CONVEX SETS - I

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UMB

Definition

Let $\mathbf{x} \in \mathbb{R}^n$ and let $U \subseteq \mathbb{R}^n$.
$\mathbf{x}$ is an *affine combination* of U if there exist $a_1, \ldots, a_k \in \mathbb{R}$ such that $\sum_{i=1}^{k} a_i = 1$ and $\mathbf{x} = a_1\mathbf{x}_1 + \cdots + a_k\mathbf{x}_k$, for $\mathbf{x}_1, \ldots, \mathbf{x}_k \in U$.
A *convex combination* of U is a linear combination $a_1\mathbf{x}_1 + \cdots + a_k\mathbf{x}_k$, where $\mathbf{x}_1, \ldots, \mathbf{x}_k \in U$, $a_i \geq 0$ for $1 \leq i \leq k$, and $a_1 + \cdots + a_k = 1$.
The vector $\mathbf{x}$ is a *conic combination of U* if there exist $a_1, \ldots, a_k \in \mathbb{R}_{\geq 0}$ such that $\mathbf{x} = a_1\mathbf{x}_1 + \cdots + a_k\mathbf{x}_k$, for $\mathbf{x}_1, \ldots, \mathbf{x}_k \in U$.

If a combination of vectors of U is both conic and affine, then it is a convex combination.

Example

Let $A \in \mathbb{R}^{m \times n}$ be a matrix and let $\mathbf{a} \in \mathbb{R}^n$ be a convex combination of its columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$. This means that there exists a vector $\mathbf{w} \in \mathbb{R}^n$ such that

$$\mathbf{a} = w_1\mathbf{a}_1 + \cdots + w_n\mathbf{a}_n = A\mathbf{w},$$

such that $w_i \geq 0$ for $1 \leq i \leq n$ and $\sum_{i=1}^n w_i = 1$. The conditions that involve $\mathbf{w}$ can be expressed succinctly as $\mathbf{w} \geq \mathbf{0}$ and $\mathbf{w}'\mathbf{1} = 1$.

Convex Sets

Definition

Let $(V, +, \cdot)$ be a real linear space and let C be a subset of V. The set C is *convex* if, for all $\mathbf{x}, \mathbf{y} \in C$ and all $a \in [0, 1]$, we have $(1 - a)\mathbf{x} + a\mathbf{y} \in C$. In other words, every point on the line segment connecting $\mathbf{x}$ and $\mathbf{y}$ belongs to C.

For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, the set

$$\{a\mathbf{x} + (1 - a)\mathbf{y} \mid a \in [0, 1]\}$$

is known as the *segment* that joins $\mathbf{x}$ to $\mathbf{y}$ and is denoted by $\overline{\mathbf{xy}}$. Thus, C is convex if $\mathbf{x}, \mathbf{y} \in C$ implies $\overline{\mathbf{xy}} \subseteq C$.
C of V is convex if and only if it contains every convex combination of its points.

Example

Any subspace T of a real linear space V is convex; any finite subset U of V such that $|U| \leq 1$ is convex.

Example

The set $\mathbb{R}^n_{\geq 0}$ of all vectors of $\mathbb{R}^n$ having non-negative components is a convex set called the *non-negative orthant* of $\mathbb{R}^n$.

Example

The convex subsets of $(\mathbb{R}, +, \cdot)$ are the intervals of $\mathbb{R}$. Regular polygons are convex subsets of $\mathbb{R}^2$.

Example

An open sphere $C(\mathbf{x}_0, r) \subseteq \mathbb{R}^n$ is convex, where $\mathbb{R}^n$ is equipped with the Euclidean norm. Indeed, suppose that $\mathbf{x}, \mathbf{y} \in C(\mathbf{x}_0, r)$, that is, $\| \mathbf{x} - \mathbf{x}_0 \|_2 < r$ and $\| \mathbf{y} - \mathbf{x}_0 \|_2 < r$. Let $t \in [0, 1]$ and let $\mathbf{t} = t\mathbf{x} + (1 - t)\mathbf{y}$. It is easy to see that

$$\| \mathbf{x}_0 - \mathbf{u} \|_2 \leq \max\{\| \mathbf{x}_0 - \mathbf{x} \|_2, \| \mathbf{x}_0 - \mathbf{y} \|_2\} = r,$$

so $\mathbf{u} \in C(\mathbf{x}_0, r)$.

Definition

Let $S \subseteq \mathbb{R}^n$. The *convex hull* of S is the set $\text{conv}(S)$ of all convex combinations of the members of S.

Theorem

If S is a subset of $\mathbb{R}^n$, then $\text{conv}(S)$ is the smallest convex set containing S; indeed, $\text{conv}(S)$ is the intersection of all convex sets containing S.

Polytopes and Simplexes

Definition

A *polytope* is the convex hull of a finite subset $\{\mathbf{x}_1, \ldots, \mathbf{x}_k, \mathbf{x}_{k+1}\}$ in $\mathbb{R}^n$. If $\mathbf{x}_1 - \mathbf{x}_{k+1}, \ldots, \mathbf{x}_k - \mathbf{x}_{k+1}$ are linearly independent vectors, then $\text{conv}(\{\mathbf{x}_1, \ldots, \mathbf{x}_k, \mathbf{x}_{k+1}\})$ is called a *simplex* having vertices $\mathbf{x}_1, \ldots, \mathbf{x}_k, \mathbf{x}_{k+1}$.

