = \nu\left(\sum_{i=n+1}^{m} \mathbf{x}_i\right) \leqslant \sum_{i=n+1}^{m} \nu(\mathbf{x}_i) < \epsilon,$$

which proves that the sequence of partial sum is a Cauchy sequence, which implies its convergence.

Theorem 6.72. *Let (L, ν) be a linear normed space. If every absolutely convergent series is convergent, then (L, ν) is a Banach space.*

Proof. Let $(\mathbf{x}_0, \mathbf{x}_1, \ldots)$ be a Cauchy sequence in L. For every $k \in \mathbb{N}$, there exists p_k such that if $m, n \geqslant p_k$, we have $\nu(\mathbf{x}_m - \mathbf{x}_n) < 2^{-k}$.

Define $\mathbf{y}_n = \mathbf{x}_n - \mathbf{x}_{n-1}$ for $n \geqslant 1$ and $\mathbf{y}_0 = \mathbf{x}_0$. Then $\mathbf{x}_n$ is the partial sum of the sequence $(\mathbf{y}_0, \mathbf{y}_1, \ldots)$, $\mathbf{x}_n = \mathbf{y}_0 + \mathbf{y}_1 + \cdots + \mathbf{y}_n$. If $n \geqslant p_k$, $\nu(\mathbf{y}_{n+1}) = \nu(\mathbf{x}_{n+1} - \mathbf{x}_n) < 2^{-k}$, which implies that the series $\sum_{i=0}^{\infty} \mathbf{y}_n$ is absolutely convergent, so the sequence $(\mathbf{x}_0, \mathbf{x}_1, \ldots)$ is convergent. This shows that (V, ν) is a Banach space.

6.8 Norms for Matrices

In Section 5.3 we saw that the set $\mathbb{C}^{m\times n}$ is a linear space. The introduction of norms for matrices can be done by treating matrices as vectors, or by regarding matrices as representations of linear operators.

The mapping vectm introduced next allows us to treat matrices as vectors.

Definition 6.73. *The* $(m \times n)$-vectorization mapping *is the mapping* $\mathsf{vectm} : \mathbb{C}^{m\times n}\mathbb{C}\mathbb{R}^{mn}$ *defined by*

$$\mathsf{vectm}(A) = \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \\ \vdots \\ a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix},$$

obtained by reading A *column-wise.*

Using vector norms on $\mathbb{C}^{mn}$ we can define vectorial norms of matrices.

Definition 6.74. *Let* ν *be a vector norm on the space* $\mathbb{C}^{mn}$*. The* vectorial matrix norm $\mu^{(m,n)}$ on $\mathbb{C}^{m\times n}$ *is the mapping* $\mu^{(m,n)} : \mathbb{C}^{m\times n} \longrightarrow \mathbb{R}_{\geqslant 0}$ *defined by* $\mu^{(m,n)}(A) = \nu(\mathsf{vectm}(A))$*, for* $A \in \mathbb{C}^{m\times n}$*.*

Vectorial norms of matrices are defined without regard for matrix products. The link between linear transformations of finite-dimensional linear spaces and matrices suggest the introduction of an additional condition. Since every matrix $A \in \mathbb{C}^{m\times n}$ corresponds to a linear transformation $h_A : \mathbb{C}^m \longrightarrow \mathbb{C}^n$, if ν and ν' are norms on $\mathbb{C}^m$ and $\mathbb{C}^n$, respectively, it is normal to define a norm on $\mathbb{C}^{m\times n}$ as $\mu(A) = \mu(h_A)$, where $\mu = N(\nu, \nu')$ is a norm on space of linear transformations between $\mathbb{C}^m$ and $\mathbb{C}^n$.

Suppose that ν, ν' and ν'' are vector norms defined on $\mathbb{C}^m, \mathbb{C}^n$ and $\mathbb{C}^p$, respectively. In Theorem 6.68 we saw that $\mu''(gf) \leqslant \mu(f)\mu'(g)$, where $\mu = N(\nu, \nu')$, $\mu' = N(\nu', \nu'')$ and $\mu'' = N(\mu, \mu'')$, so $\mu''(AB) \leqslant \mu(A)\mu'(B)$. This leads us the following definition.

Definition 6.75. *A* consistent family of matrix norms *is a family of functions* $\mu^{(m,n)} : \mathbb{C}^{m\times n} \longrightarrow \mathbb{R}_{\geqslant 0}$*, where* $m, n \in \mathbb{P}$ *that satisfies the following conditions:*
(i) $\mu^{(m,n)}$ *is a norm on* $\mathbb{C}^{(m,n)}$ *for* $m, n \in \mathbb{P}$*, and*
(ii) $\mu^{(m,p)}(AB) \leqslant \mu^{(m,n)}(A)\mu^{(n,p)}(B)$ *for every matrix* $A \in \mathbb{C}^{m\times n}$ *and* $B \in \mathbb{C}^{n\times p}$ *(the* submultiplicative property*).*

If the format of the matrix A is clear from context or is irrelevant, then we shall write $\mu(A)$ instead of $\mu^{(m,n)}(A)$.

Example 6.76. Let $P \in \mathbb{C}^{n\times n}$ be an idempotent matrix. If μ is a matrix norm, then either $\mu(P) = 0$ or $\mu(P) \geqslant 1$.

Indeed, since P is idempotent we have $\mu(P) = \mu(P^2)$. By the submultiplicative property, $\mu(P^2) \leqslant (\mu(P))^2$, so $\mu(P) \leqslant (\mu(P))^2$. Consequently, if $\mu(P) \neq 0$, then $\mu(P) \geqslant 1$.

Some vectorial matrix norms turn out to be actual matrix norms; others fail to be matrix norms. This point is illustrated by the next two examples.

Example 6.77. Consider the vectorial matrix norm μ_1 induced by the vector norm ν_1. We have $\mu_1(A) = \sum_{i=1}^{n}\sum_{j=1}^{m} |a_{ij}|$ for $A \in \mathbb{C}^{m\times n}$. Actually, this is a matrix norm.

Indeed, for $A \in \mathbb{C}^{m\times p}$ and $B \in \mathbb{C}^{p\times n}$ we have:

$$\begin{aligned}
\mu_1(AB) &= \sum_{i=1}^{m}\sum_{j=1}^{n}\left|\sum_{k=1}^{p} a_{ik}b_{kj}\right| \leqslant \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p}|a_{ik}b_{kj}| \\
&\leqslant \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k'=1}^{p}\sum_{k''=1}^{p}|a_{ik'}||b_{k''j}| \\
&\quad \text{(because we added extra non-negative terms to the sums)} \\
&= \left(\sum_{i=1}^{m}\sum_{k'=1}^{p}|a_{ik'}|\right)\cdot\left(\sum_{j=1}^{n}\sum_{k''=1}^{p}|b_{k''j}|\right) = \mu_1(A)\mu_1(B).
\end{aligned}$$

We denote this vectorial matrix norm by the same notation as the corresponding vector norm, that is, by $\parallel A \parallel_1$.

The vectorial matrix norm μ_2 induced by the vector norm ν_2 is also a matrix norm. Indeed, using the same notations we have:

$$\begin{aligned}
(\mu_2(AB))^2 &= \sum_{i=1}^{m}\sum_{j=1}^{n}\left|\sum_{k=1}^{p} a_{ik}b_{kj}\right|^2 \leqslant \sum_{i=1}^{m}\sum_{j=1}^{n}\left(\sum_{k=1}^{p}|a_{ik}|^2\right)\left(\sum_{l=1}^{p}|b_{lj}|^2\right) \\
&\quad \text{(by Cauchy-Schwarz inequality)} \\
&\leqslant (\mu_2(A))^2(\mu_2(B))^2.
\end{aligned}$$

The vectorial norm of $A \in \mathbb{C}^{m\times n}$ $\mu_2(A) = \left(\sum_{i=1}^{n}\sum_{j=1}^{m}|a_{ij}|^2\right)^{\frac{1}{2}}$, denoted also by $\parallel A \parallel_F$, is known as the *Frobenius norm.*

It is easy to see that for real matrices we have

$$\parallel A \parallel_F^2 = \mathit{trace}(AA') = \mathit{trace}(A'A) \tag{6.15}$$

and for complex matrices the corresponding equality is

$$\parallel A \parallel_F^2 = \mathit{trace}(AA^{\mathsf{H}}) = \mathit{trace}(A^{\mathsf{H}}A). \tag{6.16}$$

Note that $\parallel A^{\mathsf{H}} \parallel_F^2 = \parallel A \parallel_F^2$ for every A.

Example 6.78. The vectorial norm μ_∞ induced by the vector norm ν_∞ is denoted by $\| A \|_\infty$ and is given by $\| A \|_\infty = \max_{i,j} |a_{ij}|$ for $A \in \mathbb{C}^{n \times n}$. This is *not* a matrix norm. Indeed, let a, b be two positive numbers and consider the matrices

$$A = \begin{pmatrix} a & a \\ a & a \end{pmatrix} \text{ and } B = \begin{pmatrix} b & b \\ b & b \end{pmatrix}.$$

We have $\| A \|_\infty = a$ and $\| B \|_\infty = b$. However, since

$$AB = \begin{pmatrix} 2ab & 2ab \\ 2ab & 2ab \end{pmatrix},$$

we have $\| AB \|_\infty = 2ab$ and the submultiplicative property of matrix norms is violated.

By regarding matrices as transformations between linear spaces, we can define norms for matrices that turn out to be matrix norms.

Definition 6.79. *Let ν_m be a norm on $\mathbb{C}^m$ and ν_n be a norm on $\mathbb{C}^n$ and let $A \in \mathbb{C}^{n \times m}$ be a matrix. The* operator norm *of A is the number $\mu^{(n,m)}(A) = \mu^{(n,m)}(h_A)$, where $\mu^{(n,m)} = N(\nu_m, \nu_n)$.*

Theorem 6.80. *Let $\{\nu_n \mid n \geqslant 1\}$ be a family of vector norms, where ν_n is a vector norm on $\mathbb{C}^n$. The family of norms $\{\mu^{(n,m)} \mid n, m \geqslant 1\}$ is consistent.*

Proof. It is easy to see that the family of norms $\{\mu^{(n,m)} \mid n, m \geqslant 1\}$ satisfies the first three conditions of Definition 6.75. For the fourth condition of Definition 6.75 and $A \in \mathbb{C}^{n \times m}$ and $B \in \mathbb{C}^{m \times p}$, we can write:

$$\begin{aligned}
\mu^{(n,p)}(AB) &= \sup\{\nu_n((AB)\mathbf{x}) \mid \nu_p(\mathbf{x}) \leqslant 1\} \\
&= \sup\{\nu_n(A(B\mathbf{x})) \mid \nu_p(\mathbf{x}) \leqslant 1\} \\
&= \sup\left\{\nu_n\left(A\frac{B\mathbf{x}}{\nu_m(B\mathbf{x})}\right)\nu_m(B\mathbf{x}) \middle| \nu_p(\mathbf{x}) \leqslant 1\right\} \\
&\leqslant \mu^{(n,m)}(A)\sup\{\nu_m(B\mathbf{x}) \middle| \nu_p(\mathbf{x}) \leqslant 1\} \\
&\quad \text{(because } \nu_m\left(\tfrac{B\mathbf{x}}{\nu(B\mathbf{x})}\right) = 1) \\
&= \mu^{(n,m)}(A)\mu^{(m,p)}(B).
\end{aligned}$$

Theorem 6.69 implies the following equivalent definitions of $\mu^{(n,m)}(A)$.

Theorem 6.81. *Let ν_n be a norm on $\mathbb{C}^n$ for $n \geqslant 1$. The following equalities hold for $\mu^{(n,m)}(A)$, where $A \in \mathbb{C}^{(n,m)}$.*

$$\begin{aligned}
\mu^{(n,m)}(A) &= \inf\{M \in \mathbb{R}_{\geqslant 0} \mid \nu_n(A\boldsymbol{x}) \leqslant M\nu_m(\boldsymbol{x}) \text{ for every } \boldsymbol{x} \in \mathbb{C}^m\} \\
&= \sup\{\nu_n(A\boldsymbol{x}) \mid \nu_m(\boldsymbol{x}) \leqslant 1\} = \max\{\nu_n(A\boldsymbol{x}) \mid \nu_m(\boldsymbol{x}) \leqslant 1\} \\
&= \max\{\nu'(f(\boldsymbol{x})) \mid \nu(\boldsymbol{x}) = 1\} = \sup\left\{\frac{\nu'(f(\boldsymbol{x}))}{\nu(\boldsymbol{x})} \mid \boldsymbol{x} \in \mathbb{C}^m - \{\boldsymbol{0}_m\}\right\}.
\end{aligned}$$

Proof. The theorem is simply a reformulation of Theorem 6.69.

Corollary 6.82. *Let μ be the matrix norm on $\mathbb{C}^{n\times n}$ induced by the vector norm ν. We have $\nu(A\boldsymbol{u}) \leqslant \mu(A)\nu(\boldsymbol{u})$ for every $\boldsymbol{u} \in \mathbb{C}^n$.*

Proof. The inequality is obviously satisfied when $\mathbf{u} = \mathbf{0}_n$. Therefore, we may assume that $\mathbf{u} \neq \mathbf{0}_n$ and let $\mathbf{x} = \frac{1}{\nu(\mathbf{u})}\mathbf{u}$. Clearly, $\nu(\mathbf{x}) = 1$ and Equality (6.14) implies that

$$\nu\left(A\frac{1}{\nu(\mathbf{u})}\mathbf{u}\right) \leqslant \mu(A)$$

for every $\mathbf{u} \in \mathbb{C}^n - \{\mathbf{0}_n\}$. This implies immediately the desired inequality.

If μ is a matrix norm induced by a vector norm on $\mathbb{R}^n$, then $\mu(I_n) = \sup\{\nu(I_n\mathbf{x}) \mid \nu(\mathbf{x}) \leqslant 1\} = 1$.

The operator matrix norm induced by the vector norm $\|\cdot\|_p$ is denoted by $|||\cdot|||_p$.

Example 6.83. To compute $|||A|||_1 = \sup\{\| A\mathbf{x} \|_1 \mid \| \mathbf{x} \|_1 \leqslant 1\}$, where $A \in \mathbb{R}^{n\times n}$, suppose that the columns of A are the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$, that is

$$\mathbf{a}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix}.$$

Let $\mathbf{x} \in \mathbb{R}^n$ be a vector whose components are $x_1, \ldots, x_n$. Then, $\mathbf{Ax} = x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n$, so

$$\begin{aligned} \| A\mathbf{x} \|_1 &= \| x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n \|_1 \leqslant \sum_{j=1}^n |x_j| \, \| \mathbf{a}_j \|_1 \\ &\leqslant \max_j \| \mathbf{a}_j \|_1 \sum_{j=1}^n |x_j| = \max_j \| \mathbf{a}_j \|_1 \cdot \| \mathbf{x} \|_1 . \end{aligned}$$

Thus, $|||A|||_1 \leqslant \max_j \| \mathbf{a}_j \|_1$.

Let $\mathbf{e}_j$ be the vector whose components are 0 with the exception of its j^{th} component that is equal to 1. Clearly, we have $\| \mathbf{e}_j \|_1 = 1$ and $\mathbf{a}_j = A\mathbf{e}_j$. This, in turn implies $\| \mathbf{a}_j \|_1 = \| A\mathbf{e}_j \|_1 \leqslant |||A|||_1$ for $1 \leqslant j \leqslant n$. Therefore, $\max_j \| \mathbf{a}_j \|_1 \leqslant |||A|||_1$, so

$$|||A|||_1 = \max_j \| \mathbf{a}_j \|_1 = \max_j \sum_{i=1}^n |a_{ij}|.$$

In other words, $|||A|||_1$ equals the maximum column sum of the absolute values.

Example 6.84. Let $A \in \mathbb{R}^{n\times n}$. We have

$$\begin{aligned}\parallel A\mathbf{x} \parallel_\infty &= \max_{1\leqslant i\leqslant n}\left|\sum_{j=1}^{n} a_{ij}x_j\right| \leqslant \max_{1\leqslant i\leqslant n}\sum_{j=1}^{n}|a_{ij}x_j| \\ &\leqslant \max_{1\leqslant i\leqslant n} \parallel \mathbf{x} \parallel_\infty \sum_{j=1}^{n}|a_{ij}|.\end{aligned}$$

Consequently, if $\parallel \mathbf{x} \parallel_\infty \leqslant 1$ we have $\parallel A\mathbf{x} \parallel_\infty \leqslant \max_{1\leqslant i\leqslant n}\sum_{j=1}^{n}|a_{ij}|$. Thus, $|\!|\!| A |\!|\!|_\infty \leqslant \max_{1\leqslant i\leqslant n}\sum_{j=1}^{n}|a_{ij}|$.

The converse inequality is immediate if $A = O_{n,n}$. Therefore, assume that $A \neq O_{n\times n}$, and let $(a_{p1},\ldots,a_{pn})$ be any row of A that has at least one element distinct from 0. Define the vector $\mathbf{z} \in \mathbb{R}^n$ by

$$z_j = \begin{cases}\frac{|a_{pj}|}{a_{pj}} & \text{if } a_{pj} \neq 0,\\ 1 & \text{otherwise,}\end{cases}$$

for $1 \leqslant j \leqslant n$. It is clear that $z_j \in \{-1,1\}$ for every j, $1 \leqslant j \leqslant n$ and, therefore, $\parallel \mathbf{z} \parallel_\infty = 1$. Moreover, we have $|a_{pj}| = a_{pj}z_j$ for $1 \leqslant j \leqslant n$. Therefore, we can write:

$$\begin{aligned}\sum_{j=1}^{n}|a_{pj}| = \sum_{j=1}^{n}a_{pj}z_j \leqslant \left|\sum_{j=1}^{n}a_{pj}z_j\right| \leqslant \max_{1\leqslant i\leqslant n}\left|\sum_{j=1}^{n}a_{ij}z_j\right| \\ = \parallel A\mathbf{z} \parallel_\infty \leqslant \max\{\parallel A\mathbf{x} \parallel_\infty \mid \parallel \mathbf{x} \parallel_\infty \leqslant 1\} = |\!|\!| A |\!|\!|_\infty.\end{aligned}$$

Since this holds for every row of A, it follows that $\max_{1\leqslant i\leqslant n}\sum_{j=1}^{n}|a_{ij}| \leqslant |\!|\!| A |\!|\!|_\infty$, which proves that $|\!|\!| A |\!|\!|_\infty = \max_{1\leqslant i\leqslant n}\sum_{j=1}^{n}|a_{ij}|$. In other words, $|\!|\!| A |\!|\!|_\infty$ equals the maximum row sum of the absolute values.

Example 6.85. Let $D = \mathsf{diag}(d_1,\ldots,d_n) \in \mathbb{C}^{n\times n}$ be a diagonal matrix. If $\mathbf{x} \in \mathbb{C}^n$ we have

$$D\mathbf{x} = \begin{pmatrix} d_1x_1 \\ \vdots \\ d_nx_n \end{pmatrix},$$

so

$$\begin{aligned}|\!|\!| D |\!|\!|_2 &= \max\{\parallel D\mathbf{x} \parallel_2 \mid \parallel \mathbf{x} \parallel_2 = 1\} \\ &= \max\{\sqrt{(d_1x_1)^2 + \cdots + (d_nx_n)^2} \mid x_1^2 + \cdots + x_n^2 = 1\} \\ &= \max\{|d_i| \mid 1 \leqslant 1 \leqslant n\}.\end{aligned}$$

Norms that are invariant with respect to multiplication by unitary matrices are known as *unitarily invariant norms*.

