Homework 1

Posted: September 18, 2023
Due: October 2, 2023 at 4:00pm

1. Let $f(x)$ be the greatest number n such that $n^2 < x$. Write a program in $\mathcal{S}$ that computes f.

2. Let $\mathcal{P}$ be the program

$$\begin{array}{ll} & Y \leftarrow X_1 \\ [A] & \text{IF } X_2 = 0 \text{ GOTO } E \\ & Y \leftarrow Y + 1 \\ & Y \leftarrow Y + 1 \\ & X_2 \leftarrow X_2 - 1 \\ & \text{GOTO } A \end{array}$$

   What is $\Psi_{\mathcal{P}}^{(1)}(r_1)$? $\Psi_{\mathcal{P}}^{(2)}(r_1, r_2)$? $\Psi_{\mathcal{P}}^{(3)}(r_1, r_2, r_3)$?

3. A unary function $f(x)$ is said to be *partially n-computable* if it is computed by some $\mathcal{S}$ program $\mathcal{P}$ such that $\mathcal{P}$ has no more than n instructions, every variable in $\mathcal{P}$ is among $X, Y, Z_1, \ldots, Z_n$ and every label in $\mathcal{P}$ is among $A_1, \ldots, A_n, E$. Prove that if a unary function $f : \mathbb{N} \longrightarrow \mathbb{N}$ is computed by a program with no more than n instructions, then f is partially n-computable.

4. Let $P(x)$ be a computable predicate. Show that the function f defined as

$$f(x_1, x_2) = \begin{cases} x_1 + x_2 & \text{if } P(x_1 + x_2) \\ \uparrow & \text{otherwise} \end{cases}$$

   is partially computable.

5. Let $f(x)$ be a partially computable but not total function, let M be a *finite* set of numbers such that $f(m) \uparrow$ (which means that $f(m)$ is undefined) for all $m \in M$ and let g be an arbitrary partially computable function. Show that $h(x)$ defined as

$$h(x) = \begin{cases} g(x) & \text{if } x \in M, \\ f(x) & \text{otherwise} \end{cases}$$

is partially computable.

Explain your solutions in full and be careful with the grammar!