Theorem 6.86. *Let $U \in \mathbb{C}^{n\times n}$ be a unitary matrix. The following statements hold:*
(i) $\parallel U\boldsymbol{x} \parallel_2 = \parallel \boldsymbol{x} \parallel_2$ *for every* $\boldsymbol{x} \in \mathbb{C}^n$*;*
(ii) $|||UA|||_2 = |||A|||_2$ *for every* $A \in \mathbb{C}^{n\times p}$*;*
(iii) $\parallel UA \parallel_F = \parallel A \parallel_F$ *for every* $A \in \mathbb{C}^{n\times p}$*.*

Proof. For the first part of the theorem note that $\parallel U\mathbf{x} \parallel_2^2 = (U\mathbf{x})^{\mathsf{H}}U\mathbf{x} = \mathbf{x}^{\mathsf{H}}U^{\mathsf{H}}U\mathbf{x} = \mathbf{x}^{\mathsf{H}}\mathbf{x} = \parallel \mathbf{x} \parallel_2^2$, because $U^{\mathsf{H}}A = I_n$.

The proof of the second part is shown next:

$$\begin{aligned}|||UA|||_2 &= \max\{\parallel (UA)\mathbf{x} \parallel_2 \mid \parallel \mathbf{x} \parallel_2 = 1\} = \max\{\parallel U(A\mathbf{x}) \parallel_2 \mid \parallel \mathbf{x} \parallel_2 = 1\} \\ &= \max\{\parallel A\mathbf{x} \parallel_2 \mid \parallel \mathbf{x} \parallel_2 = 1\} = |||A|||_2.\end{aligned}$$

For the Frobenius norm note that

$$\parallel UA \parallel_F = \sqrt{\mathit{trace}((UA)^{\mathsf{H}}UA)} = \sqrt{\mathit{trace}(A^{\mathsf{H}}U^{\mathsf{H}}UA)} = \sqrt{\mathit{trace}(A^{\mathsf{H}}A)} = \parallel A \parallel_F,$$

by Equality (6.15).

Corollary 6.87. *If $U \in \mathbb{C}^{n\times n}$ is a unitary matrix, then $|||U|||_2 = 1$.*

Proof. Since $|||U|||_2 = \sup\{\parallel U\mathbf{x} \parallel_2 \mid \parallel \mathbf{x} \parallel_2 \leqslant 1\}$, by Part (ii) of Theorem 6.86,

$$|||U|||_2 = \sup\{\parallel \mathbf{x} \parallel_2 \mid \parallel \mathbf{x} \parallel_2 \leqslant 1\} = 1.$$

Corollary 6.88. *Let $A, U \in \mathbb{C}^{n\times n}$. If U is an unitary matrix, then*

$$\parallel U^{\mathsf{H}}AU \parallel_F = \parallel A \parallel_F .$$

Proof. Since U is a unitary matrix, so is U^{H}. By Part (iii) of Theorem 6.86,

$$\parallel U^{\mathsf{H}}AU \parallel_F = \parallel AU \parallel_F = \parallel U^{\mathsf{H}}A^{\mathsf{H}} \parallel_F^2 = \parallel A^{\mathsf{H}} \parallel_F^2 = \parallel A \parallel_F^2,$$

which proves the corollary.

Example 6.89. Let $S = \{\mathbf{x} \in \mathbb{R}^n \mid \parallel \mathbf{x} \parallel_2 = 1\}$ be the surface of the sphere in $\mathbb{R}^n$. The image of S under the linear transformation h_U that corresponds to the unitary matrix U is S itself. Indeed, by Theorem 6.86, $\parallel h_U(\mathbf{x}) \parallel_2 = \parallel \mathbf{x} \parallel_2 = 1$, so $h_U(\mathbf{x}) \in S$ for every $\mathbf{x} \in S$. Also, note that h_U restricted to S is a bijection because $h_{U^{\mathsf{H}}}(h_U(\mathbf{x})) = \mathbf{x}$ for every $\mathbf{x} \in \mathbb{R}^n$.

Theorem 6.90. *Let $A \in \mathbb{R}^{n\times n}$. We have $|||A|||_2 \leqslant \parallel A \parallel_F$.*

Proof. Let $\mathbf{x} \in \mathbb{R}^n$. We have

$$A\mathbf{x} = \begin{pmatrix} \mathbf{r}_1\mathbf{x} \\ \vdots \\ \mathbf{r}_n\mathbf{x} \end{pmatrix},$$

where $\mathbf{r}_1, \ldots, \mathbf{r}_n$ are the rows of the matrix A. Thus,

$$\frac{\| A\mathbf{x} \|_2}{\| \mathbf{x} \|_2} = \frac{\sqrt{\sum_{i=1}^{n} (\mathbf{r}_i \mathbf{x})^2}}{\| \mathbf{x} \|_2}.$$

By Cauchy-Schwarz inequality we have $(\mathbf{r}_i \mathbf{x})^2 \leqslant \| \mathbf{r}_i \|_2^2 \| \mathbf{x} \|_2^2$, so

$$\frac{\| A\mathbf{x} \|_2}{\| \mathbf{x} \|_2} \leqslant \sqrt{\sum_{i=1}^{n} \| \mathbf{r}_i \|_2^2} = \| A \|_F \,.$$

This implies $|||A|||_2 \leqslant \| A \|_F$.

6.9 Projection on Subspaces

If U, W are two complementary subspaces of $\mathbb{C}^n$, then there exist idempotent endomorphisms g and h of $\mathbb{C}^n$ such that $W = \mathsf{Ker}(g)$, $U = \mathsf{Img}(g)$ and $U = \mathsf{Ker}(h)$ and $W = \mathsf{Img}(h)$. The vector $g(\mathbf{x})$ is the *oblique projection of* $\mathbf{x}$ *on* U *along the subspace* W and $h(\mathbf{x})$ is the *oblique projection* of $\mathbf{x}$ on W along the subspace U.

If g and h are represented by the matrices B_U and B_W, respectively, it follows that these matrices are idempotent, $g(\mathbf{x}) = B_U\mathbf{x} \in U$, and $h(\mathbf{x}) = B_W\mathbf{x} \in W$ for $\mathbf{x} \in \mathbb{C}^n$. Also, $B_U B_W = B_W B_U = O_{n,n}$.

Let U and W be two complementary subspaces of $\mathbb{C}^n$, $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be a basis for U, and let $\{\mathbf{w}_1, \ldots, \mathbf{w}_q\}$ be a basis for W, where $p + q = n$. Clearly, $\{\mathbf{u}_1, \ldots, \mathbf{u}_p, \mathbf{w}_1, \ldots, \mathbf{w}_q\}$ is a basis for $\mathbb{C}^n$ and every $\mathbf{x} \in \mathbb{C}^n$ can be written as

$$\mathbf{x} = x_1\mathbf{u}_1 + \cdots + x_p\mathbf{u}_p + x_{p+1}\mathbf{w}_1 + \cdots + x_{p+q}\mathbf{w}_q.$$

Let $B \in \mathbb{C}^{n \times n}$ be the matrix $B = (\mathbf{u}_1 \ \cdots \ \mathbf{u}_p \ \mathbf{w}_1 \ \cdots \ \mathbf{w}_q)$, which is clearly invertible. Note that

$$B_U\mathbf{u}_i = \mathbf{u}_i, B_U\mathbf{w}_j = \mathbf{0}_n, B_W\mathbf{u}_i = \mathbf{0}_n, B_W\mathbf{w}_j = \mathbf{w}_j,$$

for $1 \leqslant i \leqslant p$ and $1 \leqslant j \leqslant q$. Therefore, we have

$$B_U B = (\mathbf{u}_1 \ \cdots \ \mathbf{u}_p \ \mathbf{0}_n \ \cdots \ \mathbf{0}_n) = B \begin{pmatrix} I_p & O_{p,n-p} \\ O_{n-p,p} & O_{n-p,n-p} \end{pmatrix},$$

so

$$B_U = B \begin{pmatrix} I_p & O_{p,n-p} \\ O_{n-p,p} & O_{n-p,n-p} \end{pmatrix} B^{-1}.$$

Similarly, we can show that

$$B_W = B \begin{pmatrix} O_{n-q,n-q} & O_{n-q,q} \\ O_{q,n-q} & I_q \end{pmatrix} B^{-1}.$$

Note that $B_U + B_W = BI_nB^{-1} = I_n$. Thus, the oblique projection on U along W is given by

$$g(\mathbf{x}) = B_U\mathbf{x} = B\begin{pmatrix} I_p & O_{p,n-p} \\ O_{n-p,p} & O_{n-p,n-p} \end{pmatrix} B^{-1}\mathbf{x}.$$

The similar oblique projection on W along U is

$$h(\mathbf{x}) = B_U\mathbf{x} = B\begin{pmatrix} O_{n-q,n-q} & O_{n-q,q} \\ O_{q,n-q} & I_q \end{pmatrix} B^{-1}\mathbf{x},$$

for $\mathbf{x} \in \mathbb{C}^n$. Observe that $g(\mathbf{x}) + h(\mathbf{x}) = \mathbf{x}$, so the projection on W along U is $h(\mathbf{x}) = \mathbf{x} - g(\mathbf{x}) = (I_n - B_U)\mathbf{x}$.

A special important type of projections involves pairs of orthogonal subspaces. Let U be a subspace of $\mathbb{C}^n$ with $\dim U = p$ and let $B_U = \{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be an orthonormal basis of U. Taking into account that $(U^\perp)^\perp = U$, by Theorem 5.33 there exists an idempotent endomorphism g of $\mathbb{C}^n$ such that $U = \mathsf{Img}(g)$ and $U^\perp = \mathsf{Ker}(g)$. The proof of Theorem 6.39 shows that this endomorphism is defined by $g(\mathbf{x}) = (\mathbf{x}, \mathbf{u}_1)\mathbf{u}_1 + \cdots + (\mathbf{x}, \mathbf{u}_m)\mathbf{u}_m$.

Definition 6.91. *Let $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m\}$ be an orthonormal basis of an m-dimensional subspace of U of $\mathbb{C}^n$. The* orthogonal projection of the vector $\mathbf{x} \in \mathbb{C}^n$ on the subspace U *is the vector $\mathsf{proj}_U(\boldsymbol{x}) = (\boldsymbol{x}, \boldsymbol{u}_1)\boldsymbol{u}_1 + \cdots + (\boldsymbol{x}, \boldsymbol{u}_m)\boldsymbol{u}_m$.*

If $\{\mathbf{z}_1, \ldots, \mathbf{z}_m\}$ is an orthogonal basis of the space U, then an orthonormal basis of the same subspace is $\{\frac{1}{\|\mathbf{z}_1\|}\mathbf{z}_1, \ldots, \frac{1}{\|\mathbf{z}_m\|}\mathbf{z}_m\}$. Thus, if $\mathbf{u}_i = \frac{1}{\|\mathbf{z}_i\|}\mathbf{z}_i$ for $1 \leqslant i \leqslant m$, we have $(\mathbf{x}, \mathbf{u}_i)\mathbf{u}_i = \frac{(\mathbf{x}, \mathbf{z}_i)}{\|\mathbf{z}_i\|^2}\mathbf{z}_i$ for $1 \leqslant i \leqslant m$, and the orthogonal projection $\mathsf{proj}_U(\mathbf{x})$ can be written as

$$\mathsf{proj}_U(\mathbf{x}) = \frac{(\mathbf{x}, \mathbf{z}_1)}{\| \mathbf{z}_1 \|^2}\mathbf{z}_1 + \cdots + \frac{(\mathbf{x}, \mathbf{z}_m)}{\| \mathbf{z}_m \|^2}\mathbf{z}_m. \tag{6.17}$$

In particular, the projection of $\mathbf{x}$ on the 1-dimensional subspace generated by $\mathbf{z}$ is denoted by $\mathsf{proj}_{\mathbf{z}}(\mathbf{x})$ and is given by

$$\mathsf{proj}_{\mathbf{z}}(\mathbf{x}) = \frac{(\mathbf{x}, \mathbf{z})}{\| \mathbf{z} \|^2}\mathbf{z} = \frac{\mathbf{x}^{\mathsf{H}}\mathbf{z}}{\mathbf{z}^{\mathsf{H}}\mathbf{z}}\mathbf{z}. \tag{6.18}$$

Theorem 6.92. *Let U be an m-dimensional subspace of $\mathbb{R}^n$ and let $\boldsymbol{x} \in \mathbb{R}^n$. The vector $\boldsymbol{y} = \boldsymbol{x} - \mathsf{proj}_U(\boldsymbol{x})$ belongs to the subspace $U^\perp$.*

Proof. Let $B_U = \{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ be an orthonormal basis of U. Note that

$$\begin{aligned}(\mathbf{y}, \mathbf{u}_j) &= (\mathbf{x}, \mathbf{u}_j) - \left(\sum_{i=1}^m (\mathbf{x}, \mathbf{u}_i)\mathbf{u}_i, \mathbf{u}_j\right) \\ &= (\mathbf{x}, \mathbf{u}_j) - \sum_{i=1}^m (\mathbf{x}, \mathbf{u}_i)(\mathbf{u}_i, \mathbf{u}_j) = 0,\end{aligned}$$

due to the orthogonality of the basis B_U. Therefore, $\mathbf{y}$ is orthogonal on every linear combination of B_U, that is on the subspace U.

Theorem 6.93. *Let U be an m-dimensional subspace of $\mathbb{C}^n$ having the orthonormal basis $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m\}$. The orthogonal projection proj_U is given by $\mathsf{proj}_U(\boldsymbol{x}) = B_U B_U^{\mathsf{H}} \boldsymbol{x}$ for $\boldsymbol{x} \in \mathbb{C}^n$, where $B_U \in \mathbb{R}^{n\times m}$ is the matrix $B_U = (\boldsymbol{u}_1 \ \cdots \ \boldsymbol{u}_m) \in \mathbb{C}^{n\times m}$.*

Proof. We can write

$$\mathsf{proj}_U(\mathbf{x}) = \sum_{i=1}^{m} \mathbf{u}_i(\mathbf{u}_i^{\mathsf{H}}\mathbf{x}) = (\mathbf{u}_1 \ \cdots \ \mathbf{u}_m) \begin{pmatrix} \mathbf{u}_1^{\mathsf{H}} \\ \vdots \\ \mathbf{u}_m^{\mathsf{H}} \end{pmatrix} \mathbf{x} = B_U B_U^{\mathsf{H}} \mathbf{x}.$$

Since the basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ is orthonormal, we have $B_U^{\mathsf{H}} B_U = I_m$. Observe that the matrix $B_U B_U^{\mathsf{H}} \in \mathbb{C}^{n\times n}$ is symmetric and idempotent because

$$(B_U B_U^{\mathsf{H}})(B_U B_U^{\mathsf{H}}) = B_U (B_U^{\mathsf{H}} B_U) B_U^{\mathsf{H}} = B_U B_U^{\mathsf{H}}.$$

Corollary 6.94. *Let U be an m-dimensional subspace of $\mathbb{C}^n$ having the orthonormal basis $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m\}$. We have*

$$\mathsf{proj}_U(\boldsymbol{x}) = \sum_{i=1}^{m} \mathsf{proj}_{\boldsymbol{u}_i}(\boldsymbol{x}).$$

Proof. This statement follows directly from Theorem 6.93.

For an m-dimensional subspace U of $\mathbb{C}^n$ we denote by $P_U = B_U B_U^{\mathsf{H}} \in \mathbb{C}^{n\times n}$, where B_U is a matrix of an orthonormal basis of U as defined before. P_U is the *projection matrix* of the subspace U.

Corollary 6.95. *For every non-zero subspace U, the matrix P_U is a Hermitian matrix, and therefore, a self-adjoint matrix.*

Proof. Since $P_U = B_U B_U^{\mathsf{H}}$ where B_U is a matrix of an orthonormal basis of the subspace S, it is immediate that $P_U^{\mathsf{H}} = P_U$.

The self-adjointness of P_U means that $(\mathbf{x}, P_U\mathbf{y}) = (P_U\mathbf{x}, \mathbf{y})$ for every $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$.

Corollary 6.96. *Let U be an m-dimensional subspace of $\mathbb{C}^n$ having the orthonormal basis $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m\}$.*

If $B_U = (\boldsymbol{u}_1 \ \cdots \ \boldsymbol{u}_m) \in \mathbb{C}^{n\times m}$, then for every $\boldsymbol{x} \in \mathbb{C}$ we have the decomposition $\boldsymbol{x} = P_U\boldsymbol{x} + P_{U^\perp}\boldsymbol{x}$, where $P_U = B_U B_U^{\mathsf{H}}$ and $P_{U^\perp} = I_n - P_U = I_n - B_U B_{U^\perp}$.

Proof. This statement follows immediately from Theorem 6.93.

It is possible to give a direct argument for the independence of the projection matrix P_U relative to the choice of orthonormal basis in U.

Theorem 6.97. *Let U be an m-dimensional subspace of $\mathbb{C}^n$ having the orthonormal bases $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m\}$ and $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m\}$ and let $B_U = (\boldsymbol{u}_1 \ \cdots \ \boldsymbol{u}_m) \in \mathbb{C}^{n\times m}$ and $\tilde{B}_U = (\boldsymbol{v}_1 \ \cdots \ \boldsymbol{v}_m) \in \mathbb{C}^{n\times m}$. The matrix $B_U^{\mathsf{H}}\tilde{B}_U \in \mathbb{C}^{m\times m}$ is unitary and $\tilde{B}_U\tilde{B}_U^{\mathsf{H}} = B_U B_U^{\mathsf{H}}$.*

Proof. Since the both sets of columns of B_U and $\tilde{B}_U$ are bases for U, there exists a unique square matrix $Q \in \mathbb{C}^{m\times m}$ such that $B_U = \tilde{B}_U Q$. The orthonormality of B_U and $\tilde{B}_U$ implies $B_U^{\mathsf{H}} B_U = \tilde{B}_U^{\mathsf{H}}\tilde{B}_U = I_m$. Thus, we can write $I_m = B_U^{\mathsf{H}} B_U = Q^{\mathsf{H}}\tilde{B}_U^{\mathsf{H}}\tilde{B}_U Q = Q^{\mathsf{H}} Q$, which shows that Q is unitary. Furthermore, $B_U^{\mathsf{H}}\tilde{B}_U = Q^{\mathsf{H}}\tilde{B}_U^{\mathsf{H}}\tilde{B}_U = Q^{\mathsf{H}}$ is unitary and $B_U B_U^{\mathsf{H}} = \tilde{B}_U Q Q^{\mathsf{H}}\tilde{B}_U^{\mathsf{H}} = \tilde{B}_U\tilde{B}_U^{\mathsf{H}}$.

In Example 6.76 we have shown that if P is an idempotent matrix, then for every matrix norm μ we have $\mu(P) = 0$ or $\mu(P) \geqslant 1$. For orthogonal projection matrices of the form P_U, where U is a non-zero subspace we have $|||P_U|||_2 = 1$. Indeed, we can write $\mathbf{x} = (\mathbf{x} - \mathsf{proj}_U(\mathbf{x})) + \mathsf{proj}_U(\mathbf{x})$, so

$$\| \mathbf{x} \|_2^2 = \| \mathbf{x} - \mathsf{proj}_U(\mathbf{x}) \|_2^2 + \| \mathsf{proj}_U(\mathbf{x}) \|_2^2 \geqslant \| \mathsf{proj}_U(\mathbf{x}) \|_2^2 .$$

Thus, $\| \mathbf{x} \|_2 \geqslant \| P_U(\mathbf{x}) \|$ for any $\mathbf{x} \in \mathbb{C}^n$, which implies

$$|||P_U|||_2 = \sup\left\{ \frac{\| P_U\mathbf{x} \|_2}{\| \mathbf{x} \|_2} \middle| \mathbf{x} \in \mathbb{C}^n - \{\mathbf{0}\}\right\} \leqslant 1.$$

This implies $|||P_U|||_2 = 1$.

The next theorem shows that the best approximation of a vector $\mathbf{x}$ is a subspace U (in the sense of Euclidean distance) is the orthogonal projection on $\mathbf{x}$ on U.

Theorem 6.98. *Let U be an m-dimensional subspace of $\mathbb{C}^n$ and let $\boldsymbol{x} \in \mathbb{C}^n$. The minimal value of $d_2(\boldsymbol{x}, \boldsymbol{u})$, the Euclidean distance between $\boldsymbol{x}$ and an element $\boldsymbol{u}$ of the subspace U is achieved when $\boldsymbol{u} = \mathsf{proj}_U(\boldsymbol{x})$.*

Proof. We saw that $\mathbf{x}$ can be uniquely written as $\mathbf{x} = \mathbf{y} + \mathsf{proj}_U(\mathbf{x})$, where $\mathbf{y} \in U^{\perp}$. Let now $\mathbf{u}$ be an arbitrary member of U. We have

$$d_2(\mathbf{x}, \mathbf{u})^2 = \| \mathbf{x} - \mathbf{u} \|_2^2 = \| (\mathbf{x} - \mathsf{proj}_U(\mathbf{x})) + (\mathsf{proj}_U(\mathbf{x}) - \mathbf{u}) \|_2^2 .$$

Since $\mathbf{x} - \mathsf{proj}_U(\mathbf{x}) \in U^{\perp}$ and $\mathsf{proj}_U(\mathbf{x}) - \mathbf{u} \in U$, it follows that these vectors are orthogonal. Thus, we can write

$$d_2(\mathbf{x}, \mathbf{u})^2 = \| (\mathbf{x} - \mathsf{proj}_U(\mathbf{x})) \|_2^2 + \| (\mathsf{proj}_U(\mathbf{x}) - \mathbf{u}) \|_2^2,$$

which implies that $d_2(\mathbf{x}, \mathbf{u}) \geqslant d_2(\mathbf{x} - \mathsf{proj}_U(\mathbf{x}))$.

The orthogonal projections associated with subspaces allow us to define a metric on the collection of subspaces of $\mathbb{C}^n$. Indeed, if S and T are two subspaces of $\mathbb{C}^n$ we define $d_F(S,T) = \| P_S - P_T \|_F$. When using the vector norm $\| \cdot \|_2$ and the metric induced by this norm on $\mathbb{C}^n$ we denote the corresponding metric on subspaces by d_2.

Example 6.99. Let $\mathbf{u}, \mathbf{w}$ be two distinct unit vectors in the linear space L. The orthogonal projection matrices of $\langle \mathbf{u} \rangle$ and $\langle \mathbf{w} \rangle$ are $\mathbf{uu}'$ and $\mathbf{ww}'$, respectively. Thus,

$$d_F(\langle \mathbf{u} \rangle, \langle \mathbf{w} \rangle) = \| \mathbf{uu}' - \mathbf{vv}' \|_F .$$

Suppose now that $L = \mathbb{R}^2$. Since $\mathbf{u}$ and $\mathbf{w}$ are unit vectors in $\mathbb{R}^2$ there exist $\alpha, \beta \in [0, 2\pi]$ such that

$$\mathbf{u} = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} \text{ and } \mathbf{w} = \begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix}.$$

Thus, we can write

$$\mathbf{uu}' - \mathbf{vv}' = \begin{pmatrix} \cos^2\alpha - \cos^2\beta & \cos\alpha\sin\alpha - \cos\beta\sin\beta \\ \cos\alpha\sin\alpha - \cos\beta\sin\beta & \sin^2\alpha - \sin^2\beta \end{pmatrix}.$$

and $d_F(\langle \mathbf{u} \rangle, \langle \mathbf{w} \rangle) = \sqrt{2} |\sin(\alpha - \beta)|$.

We could use any matrix norm in the definition of the distance between subspaces. For example, we could replace the Frobenius norm by $|\!|\!| \cdot |\!|\!|_1$ or by $|\!|\!| \cdot |\!|\!|_2$.

Let S be a subspace of $\mathbb{C}^n$ and let $\mathbf{x} \in \mathbb{C}^n$. The *distance between* $\boldsymbol{x}$ *and* S *defined by the norm* $\| \cdot \|$ is

$$d(\mathbf{x}, S) = \| \mathbf{x} - \mathsf{proj}_S(\mathbf{x}) \| = \| \mathbf{x} - P_S\mathbf{x} \| = \| (I - P_S)\mathbf{x} \| .$$

Theorem 6.100. *Let S and T be two non-zero subspaces of $\mathbb{C}^n$ and let*

$$\begin{aligned} \delta_S &= \max\{d_2(\boldsymbol{x}, T) \mid \boldsymbol{x} \in S, \| \boldsymbol{x} \|_2 = 1\}, \\ \delta_T &= \max\{d_2(\boldsymbol{x}, S) \mid \boldsymbol{x} \in T, \| \boldsymbol{x} \|_2 = 1\}. \end{aligned}$$

We have $d_2(S,T) = \max\{\delta_S, \delta_T\}$.

Proof. If $\mathbf{x} \in S$ and $\| \mathbf{x} \|_2 = 1$ we have

$$\begin{aligned} d_2(\mathbf{x}, T) &= \| \mathbf{x} - P_T\mathbf{x} \|_2 = \| P_S\mathbf{x} - P_T\mathbf{x} \|_2 \\ &= \| (P_S - P_T)\mathbf{x} \|_2 \leqslant |\!|\!| P_S - P_T |\!|\!|_2. \end{aligned}$$

Therefore, $\delta_S \leqslant |\!|\!| P_S - P_T |\!|\!|_2$. Similarly, $\delta_T \leqslant |\!|\!| P_S - P_T |\!|\!|_2$, so $\max\{\delta_S, \delta_T\} \leqslant d_2(S,T)$.

Note that

$$\delta_S = \max\{\| (I - P_T)\mathbf{x} \|_2 \mid \mathbf{x} \in S, \| \mathbf{x} \|_2 = 1\},$$
$$\delta_T = \max\{\| (I - P_S)\mathbf{x} \|_2 \mid \mathbf{x} \in T, \| \mathbf{x} \|_2 = 1\},$$

so, taking into account that $P_S\mathbf{x} \in S$ and $P_T\mathbf{x} \in T$ for every $\mathbf{x} \in \mathbb{C}^n$ we have

$$\| (I - P_S)P_T\mathbf{x} \|_2 \leqslant \delta_S \| P_T\mathbf{x} \|_2, \| (I - P_T)P_S\mathbf{x} \|_2 \leqslant \delta_T \| P_S\mathbf{x} \|_2 .$$

We have

$$\begin{aligned}
\| P_T(I - P_S)\mathbf{x} \|_2^2 &= (P_T(I - P_S)\mathbf{x}, P_T(I - P_S)\mathbf{x}) \\
&= ((P_T)^2(I - P_S)\mathbf{x}, (I - P_S)\mathbf{x}) \\
&= (P_T(I - P_S)\mathbf{x}, (I - P_S)\mathbf{x}) \\
&= (P_T(I - P_S)\mathbf{x}, (I - P_S)^2\mathbf{x}) \\
&= ((I - P_S)P_T(I - P_S)\mathbf{x}, (I - P_S)\mathbf{x})
\end{aligned}$$

because both P_S and $I - P_S$ are idempotent and self-adjoint. Therefore,

$$\begin{aligned}
\| P_T(I - P_S)\mathbf{x} \|_2^2 &\leqslant \| (I - P_S)P_T(I - P_S)\mathbf{x} \|_2 \| (I - P_S)\mathbf{x} \|_2 \\
&\leqslant \delta_T \| P_T(I - P_S)\mathbf{x} \|_2 \| (I - P_S)\mathbf{x} \|_2 .
\end{aligned}$$

This allows us to infer that

$$\| P_T(I - P_S)\mathbf{x} \|_2 \leqslant \delta_T \| (I - P_S)\mathbf{x} \|_2 .$$

We discuss now four fundamental subspaces associated to a matrix $A \in \mathbb{C}^{m \times n}$. The range and the null space of A, $\mathsf{Ran}(A) \subseteq \mathbb{C}^m$ and $\mathsf{NullSp}(A) \subseteq \mathbb{C}^n$ have been already discussed. We add now two new subspaces: $\mathsf{Ran}(A^{\mathsf{H}}) \subseteq \mathbb{C}^n$ and $\mathsf{NullSp}(A^{\mathsf{H}}) \subseteq \mathbb{C}^m$.

Theorem 6.101. *For every matrix $A \in \mathbb{C}^{m \times n}$ we have $(\mathsf{Ran}(A))^{\perp} = \mathsf{NullSp}(A^{\mathsf{H}})$.*

Proof. The statement follows from the equivalence of the following statements:
(i) $\mathbf{x} \in (\mathsf{Ran}(A))^{\perp}$;
(ii) $(\mathbf{x}, A\mathbf{y}) = 0$ for all $y \in \mathbb{C}^n$;
(iii) $\mathbf{x}^{\mathsf{H}}A\mathbf{y} = 0$ for all $y \in \mathbb{R}^n$;
(iv) $\mathbf{y}^{\mathsf{H}}A^{\mathsf{H}}\mathbf{x} = 0$ for all $y \in \mathbb{R}^n$;
(v) $A^{\mathsf{H}}\mathbf{x} = \mathbf{0}$;
(vi) $\mathbf{x} \in \mathsf{NullSp}(A^{\mathsf{H}})$.

Corollary 6.102. *For every matrix $A \in \mathbb{C}^{m \times n}$ we have $(\mathsf{Ran}(A^{\mathsf{H}}))^{\perp} = \mathsf{NullSp}(A)$.*

Proof. This statement follows from Theorem 6.101 by replacing A by A^{H}.

Corollary 6.103. *For every matrix* $A \in \mathbb{C}^{m\times n}$ *we have*

$$\mathbb{C}^m = \mathit{Ran}(A) \boxplus \mathit{NullSp}(A^{\mathsf{H}})$$
$$\mathbb{C}^n = \mathit{NullSp}(A) \boxplus \mathit{Ran}(A^{\mathsf{H}}).$$

Proof. By Theorem 6.39 we have $\mathbb{C}^m = \mathsf{Ran}(A) \boxplus \mathsf{Ran}(A)^{\perp}$ and $\mathbb{C}^n = \mathsf{NullSp}(A) \boxplus \mathsf{NullSp}(A)^{\perp}$. Taking into account Theorem 6.101 and Corollary 6.103 we obtain the desired equalities.

6.10 Positive Definite and Positive Semidefinite Matrices

Definition 6.104. *A matrix* $A \in \mathbb{C}^{n\times n}$ *is* positive definite *if* $\boldsymbol{x}^{\mathsf{H}}A\boldsymbol{x}$ *is a real positive number for every* $\boldsymbol{x} \in \mathbb{C}^n - \{\boldsymbol{0}\}$.

Theorem 6.105. *If* $A \in \mathbb{C}^{n\times n}$ *is positive definite, then* A *is Hermitian.*

Proof. Let $A \in \mathbb{C}^{n\times n}$ be a matrix. Since $\mathbf{x}^{\mathsf{H}}A\mathbf{x}$ is a real number it follows that it equals its conjugate, so $\mathbf{x}^{\mathsf{H}}A\mathbf{x} = \mathbf{x}^{\mathsf{H}}A^{\mathsf{H}}\mathbf{x}$ for every $\mathbf{x} \in \mathbb{C}^n$. There exists a unique pair of Hermitian matrices H_1 and H_2 such that $A = H_1 + iH_2$, which implies $A^{\mathsf{H}} = H_1^{\mathsf{H}} - iH_2^{\mathsf{H}}$. Thus, we have

$$\mathbf{x}^{\mathsf{H}}(H_1 + iH_2)\mathbf{x} = \mathbf{x}^{\mathsf{H}}(H_1^{\mathsf{H}} - iH_2^{\mathsf{H}})\mathbf{x} = \mathbf{x}^{\mathsf{H}}(H_1 - iH_2)\mathbf{x},$$

because H_1 and H_2 are Hermitian. This implies $\mathbf{x}^{\mathsf{H}}H_2\mathbf{x} = 0$ for every $\mathbf{x} \in \mathbb{C}^n$, which, in turn, implies $H_2 = O_{n,n}$. Consequently, $A = H_1$, so A is indeed Hermitian.

Definition 6.106. *A matrix* $A \in \mathbb{C}^{n\times n}$ *is* positive semidefinite *if* $\boldsymbol{x}^{\mathsf{H}}A\boldsymbol{x}$ *is a non-negative real number for every* $\boldsymbol{x} \in \mathbb{C}^n - \{\boldsymbol{0}\}$.

Positive definiteness (positive semidefiniteness) is denoted by $A \succ 0$ ($A \succeq 0$, respectively).

The definition of positive definite (semidefinite) matrix can be specialized for real matrices as follows.

Definition 6.107. *A symmetric matrix* $A \in \mathbb{R}^{n\times n}$ *is* positive definite *if* $\boldsymbol{x}'A\boldsymbol{x} > 0$ *for every* $\boldsymbol{x} \in \mathbb{R}^n - \{\boldsymbol{0}\}$.

If A *satisfies the weaker inequality* $\boldsymbol{x}'A\boldsymbol{x} \geqslant 0$ *for every* $\boldsymbol{x} \in \mathbb{R}^n - \{\boldsymbol{0}\}$*, then we say that* A *is* positive semidefinite.

$A \succ 0$ *denotes that* A *is positive definite and* $A \succeq 0$ *means that* A *is positive semidefinite.*

Note that in the case of real-valued matrices we need to require explicitely the symmetry of the matrix because, unlike the complex case, the inequality $\mathbf{x}'A\mathbf{x} > 0$ for $\mathbf{x} \in \mathbb{R}^n - \{\mathbf{0}_n\}$ does *not* imply the symmetry of A. For example, consider the matrix

$$A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix},$$

where $a, b \in \mathbb{R}$ and $a > 0$. We have

$$\mathbf{x}' A \mathbf{x} = (x_1 \; x_2) \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = a(x_1^2 + x_2^2) > 0$$

if $\mathbf{x} \neq \mathbf{0}_2$.

Example 6.108. The symmetric real matrix

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

is positive definite if and only if $a > 0$ and $b^2 - ac < 0$. Indeed, we have $\mathbf{x}' A \mathbf{x} > 0$ for every $\mathbf{x} \in \mathbb{R}^2 - \{\mathbf{0}\}$ if and only if $ax_1^2 + 2bx_1x_2 + cx_2^2 > 0$, where $\mathbf{x}' = (x_1 \;\; x_2)$; elementary algebra considerations lead to $a > 0$ and $b^2 - ac < 0$.

A positive definite matrix is non-singular. Indeed, if $A\mathbf{x} = \mathbf{0}$, where $A \in \mathbb{R}^{n \times n}$ is positive definite, then $\mathbf{x}^{\mathsf{H}} A \mathbf{x} = 0$, so $\mathbf{x} = \mathbf{0}$. By Corollary 5.91, A is non-singular.

Example 6.109. If $A \in \mathbb{C}^{m \times n}$, then the matrices $A^{\mathsf{H}} A \in \mathbb{C}^{n \times n}$ and $AA^{\mathsf{H}} \in \mathbb{C}^{m \times m}$ are positive semidefinite. For $\mathbf{x} \in \mathbb{C}^n$ we have

$$\mathbf{x}^{\mathsf{H}}(A^{\mathsf{H}} A)\mathbf{x} = (\mathbf{x}^{\mathsf{H}} A^{\mathsf{H}})(A\mathbf{x}) = (A\mathbf{x})^{\mathsf{H}}(A\mathbf{x}) = \| A\mathbf{x} \|_2^2 \geqslant 0.$$

The argument for AA^{H} is similar.

If $rank(A) = n$, then the matrix $A^{\mathsf{H}} A$ is positive definite because $\mathbf{x}^{\mathsf{H}}(A^{\mathsf{H}} A)\mathbf{x} = 0$ implies $A\mathbf{x} = \mathbf{0}$, which, in turn, implies $\mathbf{x} = \mathbf{0}$.

Theorem 6.110. *If $A \in \mathbb{C}^{n \times n}$ is a positive definite matrix, then any principal submatrix $B = A \begin{bmatrix} i_1 & \cdots & i_k \\ i_1 & \cdots & i_k \end{bmatrix}$ is a positive definite matrix.*

Proof. Let $\mathbf{x} \in \mathbb{C}^n - \{\mathbf{0}\}$ be a vector such that all components located on positions other than $i_1, \ldots, i_k$ equal 0 and let $\mathbf{y} = \mathbf{x} \begin{bmatrix} i_1 & \cdots & i_k \\ & 1 & \end{bmatrix} \in \mathbb{C}^k$ be the vector obtained from $\mathbf{x}$ by retaining only the components located on positions $i_1, \ldots, i_k$. Since $\mathbf{y}^{\mathsf{H}} B \mathbf{y} = \mathbf{x}^{\mathsf{H}} A \mathbf{x} > 0$ it follows that $B \succ 0$.

Corollary 6.111. *If $A \in \mathbb{C}^{n \times n}$ is a positive definite matrix, then any diagonal element a_{ii} is a real positive number for $1 \leqslant i \leqslant n$.*

Proof. This statement follows immediately from Theorem 6.110 by observing that every diagonal element of A is an 1×1 principal submatrix of A.

Theorem 6.112. *If $A, B \in \mathbb{C}^{n\times n}$ are two positive semidefinite matrices and a, b are two non-negative numbers, then $aA + bB \succeq 0$.*

Proof. The statement holds because $\mathbf{x}^{\mathsf{H}}(aA + bB)\mathbf{x} = a\mathbf{x}^{\mathsf{H}}A\mathbf{x} + b\mathbf{x}^{\mathsf{H}}B\mathbf{x} \geqslant 0$, due to the fact that A and B are positive semidefinite.

Theorem 6.113. *Let $A \in \mathbb{C}^{n\times n}$ be a positive definite matrix and let $S \in \mathbb{C}^{n\times m}$. The matrix $S^{\mathsf{H}}AS$ is positive semidefinite and has the same rank as S. Moreover, if $rank(S) = m$, then $S^{\mathsf{H}}AS$ is positive definite.*

Proof. Since A is positive definite, it is Hermitian and $(S^{\mathsf{H}}AS)^{\mathsf{H}} = S^{\mathsf{H}}AS$ implies that $S^{\mathsf{H}}AS$ is a Hermitian matrix.

Let $\mathbf{x} \in \mathbb{C}^m$. We have $\mathbf{x}^{\mathsf{H}}S^{\mathsf{H}}AS\mathbf{x} = (S\mathbf{x})^{\mathsf{H}}A(S\mathbf{x}) \geqslant 0$ because A is positive definite. Thus, the matrix $S^{\mathsf{H}}AS$ is positive semidefinite.

If $S\mathbf{x} = \mathbf{0}$, then $S^{\mathsf{H}}AS\mathbf{x} = \mathbf{0}$; conversely, if $S^{\mathsf{H}}AS\mathbf{x} = \mathbf{0}$, then $\mathbf{x}^{\mathsf{H}}S^{\mathsf{H}}AS\mathbf{x} = 0$, so $S\mathbf{x} = \mathbf{0}$. This allows us to conclude that $\mathsf{NullSp}(S) = \mathsf{NullSp}(S^{\mathsf{H}}AS)$. Therefore, by Equality (5.6), we have $rank(S) = rank(S^{\mathsf{H}}AS)$.

Suppose now that $rank(S) = m$ and that $\mathbf{x}^{\mathsf{H}}S^{\mathsf{H}}AS\mathbf{x} = 0$. Since A is positive definite we have $S\mathbf{x} = \mathbf{0}$ and this implies $\mathbf{x} = \mathbf{0}$, because of the assumption made relative to $rank(S)$. Consequently, $S^{\mathsf{H}}AS$ is positive definite.

Corollary 6.114. *Let $A \in \mathbb{C}^{n\times n}$ be a positive definite matrix and let $S \in \mathbb{C}^{n\times n}$. If S is non-singular, then so is $S^{\mathsf{H}}AS$.*

Proof. This is an immediate consequence of Theorem 6.113.

Theorem 6.115. *A Hermitian matrix $B \in \mathbb{C}^{n\times n}$ is positive definite if and only if the mapping $f : \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}$ given by $f(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x}^{\mathsf{H}}B\boldsymbol{y}$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^n$ defines an inner product on $\mathbb{C}^n$.*

Proof. Suppose that B defines an inner product on $\mathbb{C}^n$. Then, by Property (iii) of Definition 6.23 we have $f(\mathbf{x}, \mathbf{x}) > 0$ for $\mathbf{x} \neq \mathbf{0}$, which amounts to the positive definiteness of B.

Conversely, if B is positive definite, then f satisfies the condition from Definition 6.23. We show here only that f has the conjugate symmetry property.

We can write $\overline{f(\mathbf{y}, \mathbf{x})} = \overline{\mathbf{y}^{\mathsf{H}}B\mathbf{x}} = \mathbf{y}'\overline{B}\overline{\mathbf{x}}$, for $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$. Since B is Hermitian, $\overline{B} = \overline{B^{\mathsf{H}}} = B'$, so $\overline{f(\mathbf{y}, \mathbf{x})} = \mathbf{y}'B'\overline{\mathbf{x}}$. Observe that $\mathbf{y}'B'\overline{\mathbf{x}}$ is a number (that is, an 1×1 matrix), so $(\mathbf{y}'B'\overline{\mathbf{x}})' = \mathbf{x}^{\mathsf{H}}B\mathbf{y} = f(\mathbf{x}, \mathbf{y})$.

Corollary 6.116. *A symmetric matrix $B \in \mathbb{R}^{n\times n}$ is positive definite if and only if the mapping $f : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ given by $f(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x}'B\boldsymbol{y}$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ defines an inner product on $\mathbb{R}^n$.*

Proof. This follows immediately from Theorem 6.115.

Definition 6.117. *Let $L = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m)$ be a sequence of vectors in $\mathbb{R}^n$. The* Gram matrix of L *is the matrix $G_L = (g_{ij}) \in \mathbb{R}^{m \times m}$ defined by $g_{ij} = \boldsymbol{v}_i' \boldsymbol{v}_j$ for $1 \leqslant i, j \leqslant m$.*

Note that if $A_L = (\mathbf{v}_1 \ \cdots \ \mathbf{v}_m) \in \mathbb{R}^{n \times m}$, then $G_L = A_L' A_L$. Also, note that G_L is a symmetric matrix.

Theorem 6.118. *Let $L = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m)$ be a sequence of m vectors in $\mathbb{R}^n$, where $m \leqslant n$. If L is linearly independent, then the Gram matrix G_L is positive definite.*

Proof. Suppose that L is linearly independent. Let $\mathbf{x} \in \mathbb{R}^m$. We have $\mathbf{x}' G_L \mathbf{x} = \mathbf{x}' A_L' A_L \mathbf{x} = (A_L \mathbf{x})' A_L \mathbf{x} = \parallel A_L \mathbf{x} \parallel_2^2$. Therefore, if $\mathbf{x}' G_L \mathbf{x} = 0$, we have $A_L \mathbf{x} = \mathbf{0}$, which is equivalent to $x_1 \mathbf{v}_1 + \cdots + x_n \mathbf{v}_n = 0$. Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is linearly independent it follows that $x_1 = \cdots = x_m = 0$, so $\mathbf{x} = \mathbf{0}$. Thus, A is indeed, positive definite.

The Gram matrix of an arbitrary sequence of vectors is positive semidefinite, as the reader can easily verify.

Definition 6.119. *Let $L = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m)$ be a sequence of m vectors in $\mathbb{R}^n$, where $m \leqslant n$. The* Gramian *of L is the number* $\det(G_L)$.

Theorem 6.120. *If $L = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m)$ is a sequence of m vectors in $\mathbb{R}^n$. Then, L is linearly independent if and only if* $\det(G_L) \neq 0$.

Proof. Suppose that $\det(G_L) \neq 0$ and that L is not linearly independent. In other words, the numbers $a_1, \ldots, a_m$ exist such that at least one of them is not 0 and $a_1 \mathbf{x}_1 + \cdots + a_m \mathbf{x}_m = \mathbf{0}$. This implies the equalities

$$a_1(\mathbf{x}_1, \mathbf{x}_j) + \cdots + a_m(\mathbf{x}_m, \mathbf{x}_j) = \mathbf{0},$$

for $1 \leqslant j \leqslant m$, so the system $G_L \mathbf{a} = \mathbf{0}$ has a non-trivial solution in $a_1, \ldots, a_m$. This implies $\det(G_L) = 0$, which contradicts the initial assumption.

Conversely, suppose that L is linearly independent and $\det(G_L) = 0$. Then, the linear system

$$a_1(\mathbf{x}_1, \mathbf{x}_j) + \cdots + a_m(\mathbf{x}_m, \mathbf{x}_j) = \mathbf{0},$$

for $1 \leqslant j \leqslant m$, has a non-trivial solution in $a_1, \ldots, a_m$. If $\mathbf{w} = a_1 \mathbf{x}_1 + \cdots a_m \mathbf{x}_m$, this amounts to $(\mathbf{w}, \mathbf{x}_i) = 0$ for $1 \leqslant i \leqslant n$. This implies $(\mathbf{w}, \mathbf{w}) = \parallel \mathbf{w} \parallel_2^2 = 0$, so $\mathbf{w} = 0$, which contradicts the linear independence of L.

Theorem 6.121. (Cholesky's Decomposition Theorem) *Let $A \in \mathbb{C}^{n \times n}$ be a Hermitian positive definite matrix. There exists a unique upper triangular matrix R with real, positive diagonal elements such that $A = R^{\mathsf{H}} R$.*

Proof. The argument is by induction on $n \geqslant 1$. The base step, $n = 1$, is immediate.

Suppose that a decomposition exists for all Hermitian positive matrices of order n, and let $A \in \mathbb{C}^{(n+1)\times(n+1)}$ be a symmetric and positive definite matrix. We can write

$$A = \begin{pmatrix} a_{11} & \mathbf{a}^{\mathsf{H}} \\ \mathbf{a} & B \end{pmatrix},$$

where $B \in \mathbb{C}^{n\times n}$. By Theorem 6.110, $a_{11} > 0$ and $B \in \mathbb{C}^{n\times n}$ is a Hermitian positive definite matrix. It is easy to verify the identity:

$$A = \begin{pmatrix} \sqrt{a_{11}} & \mathbf{0} \\ \frac{1}{\sqrt{a_{11}}}\mathbf{a} & I_n \end{pmatrix} \begin{pmatrix} 1 & \mathbf{0}' \\ \mathbf{0} & B - \frac{1}{a_{11}}\mathbf{a}\mathbf{a}^{\mathsf{H}} \end{pmatrix} \begin{pmatrix} \sqrt{a_{11}} & \frac{1}{\sqrt{a_{11}}}\mathbf{a}^{\mathsf{H}} \\ 0 & I_n \end{pmatrix}. \tag{6.19}$$

Let $R_1 \in \mathbb{C}^{n\times n}$ be the upper triangular non-singular matrix

$$R_1 = \begin{pmatrix} \sqrt{a_{11}} & \frac{1}{\sqrt{a_{11}}}\mathbf{a}^{\mathsf{H}} \\ 0 & I_n \end{pmatrix}.$$

This allows us to write

$$A = R_1^{\mathsf{H}} \begin{pmatrix} 1 & \mathbf{0}' \\ \mathbf{0} & A_1 \end{pmatrix} R_1,$$

where $A_1 = B - \frac{1}{a_{11}}\mathbf{a}\mathbf{a}^{\mathsf{H}}$. Since

$$\begin{pmatrix} 1 & \mathbf{0}' \\ \mathbf{0} & A_1 \end{pmatrix} = (R_1^{-1})^{\mathsf{H}} A R_1^{-1},$$

by Theorem 6.113, the matrix

$$\begin{pmatrix} 1 & \mathbf{0}' \\ \mathbf{0} & A_1 \end{pmatrix}$$

is positive definite, which allows us to conclude that the matrix $A_1 = B - \frac{1}{a_{11}}\mathbf{a}\mathbf{a}^{\mathsf{H}} \in \mathbb{C}^{n\times n}$ is a Hermitian positive definite matrix.

By the inductive hypothesis, A_1 can be factored as

$$A_1 = P^{\mathsf{H}} P,$$

where P is an upper triangular matrix. This allows us to write

$$\begin{pmatrix} 1 & \mathbf{0}' \\ \mathbf{0} & A_1 \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{0}' \\ \mathbf{0} & P^{\mathsf{H}} \end{pmatrix} \begin{pmatrix} 1 & \mathbf{0}' \\ \mathbf{0} & P \end{pmatrix}$$

Thus,

$$A = R_1^{\mathsf{H}} \begin{pmatrix} 1 & \mathbf{0}' \\ \mathbf{0} & P^{\mathsf{H}} \end{pmatrix} \begin{pmatrix} 1 & \mathbf{0}' \\ \mathbf{0} & P \end{pmatrix} R_1$$

If R is defined as

$$R = \begin{pmatrix} 1 & \mathbf{0}' \\ \mathbf{0} & P \end{pmatrix} R_1 = \begin{pmatrix} 1 & \mathbf{0}' \\ \mathbf{0} & P \end{pmatrix} \begin{pmatrix} \sqrt{a_{11}} & \frac{1}{\sqrt{a_{11}}}\mathbf{a}^{\mathsf{H}} \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} \sqrt{a_{11}} & \frac{1}{\sqrt{a_{11}}}\mathbf{a}^{\mathsf{H}} \\ 0 & P \end{pmatrix},$$

then $A = R^{\mathsf{H}} R$ and R is clearly an upper triangular matrix.

We refer to the matrix R as the *Cholesky factor* of A.

Corollary 6.122. *If $A \in \mathbb{C}^{n\times n}$ is a Hermitian positive definite matrix, then* $\det(A) > 0$.

Proof. By Corollary 5.130, $\det(A)$ is a real number. By Theorem 6.121, $A = R^{\mathsf{H}}R$, where R is an upper triangular matrix with real, positive diagonal elements, so $\det(A) = \det(R^{\mathsf{H}})\det(R) = (\det(R))^2$. Since $\det(R)$ is the product of its diagonal elements, $\det(R)$ is a real, positive number, which implies $\det(A) > 0$.

Example 6.123. Let A be the symmetric matrix

$$A = \begin{pmatrix} 3 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}.$$

We leave to the reader to verify that this matrix is indeed positive definite starting from Definition 6.104.

By Equality (6.19), the matrix A can be written as

$$A = \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \\ \frac{2}{\sqrt{3}} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 & \frac{2}{\sqrt{3}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

because

$$A_1 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 0 \\ 2 \end{pmatrix}\begin{pmatrix} 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & \frac{2}{3} \end{pmatrix}.$$

Applying the same equality to A_1 we have

$$A_1 = \begin{pmatrix} \sqrt{2} & 0 \\ \frac{1}{\sqrt{2}} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{6} \end{pmatrix} \begin{pmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & 1 \end{pmatrix}.$$

Since the matrix $(\frac{1}{6})$ can be factored directly we have

$$\begin{aligned} A_1 &= \begin{pmatrix} \sqrt{2} & 0 \\ \frac{1}{\sqrt{2}} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{2} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{6}} \end{pmatrix}. \end{aligned}$$

In turn, this implies

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & \frac{2}{3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{6}} \end{pmatrix},$$

which produces the Cholesky final decomposition of A:

$$A = \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \\ \frac{2}{\sqrt{3}} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 & \frac{2}{\sqrt{3}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 & \frac{2}{\sqrt{3}} \\ 0 & \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{6}} \end{pmatrix}.$$

Cholesky's Decomposition Theorem can be extended to positive semidefinite matrices.

Theorem 6.124. (Cholesky's Decomposition Theorem for Positive Semidefinite Matrices) *Let $A \in \mathbb{C}^{n \times n}$ be a Hermitian positive semidefinite matrix. There exists an upper triangular matrix R with real, non-negative diagonal elements such that $A = R^{\mathsf{H}} R$.*

Proof. The argument is similar to the one used for Theorem 6.121 and is omitted.

Observe that for positive semidefinite matrices, the diagonal elements of R are non-negative numbers and the uniqueness of R does not longer hold.

Example 6.125. Let $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. Since $\mathbf{x}' A \mathbf{x} = (x_1 - x_2)$ it is clear that A is a positive semidefinite but not a positive definite matrix. Let R be a matrix of the form

$$R = \begin{pmatrix} r_1 & r \\ 0 & r_2 \end{pmatrix}$$

such that $A = R'R$. It is easy to see that the last equality is equivalent to $r_1^2 = r_2^2 = 1$ and $rr_1 = -1$. Thus, we have for distinct Cholesky factors: matrices

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}.$$

Theorem 6.126. *A Hermitian matrix $A \in \mathbb{C}^{n \times n}$ is positive definite if and only if all its leading principal minors are positive.*

Proof. By Theorem 6.110, if A is positive definite, then every principal submatrix is positive definite, so by Corollary 6.122, each principal minor of A is positive.

Conversely, suppose that $A \in \mathbb{C}^{n \times n}$ is an Hermitian matrix having positive leading principal minors. We prove by induction on n that A is positive definite.

The base case, $n = 1$ is immediate. Suppose that the statement holds for matrices in $\mathbb{C}^{(n-1)\times(n-1)}$. Note that A can be written as

$$A = \begin{pmatrix} B & \mathbf{b} \\ \mathbf{b}^{\mathsf{H}} & a \end{pmatrix},$$

where $B \in \mathbb{C}^{(n-1)\times(n-1)}$ is a Hermitian matrix. Since the leading minors of B are the first $n-1$ leading minors of A it follows, by the inductive hypothesis, that B is positive definite. Thus, there exists a Cholesky decomposition $B = R^{\mathsf{H}}R$, where R is an upper triangular matrix with real, positive diagonal elements. Since R is invertible, let $\mathbf{w} = (R^{\mathsf{H}})^{-1}b$.

The matrix B is invertible. By Theorem 5.154, we have $\det(A) = \det(B)(a - \mathbf{b}^{\mathsf{H}}B^{-1}\mathbf{b}) > 0$. Since $\det(B) > 0$ it follows that $a \geqslant \mathbf{b}^{\mathsf{H}}B^{-1}\mathbf{b}$. We observed that if B is positive definite, then so is B^{-1}. Therefore, $a \geqslant 0$ is and we can write $a = c^2$ for some positive c. This allows us to write

$$A = \begin{pmatrix} R^{\mathsf{H}} & \mathbf{0} \\ \mathbf{w}^{sH} & c \end{pmatrix} \begin{pmatrix} R & \mathbf{w} \\ \mathbf{0}^{\mathsf{H}} & c \end{pmatrix} = C^{\mathsf{H}}C,$$

where C is the upper triangular matrix with positive e

$$C = \begin{pmatrix} R & \mathbf{w} \\ \mathbf{0}^{\mathsf{H}} & c \end{pmatrix}$$

This implies immediately the positive definiteness of A.

Let $A, B \in \mathbb{C}^{n\times n}$. We write $A \succ B$ if $A - B \succ 0$, that is, if $A - B$ is a positive definite matrix. Similarly, we write $A \succeq B$ if $A - B \succeq O$, that is, if $A - B$ is positive semidefinite.

Theorem 6.127. *Let $A_0, A_1, \ldots, A_m$ be $m+1$ matrices in $\mathbb{C}^{n\times n}$ such that A_0 is positive definite and all matrices are Hermitian. There exists $a > 0$ such that for any $t \in [-a, a]$ the matrix $B_m(t) = A_0 + A_1 t + \cdots + A_m t^m$ is positive definite.*

Proof. Since all matrices $A_0, \ldots, A_m$ are Hermitian, note that $\mathbf{x}^{\mathsf{H}}A_i\mathbf{x}$ are real numbers for $0 \leqslant i \leqslant m$. Therefore, $p_m(t) = \mathbf{x}^{\mathsf{H}}B_m(t)\mathbf{x}$ is a polynomial in t with real coefficients and $p_m(0) = \mathbf{x}^{\mathsf{H}}A_0\mathbf{x}$ is a positive number if $\mathbf{x} \neq \mathbf{0}$. Since p_m is a continuous function there exists an interval $[-a, a]$ such that $t \in [-a, a]$ implies $p_m(t) > 0$ if $\mathbf{x} \neq \mathbf{0}$. This shows that $B_m(t)$ is positive definite.

6.11 The Gram-Schmidt Orthogonalization Algorithm

The Gram-Schmidt algorithm starts with a basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ of an m-dimensional space U of $\mathbb{C}^n$ and generates an orthonormal basis $\{\mathbf{q}_1, \ldots, \mathbf{q}_m\}$ of the same subspace. Clearly, we have $m \leqslant n$.

The algorithm starts with the sequence of vectors $(\mathbf{u}_1, \ldots, \mathbf{u}_m)$ and constructs the sequence of orthonormal vectors $(\mathbf{q}_1, \ldots, \mathbf{q}_m)$ such that

$$\langle \mathbf{u}_1, \ldots, \mathbf{u}_k \rangle = \langle \mathbf{q}_1, \ldots, \mathbf{q}_k \rangle$$

for $1 \leqslant k \leqslant m$ as follows:

$$\begin{aligned}
\mathbf{z}_1 &= \mathbf{u}_1, & \mathbf{q}_1 &= \tfrac{1}{\|\mathbf{z}_1\|}\mathbf{z}_1,\\
\mathbf{z}_2 &= \mathbf{u}_2 - \mathsf{proj}_{\mathbf{z}_1}(\mathbf{u}_2), & \mathbf{q}_2 &= \tfrac{1}{\|\mathbf{z}_2\|}\mathbf{z}_2,\\
\mathbf{z}_3 &= \mathbf{u}_3 - \mathsf{proj}_{\mathbf{z}_1}(\mathbf{u}_3) - \mathsf{proj}_{\mathbf{z}_2}(\mathbf{u}_3), & \mathbf{q}_3 &= \tfrac{1}{\|\mathbf{z}_3\|}\mathbf{z}_3,\\
&\vdots & &\vdots\\
\mathbf{z}_m &= \mathbf{u}_m - \mathsf{proj}_{\mathbf{z}_1}(\mathbf{u}_m) - \cdots - \mathsf{proj}_{\mathbf{z}_{m-1}}(\mathbf{u}_m), & \mathbf{q}_m &= \tfrac{1}{\|\mathbf{z}_m\|}\mathbf{z}_m.
\end{aligned}$$

The algorithm can be written in pseudo-code as follows.

Algorithm 6.11.1: Gram-Schmidt Orthogonalization Algorithm

Data: A basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ for a subspace U of $\mathbb{C}^n$
Result: An orthonormal basis $\{\mathbf{q}_1, \ldots, \mathbf{q}_m\}$ for U

$\mathbf{z}_1 = \mathbf{u}_1$; $\mathbf{q}_1 = \frac{1}{\|\mathbf{z}_1\|}\mathbf{z}_1$;
for $k = 2$ *to* m **do**
  $\mathbf{z}_k = \mathbf{u}_k - \sum_{j=1}^{k-1} \mathsf{proj}_{\mathbf{z}_j}(\mathbf{u}_k)$;
  $\mathbf{q}_k = \frac{1}{\|\mathbf{z}_k\|}\mathbf{z}_k$;
end
return $Q = (\mathbf{q}_1 \ \cdots \ \mathbf{q}_m)$;

Lemma 6.128. *The sequence $(\boldsymbol{z}_1, \ldots, \boldsymbol{z}_m)$ constructed by the Gram-Schmidt algorithm consists of pairwise orthogonal vectors; furthermore, the sequence $(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_m)$ consists of pairwise orthonormal vectors.*

Proof. Note that

$$\mathbf{z}_k = \mathbf{u}_k - \sum_{j=1}^{k-1} \mathsf{proj}_{\langle \mathbf{z}_1, \ldots, \mathbf{z}_{k-1} \rangle}(\mathbf{u}_k)$$

This implies that $\mathbf{z}_k$ is orthogonal on the subspace $\langle \mathbf{z}_1, \ldots, \mathbf{z}_{k-1} \rangle$, that is, on all its predecessors in the sequence. The second part of the lemma follows immediately.

Lemma 6.129. *We have $\langle \boldsymbol{q}_1, \ldots, \boldsymbol{q}_k \rangle = \langle \boldsymbol{u}_1, \ldots, \boldsymbol{u}_k \rangle$ for $1 \leqslant k \leqslant m$.*

Proof. The proof is by induction on k. The base step, $k = 1$, is immediate.

Suppose that the equality holds for k. Since $\mathbf{q}_j$ and $\mathbf{z}_j$ determine the same one-dimensional subspace, we have $\mathsf{proj}_{\mathbf{z}_j}(\mathbf{u}_k) = \mathsf{proj}_{\mathbf{q}_j}(\mathbf{u}_k)$, Thus,

$$\begin{aligned}\mathbf{q}_{k+1} &= \frac{1}{\parallel \mathbf{z}_{k+1} \parallel}\left(u_{k+1} - \sum_{j=1}^{k} \mathsf{proj}_{\mathbf{q}_j}(\mathbf{u}_{k+1})\right)\\ &= \frac{1}{\parallel \mathbf{z}_{k+1} \parallel}\left(u_{k+1} - \mathsf{proj}_{\langle \mathbf{q}_1,\ldots,\mathbf{q}_k\rangle}(\mathbf{u}_{k+1})\right).\end{aligned}$$

By the inductive hypothesis,

$$\mathbf{q}_{k+1} = \frac{1}{\parallel \mathbf{z}_{k+1} \parallel}\left(u_{k+1} - \mathsf{proj}_{\langle \mathbf{u}_1,\ldots,\mathbf{u}_k\rangle}(\mathbf{u}_{k+1})\right),$$

which implies $\mathbf{q}_{k+1} \in \langle \mathbf{u}_1, \ldots, \mathbf{u}_{k+1}\rangle$. Thus, $\langle \mathbf{q}_1, \ldots, \mathbf{q}_{k+1}\rangle \subseteq \langle \mathbf{u}_1, \ldots, \mathbf{u}_{k+1}\rangle$.

For the reverse inclusion, note that

$$\begin{aligned}\mathbf{u}_{k+1} &= \mathbf{z}_{k+1} + \sum_{j=1}^{k} \mathsf{proj}_{\mathbf{z}_j}(\mathbf{u}_{k+1})\\ &= \mathbf{z}_{k+1} + \mathsf{proj}_{\langle \mathbf{z}_1,\ldots,\mathbf{z}_k\rangle}(\mathbf{u}_{k+1})\\ &= \parallel \mathbf{z}_{k+1} \parallel \mathbf{q}_{k+1} + \mathsf{proj}_{\langle \mathbf{q}_1,\ldots,\mathbf{q}_k\rangle}(\mathbf{u}_{k+1}),\end{aligned}$$

which implies that $\mathbf{u}_{k+1} \in \langle \mathbf{q}_1, \ldots, \mathbf{q}_{k+1}\rangle$. Therefore,

$$\langle \mathbf{u}_1, \ldots, \mathbf{u}_{k+1}\rangle \subseteq \langle \mathbf{q}_1, \ldots, \mathbf{q}_{k+1}\rangle,$$

which concludes the argument.

Theorem 6.130. *Let $(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_m)$ be the sequence of vectors constructed by the Gram-Schmidt algorithm starting from the basis $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m\}$ of an m-dimensional subspace U of $\mathbb{C}^n$. The set $\{\boldsymbol{q}_1, \ldots, \boldsymbol{q}_m\}$ is an orthogonal basis of U and $\langle \boldsymbol{q}_1, \ldots, \boldsymbol{q}_k\rangle = \langle \boldsymbol{u}_1, \ldots \boldsymbol{u}_k\rangle$ for $1 \leqslant k \leqslant m$.*

Proof. This statement follows immediately from Lemmas 6.128 and 6.129.

Example 6.131. Let $A \in \mathbb{R}^{3\times 2}$ be the matrix

$$A = \begin{pmatrix} 1 & 1\\ 0 & 0\\ 1 & 3\end{pmatrix}.$$

It is easy to see that $rank(A) = 2$. We have $\{\mathbf{u}_1, \mathbf{u}_2\} \subseteq \mathbb{R}^3$ and we construct an orthogonal basis for the subspace generated by these columns. The matrix W is intialized to $O_{3,2}$.

By Algorithm 6.11.1, we begin by defining $\mathbf{z}_1 = \mathbf{u}_1$ and

$$\mathbf{q}_1 = \frac{1}{\parallel \mathbf{z}_1 \parallel}\mathbf{z}_1 = \begin{pmatrix} \frac{\sqrt{2}}{2}\\ 0\\ \frac{\sqrt{2}}{2}\end{pmatrix}$$

because $\| \mathbf{z}_1 \| = \sqrt{2}$. Next, we have

$$\begin{aligned}\mathbf{z}_2 &= \mathbf{u}_2 - \mathsf{proj}_{\mathbf{z}_1}(\mathbf{u}_2) = \mathbf{u}_2 - 2\mathbf{z}_1 \\ &= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix},\end{aligned}$$

which implies

$$\mathbf{q}_2 = \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{pmatrix}.$$

Thus, the orthonormal basis we are seeking consists of the vectors

$$\begin{pmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{pmatrix} \text{ and } \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{pmatrix}.$$

Using the Gram-Schmidt algorithm we can factor a matrix as a product of a matrix having orthogonal columns and an upper triangular matrix. This useful matrix decomposition (described in the next theorem) is known as the *QR-decomposition*.

Theorem 6.132. (Reduced QR Decomposition) *Let $U \in \mathbb{C}^{n \times m}$ be a full-rank matrix, where $m \leqslant n$. There exists a matrix $Q \in \mathbb{C}^{n \times m}$ having a set of orthonormal columns and an upper triangular matrix $R \in \mathbb{C}^{m \times m}$ such that $U = QR$. Furthermore, the diagonal entries of R are non-zero.*

Proof. Let $U = (\mathbf{u}_1 \ \cdots \ \mathbf{u}_m)$. We use the same notations as above. Since $\mathbf{u}_k \in \langle \mathbf{q}_1, \ldots, \mathbf{q}_k \rangle$ for $1 \leqslant k \leqslant m$, it follows that we can write the equalities

$$\begin{aligned}\mathbf{u}_1 &= \mathbf{q}_1 r_{11}, \\ \mathbf{u}_2 &= \mathbf{q}_1 r_{12} + \mathbf{q}_2 r_{22}, \\ \mathbf{u}_3 &= \mathbf{q}_1 r_{13} + \mathbf{q}_2 r_{23} + \mathbf{q}_3 r_{33}, \\ &\vdots \\ \mathbf{u}_m &= \mathbf{q}_1 r_{1m} + \mathbf{q}_2 r_{2m} + \mathbf{q}_3 r_{3m} + \cdots + \mathbf{q}_m r_{mm}.\end{aligned}$$

In matrix form these equalities are

$$(\mathbf{u}_1, \ldots, \mathbf{u}_m) = (\mathbf{q}_1, \ldots, \mathbf{q}_m) \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1m} \\ 0 & r_{22} & \cdots & r_{2m} \\ 0 & 0 & \cdots & r_{3m} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & r_{mm} \end{pmatrix}.$$

Thus, we have $Q = (\mathbf{q}_1 \ \cdots \ \mathbf{q}_m)$ and

$$R = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1m} \\ 0 & r_{22} & \cdots & r_{2m} \\ 0 & 0 & \cdots & r_{3m} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & r_{mm} \end{pmatrix}.$$

Note that $r_{kk} \neq 0$ for $1 \leqslant k \leqslant m$. Indeed, if we were to have $r_{kk} = 0$ this would imply $\mathbf{u}_k = \mathbf{q}_1 r_{1k} + \mathbf{q}_2 r_{2k} + \mathbf{q}_3 r_{3k} + \cdots + \mathbf{q}_{k-1} r_{k-1\ k-1}$, which would contradict the equality $\langle \mathbf{q}_1, \ldots, \mathbf{q}_k \rangle = \langle \mathbf{u}_1, \ldots, \mathbf{u}_k \rangle$.

Theorem 6.133. (Full QR Decomposition) *Let $U \in \mathbb{C}^{n \times m}$ be a full-rank matrix, where $m \leqslant n$. There exists a unitary matrix $Q \in \mathbb{C}^{m \times m}$ and an upper triangular matrix $R \in \mathbb{C}^{n \times m}$ such that $U = QR$.*

Proof. Let $U = Q_1 R_1$ be the reduced QR decomposition of U, where $Q_1 = (\mathbf{q}_1 \ \cdots \ \mathbf{q}_m)$ is a matrix having an orthonormal set of columns. This set can be extended to an orthonormal basis $\{\mathbf{q}_1, \ldots, \mathbf{q}_m, \mathbf{q}_{m+1}, \ldots, \mathbf{q}_n\}$. of $\mathbb{C}^n$. If $Q = (\mathbf{q}_1; \cdots \ \mathbf{q}_m \ \mathbf{q}_{m+1} \ \cdots \ \mathbf{q}_n)$ and R is the matrix obtained from R by adding $n - m$ rows equal to $\mathbf{0}$, then $U = QR$ is the desired full decomposition of U.

Corollary 6.134. *Let $U \in \mathbb{C}^{n \times n}$ be a non-singular square matrix. There exists a unitary matrix $Q \in \mathbb{C}^{n \times n}$ and an upper triangular matrix $R \in \mathbb{C}^{n \times n}$ such that $U = QR$.*

Proof. The corollary is a direct consequence of Theorem 6.133.

Theorem 6.135. *If $L = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m)$ is a sequence of m vectors in $\mathbb{R}^n$. We have*

$$\det(G_L) \leqslant \prod_{j=1}^{m} \parallel \boldsymbol{v}_j \parallel_2^2 .$$

The equality takes place only if the vectors of L are pairwise orthogonal.

Proof. Suppose that L is linearly independent and construct the orthonormal set $\{\mathbf{y}_1, \ldots, \mathbf{y}_m\}$, where $\mathbf{y}_j = b_{j1}\mathbf{v}_1 + \cdots + b_{jj}\mathbf{v}_j$ for $1 \leqslant j \leqslant m$, using the Gram-Schmidt algorithm. Since $b_{jj} \neq 0$ it follows that we can write

$$\mathbf{v}_j = c_{j1}\mathbf{y}_1 + \cdots + c_{jj}\mathbf{y}_j$$

for $1 \leqslant j \leqslant m$ so that $(\mathbf{v}_j, \mathbf{y}_p) = 0$ if $j < p$ and $(\mathbf{v}_j, \mathbf{y}_p) = c_{jp}$ if $p \leqslant j$. Thus, we have

$$(\mathbf{v}_1, \ldots, \mathbf{v}_m) = (\mathbf{y}_1, \ldots, \mathbf{y}_m) \begin{pmatrix} (\mathbf{v}_1, \mathbf{y}_1) & (\mathbf{v}_2, \mathbf{y}_1) & \cdots & (\mathbf{v}_m, \mathbf{y}_1) \\ 0 & (\mathbf{v}_2, \mathbf{y}_2) & \cdots & (\mathbf{v}_m, \mathbf{y}_2) \\ 0 & 0 & \cdots & (\mathbf{v}_m, \mathbf{y}_3) \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & (\mathbf{v}_m, \mathbf{y}_m) \end{pmatrix}.$$

This implies

$$\begin{pmatrix} (\mathbf{v}_1, \mathbf{v}_1) & \cdots & (\mathbf{v}_1, \mathbf{v}_m) \\ \vdots & \ldots & \vdots \\ (\mathbf{v}_m, \mathbf{v}_1) & \cdots & (\mathbf{v}_m, \mathbf{v}_m) \end{pmatrix} = \begin{pmatrix} \mathbf{v}_1' \\ \vdots \\ \mathbf{v}_m' \end{pmatrix} (\mathbf{v}_1, \ldots, \mathbf{v}_m)$$

$$= \begin{pmatrix} (\mathbf{v}_1, \mathbf{y}_1) & 0 & 0 \\ (\mathbf{v}_2, \mathbf{y}_1) & (\mathbf{v}_2, \mathbf{y}_2) & 0 \\ \vdots & \vdots & \vdots \\ (\mathbf{v}_m, \mathbf{y}_1) & (\mathbf{v}_m, \mathbf{y}_2) & (\mathbf{v}_m, \mathbf{y}_m) \end{pmatrix} \begin{pmatrix} (\mathbf{v}_1, \mathbf{y}_1) & (\mathbf{v}_2, \mathbf{y}_1) & \cdots & (\mathbf{v}_m, \mathbf{y}_1) \\ 0 & (\mathbf{v}_2, \mathbf{y}_2) & \cdots & (\mathbf{v}_m, \mathbf{y}_2) \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & (\mathbf{v}_m, \mathbf{y}_m) \end{pmatrix}.$$

Therefore, we have

$$\det(G_L) = \prod_{i=1}^{m} (\mathbf{v}_i, \mathbf{y}_i)^2 \leqslant \prod_{i=1}^{m} (\mathbf{v}_i, \mathbf{v}_i)^2,$$

because $(\mathbf{v}_i, \mathbf{y}_i)^2 \leqslant (\mathbf{v}_i, \mathbf{v}_i)^2 (\mathbf{y}_i, \mathbf{y}_i)^2$ and $(\mathbf{y}_i, \mathbf{y}_i) = 1$ for $1 \leqslant i \leqslant m$.

To have $\det(G_L) = \prod_{i=1}^{m} (\mathbf{v}_i, \mathbf{v}_i)^2$ we must have $\mathbf{v}_i = k_i \mathbf{y}_i$, that is, the vectors $\mathbf{v}_i$ must be pairwise orthogonal.

Definition 6.136. *A* Hadamard matrix *is a matrix $H \in \mathbb{R}^{nn}$ such that $h_{ij} \in \{1, -1\}$ for $1 \leqslant i, j \leqslant n$ and $HH' = nI_n$.*

Example 6.137. The matrices

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

are Hadamard matrices in $\mathbb{R}^{2\times 2}$ and $\mathbb{R}^{4\times 4}$, respectively.

Corollary 6.138. *Let $A \in \mathbb{R}^{n\times n}$ be a matrix such that $|a_{ij}| \leqslant 1$ for $1 \leqslant i, j \leqslant n$. Then, $|\det(A)| \leqslant n^{\frac{n}{2}}$ and the equality holds only if A is a Hadamard matrix.*

Proof. Let $\mathbf{a}_i = (a_{i1}, \ldots, a_{in})$ be the i^{th} row of A. We have $\| \mathbf{a}_i \|_2 \leqslant \sqrt{n}$, so $|(\mathbf{a}_i, \mathbf{a}_j)| \leqslant n$ by Cauchy-Schwartz inequality, for $1 \leqslant i, j \leqslant n$.

Note that $G_L = A'A$, where L is the set of rows of A. Consequently, $\det(A)^2 = \det(G_L) \leqslant \prod_{j=1}^{n} \| \mathbf{v}_j \|_2^2$, so

$$|\det(A)| \leqslant \prod_{j=1}^{n} \parallel \mathbf{v}_j \parallel_2 \leqslant n^{\frac{n}{2}}.$$

To have the equality we must have $\parallel \mathbf{v}_j \parallel = \sqrt{n}$. This is possible only if $v_{jk} \in \{-1, 1\}$. This fact together with orthogonality of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ implies that A is a Hadamard matrix.

We saw that orthonormal sets of vectors are linearly independent. This allows us to extend orthonormal set of vectors to orthonormal bases.

Theorem 6.139. *Let L be a finite-dimensional linear space. If U is an orthonormal set of vectors, then there exists a basis T of L that consists of orthonormal vectors such that $U \subseteq T$.*

Proof. Let $U = \{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ be an orthonormal set of vectors in L. There is an extension of U, $Z = \{\mathbf{u}_1, \ldots, \mathbf{u}_m, \mathbf{u}_{m+1}, \ldots, \mathbf{u}_n\}$ to a basis of L, where $n = \dim(V)$, by Theorem 5.17. Now, apply the Gram-Schmidt algorithm to the set U to produce an orthonormal basis $W = \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ for the entire space L. It is easy to see that $\mathbf{w}_i = \mathbf{u}_i$ for $1 \leqslant i \leqslant m$, so $U \subseteq W$ and W is the orthonormal basis of L that extends the set U.

Corollary 6.140. *If A is an $(m \times n)$-matrix with $m \geqslant n$ having orthonormal set of columns, then there exists an $(m \times (m-n))$-matrix B such that $(A\ B)$ is an orthogonal (unitary) square matrix.*

Proof. This follows directly from Theorem 6.139.

Corollary 6.141. *Let U be a subspace of an n-dimensional linear space L such that $\dim(U) = m$, where $m < n$. Then $\dim(U^\perp) = n - m$.*

Proof. Let $\mathbf{u}_1, \ldots, \mathbf{u}_m$ be an orthonormal basis of U, and let

$$\mathbf{u}_1, \ldots, \mathbf{u}_m, \mathbf{u}_{m+1}, \ldots, \mathbf{u}_n$$

be its completion to an orthonormal basis for L, which exists by Theorem 6.139. Then, $\mathbf{u}_{m+1}, \ldots, \mathbf{u}_n$ is a basis of the orthogonal complement $U^\perp$, so $\dim(U^\perp) = n - m$.

Theorem 6.142. *A subspace U of $\mathbb{R}^n$ is m-dimensional if and only if is the set of solution of an homogeneous linear system $A\boldsymbol{x} = \boldsymbol{0}$, where $A \in \mathbb{R}^{(n-m)\times n}$ is a full-rank matrix.*

Proof. Suppose that U is an m-dimensional subspace of $\mathbb{R}^n$. If $\mathbf{v}_1, \ldots, \mathbf{v}_{n-m}$ is a basis of the orthogonal complement of U, then $\mathbf{v}_i'\mathbf{x} = 0$ for every $\mathbf{x} \in U$ and $1 \leqslant i \leqslant n - m$. These conditions are equivalent to the equality

$$(\mathbf{v}_1'\ \mathbf{v}_2'\ \cdots\ \mathbf{v}_{n-m}')\mathbf{x} = \mathbf{0},$$

which shows that U is the set of solution of an homogeneous linear $A\mathbf{x} = \mathbf{0}$, where $A = (\mathbf{v}'_1 \; \mathbf{v}'_2 \; \cdots \; \mathbf{v}'_{n-m})$.

Conversely, if $A \in \mathbb{R}^{(n-m)\times n}$ is a full-rank matrix, then the set of solutions of the homogeneous system $A\mathbf{x} = \mathbf{b}$ is the null subspace of A and, therefore is an m-dimensional subspace.

Exercises and Supplements

1. Let ν be a norm on $\mathbb{C}^n$. Prove that there exists a number $k \in \mathbb{R}$ such that for any vector $\mathbf{x} \in \mathbb{C}^n$ we have $\nu(\mathbf{x}) \leqslant k \sum_{i=1}^{n} |x_i|$.
2. Prove that $\nu(\mathbf{x}+\mathbf{y})^2 + \nu(\mathbf{x}-\mathbf{y})^2 \leqslant 4(\nu(\mathbf{x})^2 + \nu(\mathbf{y})^2)$ for every vector norm ν on $\mathbb{R}^n$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.
3. Let $\mathbf{a} \in \mathbb{R}^n$ be a vector such that $\mathbf{a} \geqslant \mathbf{0}_n$. Prove that
$$\left(\sum_{i=1}^{n} a_i\right)^2 \leqslant n \sum_{i=1}^{n} a_i^2.$$
4. Let (S, d) be a dissimilarity space, where d is a definite dissimilarity. Define the set
$$P(x,y,z) = \{p \in \mathbb{R}_{\geqslant 0} \mid d(x,y)^p \leqslant d(x,z)^p + d(z,y)^p\}$$
for $x, y, z \in S$. Prove that
   a) $P(x,y,z) \neq \emptyset$ for $x, y, z \in S$;
   b) if $p \in P(x,y,z)$ and $q \leqslant p$, then $q \in P(x,y,z)$;
   c) if $\sup P(x,y,z) \geqslant 1$ for all $x, y, z \in S$, then d is a metric on S;
   d) if $\sup P(x,y,z) = \infty$ for all $x, y, z \in S$, then d is an ultrametric on S.
5. Let $u, v \in \mathbb{C}$. Prove that:
   a) $|\bar{u}v - u\bar{v}|^2 = 2|u|^2|v|^2 - \bar{u}^2v^2 - u^2\bar{v}^2$;
   b) if $\mathbf{x} \in \mathbb{C}$, $\parallel \mathbf{x} \parallel = 1$, and $s = \sum_{i=1}^{n} x_i^2$, we have $\sum_{i=1}^{n} \sum_{j=1}^{n} |\bar{x}_i x_j - x_j \bar{x}_i|^2 \leqslant 2 - 2|s|^2$.
6. Let $\{\mathbf{q}_1, \ldots, \mathbf{q}_n\} \subseteq \mathbb{C}^n$ be an orthonormal set of n vectors. If $\{I, J\}$ is a partition of $\{1, \ldots, n\}$, $I = \{i_1, \ldots, i_p\}$ and $J = \{j_1, \ldots, j_q\}$, prove that the subspaces $S = \langle \mathbf{q}_{i_1}, \ldots, \mathbf{q}_{i_p} \rangle$ and $T = \langle \mathbf{q}_{j_1}, \ldots, \mathbf{q}_{j_q} \rangle$ are complementary.
7. Let $Q = (\mathbf{q}_1 \; \cdots \; \mathbf{q}_k) \in \mathbb{C}^{n\times k}$ be a matrix having a set of orthonormal columns. Prove that $I_n - QQ^{\mathsf{H}} = \prod_{j=1}^{k}(I_n - \mathbf{q}_j\mathbf{q}_j^{\mathsf{H}})$.

   Solution: The equality to be shown amounts to $I_n - \sum_{j=1}^{k} \mathbf{q}_j\mathbf{q}_j^{\mathsf{H}} = \prod_{j=1}^{k}(I_n - \mathbf{q}_j\mathbf{q}_j^{\mathsf{H}})$, and the argument is by induction on k. The base case, $k = 1$ is immediate. Suppose that the equality holds for k and let $\{\mathbf{q}_1, \ldots, \mathbf{q}_k, \mathbf{q}_{k+1}\}$ be a set of othonormal vectors. We have
$$\begin{aligned}\prod_{j=1}^{k+1}(I_n - \mathbf{q}_j\mathbf{q}_j^{\mathsf{H}}) &= \prod_{j=1}^{k}(I_n - \mathbf{q}_j\mathbf{q}_j^{\mathsf{H}})(I_n - \mathbf{q}_{k+1}\mathbf{q}_{k+1}^{\mathsf{H}}) \\ &= \left(I_n - \sum_{j=1}^{k}\mathbf{q}_j\mathbf{q}_j^{\mathsf{H}}\right)(I_n - \mathbf{q}_{k+1}\mathbf{q}_{k+1}^{\mathsf{H}}) \\ &\quad \text{(by the inductive hypothesis)} \\ &= I_n - \sum_{j=1}^{k}\mathbf{q}_j\mathbf{q}_j^{\mathsf{H}} - \mathbf{q}_{k+1}\mathbf{q}_{k+1}^{\mathsf{H}} + \left(\sum_{j=1}^{k}\mathbf{q}_j\mathbf{q}_j^{\mathsf{H}}\right)\mathbf{q}_{k+1}\mathbf{q}_{k+1}^{\mathsf{H}}.\end{aligned}$$

Since $\mathbf{q}_j^{\mathsf{H}}\mathbf{q}_{k+1} = 0$ for $1 \leqslant j \leqslant k$, the inductive step is concluded.

8. Let $X = \{x_1, \ldots, x_m\}$ be a set and let S be a subset of X. The characteristic vector of S is $\mathbf{c}_S \in \{0,1\}^m$ whose components $c_1, \ldots, c_m$ are defined by $c_i = 1$ if $x_i \in S$ and $c_i = 0$, otherwise. Prove that
   a) $\| \mathbf{c}_S \|_2 = |S|$;
   b) if $S, T \subseteq X$, then $\mathbf{c}_S' \mathbf{c}_T = |S \cap T|$;
   c) if $S \subseteq X$, then
   $$\sum\{(c_i - c_j)^2 \mid 1 \leqslant i < j \leqslant m\} = |S| \cdot |X - S|.$$
   Solution: We solve only the third part. Without loss of generality assume that $S = \{x_1, \ldots, x_p\}$. Then, $c_i = 1$ for $1 \leqslant i \leqslant p$ and $c_i = 0$ for $p+1 \leqslant i \leqslant m$. The contribution of the terms of the form $(c_i - c_j)^2$, where $1 \leqslant i \leqslant p$ equals $m - p$ and there are p such terms. Thus, $\sum\{(c_i - c_j)^2 \mid 1 \leqslant i < j \leqslant m\} = p(m-p) = |S| \cdot |X - S|$.
9. Let $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$. Prove that $\mathit{trace}(\mathbf{x}\mathbf{y}^{\mathsf{H}}) = \overline{\mathbf{x}^{\mathsf{H}}\mathbf{y}}$.
10. Let $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ and let M, P be the matrices $M = \mathbf{x}\mathbf{x}^{\mathsf{H}}$ and $P = \mathbf{y}\mathbf{y}^{\mathsf{H}}$. Prove that for the inner product on $\mathbb{C}^{n \times n}$, defined in Example 6.26, we have $(M, P) = |\mathbf{x}^{\mathsf{H}}\mathbf{y}|^2$.
11. Let $\mathbf{x} \in \mathbb{R}^n$. Prove that for every $\epsilon > 0$ there exists $\mathbf{y} \in \mathbb{R}^n$ such that the components of the vector $\mathbf{x} + \mathbf{y}$ are distinct and $\| \mathbf{y} \|_2 < \epsilon$.
    Solution: Partition the set $\{1, \ldots, n\}$ into the blocks $B_1, \ldots, B_k$ such that all components of $\mathbf{x}$ that have an index in B_j have a common value c_j. Suppose that $|B_j| = p_j$. Then, $\sum_{j=1}^k p_j = n$ and the numbers $\{c_1, c_2, \ldots, c_k\}$ are pairwise distinct. Let $d = \min_{i,j} |c_i - c_j|$. The vector $\mathbf{y}$ can be defined as follows. If $B_j = \{i_1, \ldots, i_{p_j}\}$, then
    $$y_{i_1} = \eta \cdot 2^{-1}, y_{i_2} = \eta \cdot 2^{-2}, \ldots, y_{i_{p_j}} = \eta \cdot 2^{-p},$$
    where $\eta > 0$, which makes the numbers $c_j + y_{i_1}, c_j + y_{i_2}, \ldots, c_j + y_{i_{p_j}}$ pairwise distinct. It suffices to take $\eta < d$ to ensure that the components of $\mathbf{x} + \mathbf{y}$ are pairwise distinct. Also, note that $\| \mathbf{y} \|_2^2 \leqslant \sum_{j=1}^k p_j \frac{\eta^2}{4} = \frac{n\eta^2}{4}$. It suffices to choose η such that $\eta < \min\{d, \frac{2\epsilon}{n}\}$ to ensure that $\| \mathbf{y} \|_2 < \epsilon$.
12. Prove that the norms ν_1 and ν_∞ on $\mathbb{R}^n$ are not generated by an inner product.
13. Prove that if $0 < p < 1$, ν_p is not a norm on $\mathbb{R}^n$.
14. The number $\zeta(\mathbf{x})$, defined as the number of non-zero components of the vector $\mathbf{x} \in \mathbb{R}^n$, is refered to as the *zero-norm* of $\mathbf{x}$ (even though it is not a norm in the sense of Definition 6.9) and is used in the study of linear models in machine learning (see [192]). Prove that $\zeta(\mathbf{x}) = \lim_{p \to 0}(\nu_p(\mathbf{x}))^p$ and that $\lim_{p \to 0} n^{-\frac{1}{p}} \nu_p(\mathbf{x})$ equals the geometric mean of the absolute values of the components of $\mathbf{x}$.
15. Let ν be a norm on $\mathbb{R}^n$ that satisfies the parallelogram equality. Prove that the function $p : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ given by
    $$p(\mathbf{x}, \mathbf{y}) = \frac{1}{4}\left(\nu(\mathbf{x} + \mathbf{y})^2 - \nu(\mathbf{x} - \mathbf{y})^2\right)$$
    is an inner product on $\mathbb{R}^n$.
16. Let $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\} \subseteq \mathbb{C}^n$ be a set of unit vectors such that $\mathbf{w}_i \perp \mathbf{w}_j$ for $i \neq j$ and $1 \leqslant i, j \leqslant k$. If $W_k = (\mathbf{w}_1 \ \cdots \ \mathbf{w}_k) \in \mathbb{C}^{n \times k}$, prove that
    $$I_n - W_k W_k^{\mathsf{H}} = (I_n - \mathbf{w}_k \mathbf{w}_k^{\mathsf{H}}) \cdots (I_n - \mathbf{w}_1 \mathbf{w}_1^{\mathsf{H}}).$$

17. Let $\mu^{(m,n)} : \mathbb{C}^{m\times n} \longrightarrow \mathbb{R}_{\geqslant 0}$ be a vectorial matrix norm. Prove that for every $A \in \mathbb{C}^{m\times n}$ there exists a constant $k \in \mathbb{R}$ such that $\mu^{(m,n)}(A) \leqslant k \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|$.
18. We use here the notations introduced in Theorem 6.57. Prove that if $A \in \mathbb{C}^{n\times n}$ is a Hermitian matrix such that $AU = UB$ for some $U \in \mathbb{C}^{n\times k}$ having an orthonormal set of columns, then we can write:

$$(U\ V)^{\mathsf{H}} A (UV) = \begin{pmatrix} B & O_{k,n-k} \\ O_{n-k,k} & V^{\mathsf{H}} AV \end{pmatrix}$$

for some matrix $V \in \mathbb{C}^{n\times(n-k)}$ such that $(U\ V) \in \mathbb{C}^{n\times n}$ is a unitary matrix,
19. Prove that a matrix $A \in \mathbb{C}^{n\times n}$ is normal if and only if $\| A\mathbf{x} \|_2 = \| A^{\mathsf{H}}\mathbf{x} \|_2$ for every $\mathbf{x} \in \mathbb{C}^n$.
20. Prove that for every matrix $A \in \mathbb{C}^{n\times n}$ we have $|\!|\!| A |\!|\!|_2 = |\!|\!| A^{\mathsf{H}} |\!|\!|_2$.
21. Let $A \in \mathbb{R}^{m\times n}$. Prove that there exists i, $1 \leqslant i \leqslant n$ such that $\| A\mathbf{e}_i \|_2^2 \geqslant \frac{1}{n} \| A \|_F^2$.
22. Let $U \in \mathbb{C}^{n\times n}$ be a matrix whose set of columns is orthonormal and let $V \in \mathbb{C}^{n\times n}$ be a matrix whose set of rows is orthonormal. Prove that $|\!|\!| UAV |\!|\!|_F = |\!|\!| A |\!|\!|_F$.
23. Let $M \in \mathbb{R}^{n\times n}$ be a positive definite matrix. Prove that $\mathbf{x}'M\mathbf{x} + \mathbf{y}'M\mathbf{y} \geqslant \mathbf{x}'M\mathbf{y} + \mathbf{y}'M\mathbf{x}$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.
24. Let $H \in \mathbb{C}^{n\times n}$ be a non-singular matrix. Prove that the function $f : \mathbb{C}^{n\times n} \longrightarrow \mathbb{R}_{\geqslant 0}$ defined by $f(X) = \| HXH^{-1} \|_2$ for $X \in \mathbb{C}^{n\times n}$ is a matrix norm.
25. Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{p\times q}$ be two matrices. Prove that $\| A \otimes B \|_F^2 = \mathit{trace}(A'A \otimes B'B)$.
26. Let $\mathbf{x}_0, \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Prove that if $t \in [0,1]$ and $\mathbf{u} = t\mathbf{x} + (1-t)\mathbf{y}$, then $\| \mathbf{x}_0 - \mathbf{u} \|_2 \leqslant \max\{\| \mathbf{x}_0 - \mathbf{x} \|_2, \| \mathbf{x}_0 - \mathbf{y} \|_2\}$.
27. Let $\mathbf{u}_1, \ldots, \mathbf{u}_m$ be m unit vectors in $\mathbb{R}^2$, such that $\| \mathbf{u}_i - \mathbf{u}_j \| = 1$. Prove that $m \leqslant 6$.
28. Prove that if $A \in \mathbb{C}^{n\times n}$ is an invertible matrix, then $\mu(A) \geqslant \frac{1}{\mu(A^{-1})}$ for any matrix norm μ.
29. Let $\| \cdot \|$ be an unitarily invariant norm. Prove that $\| A - I_n \| \leqslant \| A - U \| \leqslant \| A + I_n \|$ for every Hermitian matrix $A \in \mathbb{C}^{n\times n}$ and every unitary matrix U.
30. Let $A \in \mathbb{C}^{n\times n}$ be an invertible matrix and let $\| \cdot \|$ be a norm on $\mathbb{C}^n$. Prove that

$$|\!|\!| A^{-1} |\!|\!| = \frac{1}{\min\{\| A\mathbf{x} \| \mid \| \mathbf{x} \| = 1\}},$$

where $|\!|\!| \cdot |\!|\!|$ is the matrix norm generated by $\| \cdot \|$.
31. Let $Y \in \mathbb{C}^{n\times p}$ be a matrix that has an orthonormal set of columns, that is, $Y^{\mathsf{H}} Y = I_p$. Prove that:
    a) $\| Y \|_F = p$;
    b) for every matrix $R \in \mathbb{C}^{p\times q}$ we have $\| YR \|_F = \| R \|_F$.
32. Let $D \in \mathbb{R}^{n\times n}$ be a diagonal matrix such that $d_{ii} \geqslant 0$ for $1 \leqslant i \leqslant n$. Prove that if X is an orthogonal matrix, then $\mathit{trace}(XD) \leqslant \mathit{trace}(D)$.
33. Let

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

be the Pauli matrices defined in Example 6.45 Prove that
    a) the Pauli matrices are pairwise orthogonal;
    b) we have the equalities $XY = iZ$, $YZ = iX$, and $ZX = iY$;

c) if W is a Hermitian matrix in $\mathbb{C}^{2\times 2}$ such that $traceW = 0$, then W is a linear combination of the Pauli matrices.

34. Prove that if $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ are such that $\parallel \mathbf{x} \parallel_2 = \parallel \mathbf{y} \parallel_2$, then $\mathbf{x}+\mathbf{y} \perp \mathbf{x}-\mathbf{y}$.
35. If S is a subspace of $\mathbb{C}^n$, prove that $(S^\perp)^\perp = S$.
36. Prove that every permutation matrix P_ϕ is orthogonal.
37. Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix. Prove that
$$(\mathbf{x}, A\mathbf{x}) - (\mathbf{y}, A\mathbf{y}) = (A(\mathbf{x}-\mathbf{y}), \mathbf{x}+\mathbf{y})$$
for every $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.
38. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ be two unit vectors. Prove that
$$|\sin \angle(\mathbf{x}, \mathbf{y})| = \frac{\parallel \mathbf{x}+\mathbf{y} \parallel_2 \parallel \mathbf{x}-\mathbf{y} \parallel_2}{2}.$$
39. Let $\mathbf{u}$ and $\mathbf{v}$ be two unit vectors in $\mathbb{R}^n$. Prove that
    a) if $\alpha = \angle(\mathbf{u}, \mathbf{v})$, then $\parallel \mathbf{u} - \mathbf{v}\cos\alpha \parallel_2 = \sin\alpha$;
    b) $\mathbf{v}\cos\alpha$ is the closest vector in $\langle \mathbf{v} \rangle$ to $\mathbf{u}$.
40. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} \subseteq \mathbb{R}^n$ be a collection of p unit vectors such that $\angle(\mathbf{v}_i, \mathbf{v}_j) = \theta$, where $0 < \theta \leqslant \frac{\pi}{2}$ for every pair $(\mathbf{v}_i, \mathbf{v}_j)$ such that $1 \leqslant i, j \leqslant p$ and $i \neq j$. Prove that $p \leqslant \frac{n(n+1)}{2}$.
41. Let $\mathbb{C}^{n\times n}$ be the linear space of complex matrices. Prove that:
    a) the set of Hermitian matrices $\mathcal{H}$ and the set of skew-Hermitian matrices $\mathcal{K}$ in $\mathbb{C}^{n\times n}$ are subspaces of $\mathbb{C}^{n\times n}$;
    b) if $\mathbb{C}^{n\times n}$ is equipped with the inner product defined in Example 6.26, then $\mathcal{K} = \mathcal{H}^\perp$.
42. Give an example of a matrix that has positive elements but is not positive definite.
43. Let $M \in \mathbb{R}^{n\times n}$ be a positive definite matrix. Prove that $\mathbf{x}'M\mathbf{x} + \mathbf{y}'M\mathbf{y} \geqslant \mathbf{x}'M\mathbf{y} + \mathbf{y}'M\mathbf{x}$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.
44. Prove that if $A \in \mathbb{R}^{n\times n}$ is a positive definite matrix, then A is invertible and A^{-1} is also positive definite.
45. Let $A \in \mathbb{C}^{n\times n}$ be a positive definite Hermitian matrix. If $A = B + iC$, where $B, C \in \mathbb{R}^{n\times n}$, prove that the real matrix
$$D = \begin{pmatrix} B & -C \\ C & B \end{pmatrix}$$
is positive definite.
46. Let L be a real linear space and let $\parallel \cdot \parallel$ be a norm generated by an inner product defined on L. L is said to be *symmetric relative to the norm* if $\parallel a\mathbf{x} - \mathbf{y} \parallel = \parallel \mathbf{x} - a\mathbf{y} \parallel$ for $a \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in V$ such that $\parallel \mathbf{x} \parallel = \parallel \mathbf{y} \parallel = 1$.
    a) Prove that if a norm on a linear vector space L is induced by an inner product, then L is symmetric relative to that norm.
    b) Prove that L satisfies the *Ptolemy inequality* $\parallel \mathbf{x}-\mathbf{y} \parallel \parallel \mathbf{z} \parallel \leqslant \parallel \mathbf{y}-\mathbf{z} \parallel \parallel \mathbf{x} \parallel + \parallel \mathbf{z}-\mathbf{x} \parallel \parallel \mathbf{y} \parallel$, for $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ if and only if L is symmetric.
47. Let $H_\mathbf{u}$ be the Householder matrix corresponding to the unit vector $\mathbf{u} \in \mathbb{R}^n$. If $\mathbf{x} \in \mathbb{R}^n$ is written as $\mathbf{x} = \mathbf{y} + \mathbf{z}$, where $\mathbf{y} = a\mathbf{u}$ and $\mathbf{z} \perp \mathbf{u}$, then $H_\mathbf{u}\mathbf{x}$ is obtained by a reflection of $\mathbf{x}$ relative to the hyperplane that is perpendicular on $\mathbf{u}$, that is, $H_\mathbf{u}\mathbf{x} = -\mathbf{u} + \mathbf{v}$.

48. Let $Y \in \mathbb{R}^{n \times k}$ be a matrix such that $Y'Y = I_k$, where $k \leqslant n$. Prove that the matrix $I_n - YY'$ is positive semidefinite.
49. Prove that if $A, B \in \mathbb{R}^{2\times 2}$ are two rotation matrices, then $AB = BA$.
50. Let $\mathbf{u} \in \mathbb{R}^n$ be a unit vector. A *rotation with axis* $\boldsymbol{u}$ is an orthogonal matrix A such that $A\mathbf{u} = \mathbf{u}$. Prove that if $\mathbf{v} \perp \mathbf{u}$, then $A\mathbf{v} \perp \mathbf{u}$ and $A'\mathbf{v} \perp \mathbf{u}$.
51. Let $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ be three unit vectors in $\mathbb{R}^2 - \{\mathbf{0}\}$. Prove that $\angle(\mathbf{u}, \mathbf{v}) \leqslant \angle(\mathbf{u}, \mathbf{w}) + \angle(\mathbf{w}, \mathbf{v})$.
52. Let $A \in \mathbb{R}^{m\times n}$ be a matrix such that $rank(A) = n$. Prove that the R-factor of the QR-decomposition of $A = QR$ has positive diagonal elements, it equals the Cholesky factor of $A'A$, and therefore is uniquely determined.
53. Let $A \in \mathbb{C}^{n\times m}$ be a full-rank matrix such that $m \geqslant n$. Prove that A can be factored as $A = LQ$, where $L \in \mathbb{C}^{n\times n}$ and $Q \in \mathbb{C}^{n\times m}$, such that the columns of Q constitute an orthonormal basis for $\mathsf{Ran}(A^{\mathsf{H}})$, and $L = (\ell_{ij})$ is an lower triangular invertible matrix such that its diagonal elements are real non-negative numbers, that is, $\ell_{ii} \geqslant 0$ for $1 \leqslant i \leqslant n$.

Let

$$D = \begin{pmatrix} \mathbf{u}'_1 \\ \vdots \\ \mathbf{u}'_m \end{pmatrix} = (\mathbf{v}_1 \ \cdots \ \mathbf{v}_n) \in \mathbb{R}^{m\times n}$$

be a data matrix (as introduced in on page 271) and let $\mathbf{z} \in \mathbb{R}^n$. The *inertia of* D *relative to* $\mathbf{z}$ is the number $I_{\mathbf{z}}(D) = \sum_{j=1}^{m} \parallel \mathbf{u}_j - \mathbf{z} \parallel_2^2$.

54. Let

$$D = \begin{pmatrix} \mathbf{u}'_1 \\ \vdots \\ \mathbf{u}'_m \end{pmatrix}$$

be a data matrix. Prove that

$$I_{\mathbf{z}}(D) - I_{\tilde{D}}(D) = m \parallel \tilde{D} - \mathbf{z} \parallel_2^2,$$

for every $\mathbf{z} \in \mathbb{R}^n$. Conclude that the minimal value of the inertia $I_{\mathbf{z}}(D)$ is achieved for $\mathbf{z} = \tilde{D}$.

The *standard deviation of a vector* $\mathbf{v} \in \mathbb{R}^m$ is the number

$$s_{\mathbf{v}} = \sqrt{\frac{1}{m-1}\sum_{i=1}^{m}(v_i - \tilde{v})^2},$$

where $\tilde{v} = \frac{1}{m}\sum_{i=1}^{m} v_i$ is the mean of the components of $\mathbf{v}$. The *variance* is $var(\mathbf{v}) = s_{\mathbf{v}}^2$.

The *standard deviation of a data matrix* $D \in \mathbb{R}^{m\times n}$, where $D = (\mathbf{v}_1 \ \cdots \ \mathbf{v}_n)$ is the row $\mathbf{s} = (s_{\mathbf{v}_1}, \ldots, s_{\mathbf{v}_n}) \in \mathbb{R}^n$.

Let $\mathbf{u}$ and $\mathbf{w}$ be two vectors in $\mathbb{R}^m$, where $m > 1$, having the means $\tilde{u}$ and $\tilde{w}$, and the standard deviations s_u and s_v, respectively. The *covariance coefficient* of $\mathbf{u}$ and $\mathbf{w}$ is the number

$$cov(\mathbf{u}, \mathbf{w}) = \frac{1}{m-1} \sum_{i=1}^{m-1} (u_i - \tilde{u})(w_i - \tilde{w}).$$

The *correlation coefficient* of $\mathbf{u}$ and $\mathbf{w}$ is the number

$$\rho(\mathbf{u}, \mathbf{w}) = \frac{cov(\mathbf{u}, \mathbf{w})}{s_u s_w}.$$

The *covariance matrix* is of a data matrix $D \in \mathbb{R}^{m \times n}$ is

$$cov(D) = \frac{1}{m-1} \hat{D}' \hat{D} \in \mathbb{R}^{n \times n}.$$

55. Prove that for $\mathbf{v} \in \mathbb{R}^m$ we have

$$var(\mathbf{v}) = \frac{1}{m-1} \left(\| \mathbf{v} \|^2 - m\tilde{v}^2 \right).$$

56. Let $D \in \mathbb{R}^{m \times n}$ be a data matrix, where

$$D = \begin{pmatrix} \mathbf{u}'_1 \\ \vdots \\ \mathbf{u}'_m \end{pmatrix} = (\mathbf{v}_1 \ \cdots \ \mathbf{v}_n).$$

Prove that the mean square distance between column vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is equal to twice the sum of row variances, $\sum_{i=1}^m var(\mathbf{u}_i)$.

Solution: The mean square distance between the columns of D is

$$\begin{aligned}
\frac{2}{n(n-1)} \sum_{i<j} \| \mathbf{v}_i - \mathbf{v}_j \|^2 &= \frac{2}{n(n-1)} \left(\sum_{j=1}^n \| \mathbf{v}_j \|^2 - 2 \sum_{i<j} \mathbf{v}'_i \mathbf{v}_j \right) \\
&= \frac{2}{n(n-1)} \left((n-1) \| D \|_F^2 + \| D \|_F^2 - \mathbf{1}'_n D D' \mathbf{1}_n \right) \\
&= \frac{2}{n(n-1)} \left(n \| D \|_F^2 - \mathbf{1}'_n D D' \mathbf{1}_n \right).
\end{aligned}$$

Since each vector $\mathbf{u}_k$ belongs to $\mathbb{R}^n$, the the sum of row variances is

$$\sum_{k=1}^m var(\mathbf{u}_k) = \sum_{k=1}^m \frac{1}{n-1} \left(\| \mathbf{u}_k \|^2 - n\tilde{u}_k^2 \right) = \frac{1}{n-1} \| D \|_F^2 - \frac{n}{n-1} \sum_{k=1}^n \tilde{u}_k^2.$$

Taking into account that

$$\tilde{u}_k = \frac{1}{n} \mathbf{1}'_n D' \mathbf{e}_k = \frac{1}{n} e'_k D \mathbf{1}_n,$$

we have $\tilde{u}_k^2 = \frac{1}{n^2} \mathbf{1}'_n D' \mathbf{e}_k e'_k D \mathbf{1}_n$, which implies

$$\sum_{k=1}^n \tilde{u}_k^2 = \frac{1}{n^2} \mathbf{1}'_n D' \left(\sum_{k=1}^n \mathbf{e}_k \mathbf{e}'_k \right) D \mathbf{1}_n = \frac{1}{n^2} \mathbf{1}'_n D' D \mathbf{1}_n,$$

because $\sum_{k=1}^n \mathbf{e}_k \mathbf{e}'_k = I_m$. The desired equality follows immediately.

57. Prove that $-1 \leq \rho(\mathbf{u}, \mathbf{w}) \leq 1$ for any vectors $\mathbf{u}, \mathbf{w} \in \mathbb{R}^m$.
58. Prove that the covariance matrix of a data matrix $D \in \mathbb{R}^{m \times n}$ can be written as $cov(D) = \frac{1}{m-1} D' H_m D$; if D is centered, then $cov(D) = \frac{1}{m-1} D' D$.
59. Let $D \in \mathbb{R}^{m \times n}$ be a centered data matrix and let $R \in \mathbb{R}^{n \times n}$ be an orthogonal matrix. Prove that $cov(DR) = R' cov(X) R$.

Bibliographical Comments

The notion of vector inner product, which is fundamental for linear algebra, functional analysis and other mathematical disciplines was introduced by Hermann Günther Grassmann (1809-1877) is his fundamental work “Die Lineale Ausdehnungslehre, ein neuer Zweig der Mathematik”.

Almost every advanced linear algebra reference deals with inner products and norms and their applications at the level that we need. We recommend especially [137] and the two volumes [95] and [96].

7

Spectral Properties of Matrices

7.1 Introduction

The existence of directions that are preserved by linear transformations (which are referred to as eigenvectors) has been discovered by L. Euler in his study of movements of rigid bodies. This work was continued by Lagrange, Cauchy, Fourier, and Hermite. The study of eigenvectors and eigenvalues acquired increasing significance through its applications in heat propagation and stability theory. Later, Hilbert initiated the study of eigenvalue in functional analysis (in the theory of integral operators). He introduced the terms of eigenvalue and eigenvector. The term *eigenvalue* is a German-English hybrid formed from the German word *eigen* which means "own" and the English word "value". It is interesting that Cauchy referred to the same concept as *characteristic value* and the term *characteristic polynomial* of a matrix (which we introduce in Definition 7.1) was derived from this naming.

We present the notions of geometric and algebraic multiplicities of eigenvalues, examine properties of spectra of special matrices, discuss variational characterizations of spectra and the relationships between matrix norms and eigenvalues. We conclude this chapter with a section dedicated to singular values of matrices.

7.2 Eigenvalues and Eigenvectors

Let $A \in \mathbb{C}^{n \times n}$ be a square matrix. An *eigenpair* of A is a pair $(\lambda, \mathbf{x}) \in \mathbb{C} \times (\mathbb{C}^n - \{\mathbf{0}\})$ such that $A\mathbf{x} = \lambda\mathbf{x}$. We refer to λ is an *eigenvalue* and to $\mathbf{x}$ is an *eigenvector*. The set of eigenvalues of A is the *spectrum* of A and will be denoted by $\mathsf{spec}(A)$.

If $(\lambda, \mathbf{x})$ is an eigenpair of A, the linear system $A\mathbf{x} = \lambda\mathbf{x}$ has a non-trivial solution in $\mathbf{x}$. An equivalent homogeneous system is $(\lambda I_n - A)\mathbf{x} = \mathbf{0}$ and this system has a non-trivial solution only if $\det(\lambda I_n - A) = 0$.

Definition 7.1. *The* characteristic polynomial *of the matrix A is the polynomial p_A defined by $p_A(\lambda) = \det(\lambda I_n - A)$ for $\lambda \in \mathbb{C}$.*

Thus, the eigenvalues of A are the roots of the characteristic polynomial of A.

Lemma 7.2. *Let $A = (\boldsymbol{a}_1 \ \cdots \ \boldsymbol{a}_n) \in \mathbb{C}^n$ and let B be the matrix obtained from A by replacing the column $\boldsymbol{a}_j$ by $\boldsymbol{e}_j$. Then, we have*

$$\det(B) = \det\left(A\begin{bmatrix}1 & \cdots & j-1 & j+1 & \cdots & n\\ 1 & \cdots & j-1 & j+1 & \cdots & n\end{bmatrix}\right).$$

Proof. The result follows immediately by expanding B on the j^{th} column.

The result obtained in Lemma 7.2 can be easily extended as follows. If B is the matrix obtained from A by replacing the columns $\mathbf{a}_{j_1}, \ldots, \mathbf{a}_{j_k}$ by $\mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_k}$ and $\{i_1, \ldots, i_p\} = \{1, \ldots, n\} - \{j_1, \ldots, j_k\}$, then

$$\det(B) = \det\left(A\begin{bmatrix}i_1 & \cdots & i_p\\ i_1 & \cdots & i_p\end{bmatrix}\right). \tag{7.1}$$

In other words, $\det(B)$ equals a principal p-minor of A.

Theorem 7.3. *Let $A \in \mathbb{C}^{n\times n}$ be a matrix. Its characteristic polynomial p_A can be written as*

$$p_A(\lambda) = \sum_{k=0}^{n} (-1)^k a_k \lambda^{n-k},$$

where a_k is the sum of the principal minors of order k of A.

Proof. By Theorem 5.146 the determinant

$$p_A(\lambda) = \det(\lambda I_n - A) = (-1)^n \det(A - \lambda I_n)$$

can be written as a sum of 2^n determinants of matrices obtained by replacing each subset of the columns of A by the corresponding subset of columns of $-\lambda I_n$. If the subset of columns of $-\lambda I_n$ involved are $-\lambda\mathbf{e}_{j_1}, \ldots, -\lambda\mathbf{e}_{j_k}$ the result of the substitution is $(-1)^k\lambda^k \det\left(A\begin{bmatrix}i_1 \cdots i_p\\ i_1 \cdots i_p\end{bmatrix}\right)$, where $\{i_1, \ldots, i_p\} = \{1, \ldots, n\} - \{j_1, \ldots, j_k\}$. The total contribution of sets of k columns of $-\lambda I_n$ is $(-1)^k\lambda^k a_{n-k}$. Therefore,

$$p_A(\lambda) = (-1)^n \sum_{k=0}^{n} (-1)^k \lambda^k a_{n-k}.$$

Replacing k by $n-k$ as the summation index yields

$$p_A(\lambda) = (-1)^n \sum_{k=0}^{n} (-1)^{n-k} \lambda^{n-k} a_k = \sum_{k=0}^{n} (-1)^k a_k \lambda^{n-k}.$$

Definition 7.4. *Two matrices $A, B \in \mathbb{C}^{n\times n}$ are* similar *if there exists an invertible matrix $P \in \mathbb{C}^{n\times n}$ such that $B = PAP^{-1}$. This is denoted by $A \sim B$.*

If there exists a unitary matrix U such that $B = UAU^{-1}$, then A is unitarily similar *to B. This is denoted by $A \sim_u B$.*

The matrices A, B are congruent *if $B = SAS^{\mathsf{H}}$ for some non-singular matrix S. This is denoted by $A \approx B$. If $A, B \in \mathbb{R}^{n\times n}$, we say that they are* t-congruent *if $B = SAS'$ for some invertible matrix S; this is denoted by $A \approx_t B$.*

For real matrices the notions of t-congruence and congruence are identical.

It is easy to verify that $\sim, \sim_u$ and $\approx$ are equivalence relations on $\mathbb{C}^{n\times n}$ and $\approx_t$ is an equivalence on $\mathbb{R}^{n\times n}$.

Similar matrices have the same characteristic polynomial. Indeed, suppose that $B = PAP^{-1}$. We have

$$\begin{aligned} p_B(\lambda) &= \det(\lambda I_n - B) = \det(\lambda I_n - PAP^{-1}) \\ &= \det(\lambda P I_n P^{-1} - PAP^{-1}) = \det(P(\lambda I_n - A)P^{-1}) \\ &= \det(P)\det(\lambda I_n - A)\det(P^{-1}) = \det(\lambda I_n - A) = p_A(\lambda), \end{aligned}$$

because $\det(P)\det(P^{-1}) = 1$. Thus, similar matrices have the same eigenvalues.

Example 7.5. Let A be the matrix

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

We have

$$p_A = \det(\lambda I_2 - A) = (\lambda - \cos\theta)^2 + \sin^2\theta = \lambda^2 - 2\lambda\cos\theta + 1.$$

The roots of this polynomial are $\lambda_1 = \cos\theta + i\sin\theta$ and $\lambda_2 = \cos\theta - i\sin\theta$, so they are complex numbers.

We regard A as a complex matrix with real entries. If we were to consider A as a real matrix, we would not be able to find real eigenvalues for A unless θ were equal to 0.

Definition 7.6. *The* algebraic multiplicity *of the eigenvalue λ of a matrix $A \in \mathbb{C}^{n\times n}$ is the multiplicity of λ as a root of the characteristic polynomial p_A of A.*

The algebraic multiplicity of λ is denoted by $\mathsf{algm}(A, \lambda)$. If $\mathsf{algm}(A, \lambda) = 1$ we say that λ is a simple eigenvalue.

Example 7.7. Let $A \in \mathbb{R}^{2\times 2}$ be the matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$$

The characteristic polynomial of A is

$$p_A(\lambda) = \begin{vmatrix} \lambda & 1 \\ -1 & \lambda - 2 \end{vmatrix} = \lambda^2 - 2\lambda + 1.$$

Therefore, A has the eigenvalue 1 with $\mathsf{algm}(A, 1) = 2$.

Example 7.8. Let $P(a) \in \mathbb{C}^{n\times n}$ be the matrix $P(a) = (a-1)I_n + J_n$. To find the eigenvalues of $P(a)$ we need to solve the equation

$$\begin{vmatrix} \lambda - a & -1 & \cdots & -1 \\ -1 & \lambda - a & \cdots & -1 \\ \vdots & \vdots & \cdots & \vdots \\ -1 & -1 & \cdots & \lambda - a \end{vmatrix} = 0.$$

By adding the first $n-1$ columns to the last and factoring out $\lambda - (a+n-1)$, we obtain the equivalent equation

$$(\lambda - (a+n-1)) \begin{vmatrix} \lambda - a & -1 & \cdots & 1 \\ -1 & \lambda - a & \cdots & 1 \\ \vdots & \vdots & \cdots & \vdots \\ -1 & -1 & \cdots & 1 \end{vmatrix} = 0.$$

Adding the last column to the first $n-1$ columns and expanding the determinant yields the equation $(\lambda - (a+n-1))(\lambda - a + 1)^{n-1} = 0$, which shows that $P(a)$ has the eigenvalue $a+n-1$ with $\mathsf{algm}(P(a), a+n-1) = 1$ and the eigenvalue $a-1$ with $\mathsf{algm}(P(a), a-1) = n-1$.

In the special case when $a = 1$ we have $P(1) = J_{n,n}$. Thus, $J_{n,n}$ has the eigenvalue $\lambda_1 = n$ with algebraic multiplicity 1 and the eigenvalue 0 with algebraic multiplicity $n-1$.

Theorem 7.9. *The eigenvalues of Hermitian complex matrices are real numbers.*

Proof. Let $A \in \mathbb{C}^{n\times n}$ be a Hermitian matrix and let λ be an eigenvalue of A. We have $A\mathbf{x} = \lambda\mathbf{x}$ for some $\mathbf{x} \in \mathbb{C}^n - \{\mathbf{0}_n\}$, so $\mathbf{x}^{\mathsf{H}} A^{\mathsf{H}} = \overline{\lambda}\mathbf{x}^{\mathsf{H}}$. Since $A^{\mathsf{H}} = A$, we have

$$\lambda\mathbf{x}^{\mathsf{H}}\mathbf{x} = \mathbf{x}^{\mathsf{H}} A\mathbf{x} = \mathbf{x}^{\mathsf{H}} A^{\mathsf{H}}\mathbf{x} = \overline{\lambda}\mathbf{x}^{\mathsf{H}}\mathbf{x}.$$

Since $\mathbf{x} \neq \mathbf{0}$ implies $\mathbf{x}^{\mathsf{H}}\mathbf{x} \neq 0$, it follows that $\overline{\lambda} = \lambda$. Thus, λ is a real number.

Corollary 7.10. *The eigenvalues of symmetric real matrices are real numbers.*

Proof. This is a direct consequence of Theorem 7.9.

Theorem 7.11. *The eigenvectors of a complex Hermitian matrix corresponding to distinct eigenvalues are orthogonal to each other.*

Proof. Let $(\lambda, \mathbf{u})$ and $(\mu, \mathbf{v})$ be two eigenpairs of the Hermitian matrix $A \in \mathbb{C}^{n\times n}$, where $\lambda \neq \mu$. Since A is Hermitian, $\lambda, \mu \in \mathbb{R}$. Since $A\mathbf{u} = \lambda\mathbf{u}$ we have $\mathbf{v}^{\mathsf{H}} A\mathbf{u} = \lambda\mathbf{v}^{\mathsf{H}}\mathbf{u}$. The last equality can be written as $(A\mathbf{v})^{\mathsf{H}}\mathbf{u} = \lambda\mathbf{v}^{\mathsf{H}}\mathbf{u}$, or as $\mu\mathbf{v}^{\mathsf{H}}\mathbf{u} = \lambda\mathbf{v}^{\mathsf{H}}\mathbf{u}$. Since $\mu \neq \lambda$, $\mathbf{v}^{\mathsf{H}}\mathbf{u} = 0$, so $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

The statement clearly holds if we replace complex Hermitian matrices by real symmetric matrices.

Corollary 7.12. *The eigenvectors of a Hermitian matrix corresponding to distinct eigenvalues form a linearly independent set.*

Proof. This statement follows from Theorems 6.41 and 7.11.

The next statement is a result of Issai Schur (1875-1941), a mathematician born in Russia, who studied and worked in Germany.

Theorem 7.13. (Schur's Triangularization Theorem) *Let $A \in \mathbb{C}^{n\times n}$ be a square matrix. There exists an upper-triangular matrix $T \in \mathbb{C}^{n\times n}$ such that $A =\sim_u T$.*

The diagonal elements of T are the eigenvalues of A; moreover, each eigenvalue λ of A occurs in the sequence of diagonal elements of T a number of $\mathsf{algm}(A, \lambda)$ times. The columns of U are unit eigenvectors of A.

Proof. The argument is by induction on n. The base case, $n = 1$, is immediate.

Suppose that the statement holds for matrices in $\mathbb{C}^{(n-1)\times(n-1)}$ and let $A \in \mathbb{C}^{n\times n}$. If $(\lambda, \mathbf{x})$ is an eigenpair of A with $\| \mathbf{x} \|_2 = 1$, let $H_{\mathbf{v}}$ be a Householder matrix that transforms $\mathbf{x}$ into $\mathbf{e}_1$. Since we also have $H_{\mathbf{v}}\mathbf{e}_1 = \mathbf{x}$, $\mathbf{x}$ is the first column of $H_{\mathbf{v}}$ and we can write $H_{\mathbf{v}} = (\mathbf{x}\ K)$, where $K \in \mathbb{C}^{n\times(n-1)}$. Consequently,

$$H_{\mathbf{v}} A H_{\mathbf{v}} = H_{\mathbf{v}} A(\mathbf{x}\ K) = H_{\mathbf{v}}(\lambda\mathbf{x}\ H_{\mathbf{v}} AK) = (\lambda\mathbf{e}_1\ H_{\mathbf{v}} AK).$$

Since $H_{\mathbf{v}}$ is Hermitian and $H_{\mathbf{v}} = (\mathbf{x}\ K)$, it follows that

$$H_{\mathbf{v}}^{\mathsf{H}} = \begin{pmatrix} \mathbf{x}^{\mathsf{H}} \\ K^{\mathsf{H}} \end{pmatrix} = H_{\mathbf{v}}.$$

Therefore,

$$H_{\mathbf{v}} A H_{\mathbf{v}} = \begin{pmatrix} \lambda & \mathbf{x}^{\mathsf{H}} AK \\ \mathbf{0}_{n-1} & K^{\mathsf{H}} AK \end{pmatrix}.$$

Since $K^{\mathsf{H}} AK \in \mathbb{C}^{(n-1)\times(n-1)}$, by the inductive hypothesis, there exists a unitary matrix W and an upper triangular matrix S such that $W^{\mathsf{H}}(K^{\mathsf{H}} AK)W = S$. Note that the matrix

$$U = H_{\mathbf{v}} \begin{pmatrix} 1 & \mathbf{0}'_{n-1} \\ \mathbf{0}_{n-1} & W \end{pmatrix}$$

is unitary and

$$U^{\mathsf{H}} A U^{\mathsf{H}} = \begin{pmatrix} \lambda & x^{\mathsf{H}} AKW \\ \mathbf{0}_{n-1} & W^{\mathsf{H}} K^{\mathsf{H}} AKW \end{pmatrix} = \begin{pmatrix} \lambda & \mathbf{x}^{\mathsf{H}} AKW \\ \mathbf{0}_{n-1} & S \end{pmatrix}.$$

The last matrix is clearly upper triangular.

Since $A \sim_u T$, A and T have the same characteristic polynomials and, therefore, the same eigenvalues, with identical multiplicities. Note that the factorization of A can be written as $A = UDU^{\mathsf{H}}$ because $U^{-1} = U^{\mathsf{H}}$. Since $AU = UD$, each column $\mathbf{u}_i$ of U is an eigenvector of A that corresponds to the eigenvalue λ_i for $1 \leqslant i \leqslant n$.

Corollary 7.14. *If $A \in \mathbb{R}^{n\times n}$ is a matrix such that $\mathsf{spec}(A) = \{0\}$, then A is nilpotent.*

Proof. By Schur's Triangularization Theorem, A is unitarily similar to a strictly upper triangular matrix, $A = UTU^{\mathsf{H}}$, so $A^n = UT^nU^{\mathsf{H}}$. Since $\mathsf{spec}(T) = \{0\}$, we have $T^n = O$, so $A^n = O$.

Corollary 7.15. *Let $A \in \mathbb{C}^{n\times n}$ and let f be a polynomial. If $\mathsf{spec}(A) = \{\lambda_1, \ldots, \lambda_n\}$ (including multiplicities), then $\mathsf{spec}(f(A)) = \{f(\lambda_1), \ldots, f(\lambda_n)\}$.*

Proof. By Schur's Triangularization Theorem there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ and an upper-triangular matrix $T \in \mathbb{C}^{n\times n}$ such that $A = UTU^{\mathsf{H}}$ and the diagonal elements of T are the eigenvalues of A, $\lambda_1, \ldots, \lambda_n$. Therefore $f(A) = Uf(T)U^{\mathsf{H}}$, and by Theorem 5.49, the diagonal elements of $f(T)$ are $f(\lambda_1), , \ldots, f(\lambda_m)$. Since $f(A) \sim_u f(T)$, we obtain the desired conclusion because two similar matrices have the same eigenvalues with the same algebraic multiplicities.

Definition 7.16. *A matrix $A \in \mathbb{C}^{n\times n}$ is* diagonalizable *(unitarily diagonalizable) if there exists a diagonal matrix $D = \mathsf{diag}(d_1, \ldots, d_n)$ such that $A \sim D$ ($A \sim_u D$).*

Theorem 7.17. *A matrix $A \in \mathbb{C}^{n\times n}$ is diagonalizable if and only if there exists a linearly independent set $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ of n eigenvectors of A.*

Proof. Let $A \in \mathbb{C}^{n\times n}$ such that there exists a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of n eigenvectors of A that is linearly independent and let P be the matrix $(\mathbf{v}_1 \; \mathbf{v}_2 \; \cdots \; \mathbf{v}_n)$ that is clearly invertible. We have:

$$\begin{aligned} P^{-1}AP &= P^{-1}(A\mathbf{v}_1 \; A\mathbf{v}_2 \; \cdots \; A\mathbf{v}_n) = P^{-1}(\lambda_1\mathbf{v}_1 \; \lambda_2\mathbf{v}_2 \; \cdots \; \lambda_n\mathbf{v}_n) \\ &= P^{-1}P \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}. \end{aligned}$$

Therefore, we have $A = PDP^{-1}$, where

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

so $A \sim D$.

Conversely, suppose that A is diagonalizable, so $AP = PD$, where D is a diagonal matrix and P is an invertible matrix, and let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be the columns of the matrix P. We have $A\mathbf{v}_i = d_{ii}\mathbf{v}_i$ for $1 \leqslant i \leqslant n$, so each $\mathbf{v}_i$ is an eigenvector of A. Since P is invertible, its columns are linear independent.

Corollary 7.18. *If $A \in \mathbb{C}^{n\times n}$ is diagonalizable then the columns of any matrix P such that $D = P^{-1}AP$ is a diagonal matrix are eigenvectors of A. Furthermore, the diagonal entries of D are the eigenvalues that correspond to the columns of P.*

Proof. This statement follows from the proof of Theorem 7.17.

Corollary 7.19. *A matrix $A \in \mathbb{C}^{n\times n}$ is unitarily diagonalizable if and only if there exists a set $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ of n orthonormal eigenvectors of A.*

Proof. This statement follows from the proof of Theorem 7.17 by observing that if $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a set n orthonormal eigenvectors of A, then $P = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ is a unitary matrix that gives a unitary diagonalization of A. Conversely, if P is an unitary matrix such that $A = PDP^{-1}$ its set of columns consists of orthogonal unitary eigenvectors of A.

Corollary 7.20. *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix. There exists a orthonormal matrix U and a diagonal matrix T such that $A = UTU^{-1}$. The diagonal elements of T are the eigenvalues of A; moreover, each eigenvalue λ of A occurs in the sequence of diagonal elements of T a number of $\mathsf{algm}(A, \lambda)$ times.*

Proof. As the previous corollary, this follows from the proof of Theorem 7.17.

Theorem 7.21. *Let $A \in \mathbb{C}^{n\times n}$ be a block diagonal matrix,*

$$A = \begin{pmatrix} A_{11} & O & \cdots & O \\ O & A_{22} & \cdots & O \\ \vdots & \vdots & \cdots & \vdots \\ O & O & \cdots & A_{mm} \end{pmatrix}.$$

A is diagonalizable if and only if every matrix A_{ii} is diagonalizable for $1 \leqslant i \leqslant m$.

Proof. Suppose that A is a block diagonal matrix which is diagonalizable. Furthermore, suppose that $A_{ii} \in \mathbb{C}^{n_i \times n_i}$ for $1 \leqslant i \leqslant n$ and $\sum_{i=1}^{m} n_i = n$. There exists an invertible matrix $P \in \mathbb{C}^{n \times n}$ such that $P^{-1}AP$ is a diagonal matrix $D = \mathsf{diag}(\lambda_1, \ldots, \lambda_n)$. Let $\mathbf{p}_1, \ldots, \mathbf{p}_n$ be the columns of P, which are eigenvectors of A. Each vector $\mathbf{p}_i$ is divided into m blocks $\mathbf{p}_i^j$ with $1 \leqslant j \leqslant m$, where $\mathbf{p}_i^j \in \mathbb{C}^{n_j}$. Thus, P can be written as

$$P = \begin{pmatrix} \mathbf{p}_1^1 & \mathbf{p}_2^1 & \cdots & \mathbf{p}_n^1 \\ \mathbf{p}_1^2 & \mathbf{p}_2^2 & \cdots & \mathbf{p}_n^2 \\ \vdots & \vdots & \cdots & \vdots \\ \mathbf{p}_1^m & \mathbf{p}_2^m & \cdots & \mathbf{p}_n^m \end{pmatrix}.$$

The equality $A\mathbf{p}_i = \lambda_i \mathbf{p}_i$ can be expressed as

$$\begin{pmatrix} A_{11} & O & \cdots & O \\ O & A_{22} & \cdots & O \\ \vdots & \vdots & \cdots & \vdots \\ O & O & \cdots & A_{mm} \end{pmatrix} \begin{pmatrix} \mathbf{p}_i^1 \\ \mathbf{p}_i^2 \\ \vdots \\ \mathbf{p}_i^m \end{pmatrix} = \lambda_i \begin{pmatrix} \mathbf{p}_i^1 \\ \mathbf{p}_i^2 \\ \vdots \\ \mathbf{p}_i^m \end{pmatrix},$$

which shows that $A_{jj}\mathbf{p}_i^j = \lambda_i \mathbf{p}_i^j$ for $1 \leqslant j \leqslant m$. Let $M^j = (\mathbf{p}_1^j \ \mathbf{p}_2^j \ \cdots \ \mathbf{p}_n^j) \in \mathbb{C}^{n_j \times n}$. We claim that $\mathit{rank}(M^j) = n_j$. Indeed if $\mathit{rank}(M^j)$ were less than n_j, we would have fewer that n independent rows M^j for $1 \leqslant j \leqslant m$. This, however, would imply that the rank of P is less then n, which contradicts the invertibility of P. Since there are n_j linearly independent eigenvectors of A_{jj}, it follows that each block A_{jj} is diagonalizable.

Conversely, suppose that each A_{jj} is diagonalizable, that is, there exists a invertible matrix Q_j such that $Q_j^{-1}A_{jj}Q_j$ is a diagonal matrix. Then, it is immediate to verify that the block diagonal matrix

$$Q = \begin{pmatrix} Q_1 & O & \cdots & O \\ O & Q_2 & \cdots & O \\ \vdots & \vdots & \cdots & \vdots \\ O & O & \cdots & Q_m \end{pmatrix}$$

is invertible and $Q^{-1}AQ$ is a diagonal matrix.

Theorem 7.22. *Let $A \in \mathbb{C}^{n \times n}$ be a matrix and let $(\lambda_1, \boldsymbol{v}_1), \ldots, (\lambda_k, \boldsymbol{v}_k)$ be k eigenpairs of A, where $\lambda_1, \ldots, \lambda_k$ are pairwise distinct. Then, $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ is a linearly independent set.*

Proof. Suppose that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is not linearly independent. Then, there exists a linear combination of this set that equals $\mathbf{0}$, that is

$$c_1\mathbf{v}_{i_1} + \cdots + c_r\mathbf{v}_{i_r} = \mathbf{0}_n, \tag{7.2}$$

at least one coefficient is not 0, and $r > 1$. Choose this linear combination to involve the minimal number of terms. We have

$$A(c_1\mathbf{v}_{i_1} + c_2\mathbf{v}_{i_2} + \cdots + c_r\mathbf{v}_{i_r}) = c_1\lambda_{i_1}\mathbf{v}_1 + c_2\lambda_{i_2}\mathbf{v}_{i_2} + \cdots + c_r\lambda_{i_r}\mathbf{v}_{i_r} = \mathbf{0}_n.$$

By multiplying Equality (7.2) by λ_{i_1} we have

$$c_1\lambda_{i_1}\mathbf{v}_{i_1} + c_2\lambda_{i_1}\mathbf{v}_{i_2} + \cdots + c_r\lambda_{i_1}\mathbf{v}_{i_r} = \mathbf{0}_n.$$

It follows that $c_2(\lambda_{i_2} - \lambda_{i_1})\mathbf{v}_{i_2} + \cdots + c_r(\lambda_{i_r} - \lambda_{i_1})\mathbf{v}_{i_r} = \mathbf{0}_n$. Since there exists a coefficient $\lambda_{i_p} - \lambda_{i_1}$ that is non-zero, this contradicts the minimality of r.

Corollary 7.23. *If $A \in \mathbb{C}^{n\times n}$ has n distinct eigenvalues, then A is diagonalizable.*

Proof. If $\mathsf{spec}(A) = \{\lambda_1, \ldots, \lambda_n\}$ consists of n complex numbers and $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are corresponding eigenvectors, then, by Theorem 7.22, $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent. The statement follows immediately from Theorem 7.17.

7.3 Geometric and Algebraic Multiplicities of Eigenvalues

For a matrix $A \in \mathbb{C}^{n\times n}$ let $S_{A,\lambda}$ be the subspace $\mathsf{NullSp}(\lambda I_n - A)$. We refer to $S_{A,\lambda}$ as *invariant subspace of A and λ* or as the *eigenspace* of λ.

Definition 7.24. *Let $A \in \mathbb{C}^{n\times n}$ and let $\lambda \in \mathsf{spec}(A)$. The* geometric multiplicity *of λ is the dimension $\mathsf{geomm}(A, \lambda)$ of $S_{A,\lambda}$.*

Example 7.25. The geometric multiplicity of 1 as an eigenvalue of the matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix},$$

considered in Example 7.7, is 1. Indeed if $\mathbf{x}$ is an eigenvector that corresponds to this value we have $-x_2 = x_1$ and $x_1 + 2x_2 = x_2$, which means that any such eigenvector has the form $a\mathbf{1}_2$. Thus, $\dim(S_{A,1}) = 1$.

The definition of the geometric multiplicity of $\lambda \in \mathsf{spec}(A)$ implies

$$\mathsf{geomm}(A, \lambda) = \dim(\mathsf{NullSp}(A - \lambda I_n)) = n - \mathit{rank}(A - \lambda I_n). \tag{7.3}$$

Theorem 7.26. *Let $A \in \mathbb{R}^{n\times n}$. We have $0 \in \mathsf{spec}(A)$ if and only if A is a singular matrix. Moreover, in this case, $\mathsf{geomm}(A, 0) = n - \mathit{rank}(A) = \dim(\mathsf{NullSp}(A))$. If $\mathsf{algm}(A, 0) = 1$, then $\mathit{rank}(A) = n - 1$.*

Proof. The statement is an immediate consequence of Equality (7.3).