There is no simplex in $\mathbb{R}^n$ with more than $n+1$ vertices.

Carathéodory's Theorem

Theorem

Let S be a subset of $\mathbb{R}^n$. If $\mathbf{x} \in conv(S)$, then $\mathbf{x} \in conv(\{\mathbf{x}_1, \ldots, \mathbf{x}_{n+1}\})$; in other words, any point in $conv(S)$ can be represented as a convex combination of at most $n+1$ points in S.

Proof

If $\mathbf{x} \in \text{conv}(S)$, then $\mathbf{x} = \sum_{j=1}^{k} a_j \mathbf{x}_j$. Suppose that $k > n+1$. Then, the $k-1$ vectors $\mathbf{x}_1 - \mathbf{x}_k, \ldots, \mathbf{x}_{k-1} - \mathbf{x}_k$ are linearly dependent, so there exist $b_1, \ldots, b_{k-1}$ not all 0, such that

$$b_1(\mathbf{x}_1 - \mathbf{x}_k) + \cdots + b_{k-1}(\mathbf{x}_{k-1} - \mathbf{x}_k) = \mathbf{0},$$

or

$$b_1\mathbf{x}_1 + \cdots + b_{k-1}\mathbf{x}_{k-1} + b_k\mathbf{x}_k = \mathbf{0},$$

where b_k is such that $b_1 + \cdots + +b_{k-1} + b_k = 0$, and not all $b_1, \ldots, b_k$ are 0. Then,

$$\mathbf{x} = \sum_{j=1}^{k} a_j\mathbf{x}_j + \mathbf{0} = \sum_{j=1}^{k} a_j\mathbf{x}_j - c\sum_{j=1}^{k} b_j\mathbf{x}_j = \sum_{j=1}^{k}(a_j - cb_j)\mathbf{x}_j.$$

Choose $c = \min\{a_j/b_j \mid b_j > 0, 1 \leq j \leq k\} = a_i/b_i$.

Proof (cont'd)

Note that with this choice of c, $a_j - cb_j \geqslant 0$ if $j \neq i$ and $a_i - cb_i = 0$. Thus, $\mathbf{x}$ is a convex combination of at most $k - 1$ points. The argument can be repeated until $\mathbf{x}$ can be expressed as a combination of $n + 1$ points in S.

Closure, Interior, and Border of a Set

Let $S \subseteq \mathbb{R}^n$ and let $\mathbf{x} \in \mathbb{R}^n$. $\mathbf{x}$ is in

- the closure of S, $\mathbf{K}(S)$, if $C(\mathbf{x}, \epsilon) \cap S \neq \emptyset$ for every $\epsilon > 0$;
- the interior of S, $\mathbf{I}(S)$, if $C(\mathbf{x}, \epsilon) \subset S$ for some $\epsilon > 0$;
- the boundary of S, ∂S, if $C(\mathbf{x}, \epsilon) \cap S \neq \emptyset$ and $C(\mathbf{x}, \epsilon) \cap (\mathbb{R}^n - S) \neq \emptyset$ for every $\epsilon > 0$.

A set S is:

- closed, if $S = \mathbf{K}(S)$;
- open, if $S = \mathbf{I}(S)$;

Example

- the set $B = \{x_1^2 + x_2^2 \leqslant 1\}$ is closed;
- the set $C = \{x_1^2 + x_2^2 < 1\}$ is open;
- the boundary of B is $\partial B = \{x_1^2 + x_2^2 = 1\}$.

Theorem

Let $S \subseteq \mathbb{R}^n$ *be a convex set with* $\mathbf{I}(S) \neq \emptyset$ *and let* $\mathbf{x} \in \mathbf{K}(S)$ *and* $\mathbf{y} \in \mathbf{I}(S)$*. Then,* $a\mathbf{x} + (1-a)\mathbf{y} \in \mathbf{I}(S)$ *for every* $a \in (0,1)$*.*

Corollary

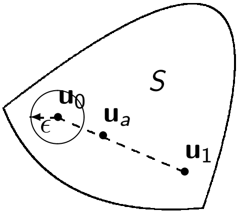
- if $S \subseteq \mathbb{R}^n$ is convex, then $\mathbf{I}(S)$ is convex;
- if $S \subseteq \mathbb{R}^n$ is convex and $\mathbf{I}(S) \neq \emptyset$, then $\mathbf{K}(S)$ is convex;
- if $S \subseteq \mathbb{R}^n$ is convex and $\mathbf{I}(S) \neq \emptyset$, then $\mathbf{K}(\mathbf{I}(S)) = \mathbf{K}(S)$ and $\mathbf{I}(\mathbf{K}(S)) = \mathbf{I}(S)$.

Proof of Part 1

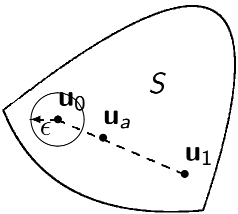
Lemma

If $S \subseteq \mathbb{R}^n$ is convex and $\mathbf{u}_0 \in \mathbf{I}(S)$, then for every $\mathbf{u}_1 \in S$ and $0 \leqslant a < 1$, we have $\mathbf{u}_a = (1-a)\mathbf{u}_0 + a\mathbf{u}_1 \in \mathbf{I}(S)$.

Proof: Since $\mathbf{u}_0$ is an interior point there exists $\epsilon > 0$ such that $C(\mathbf{u}_0, \epsilon) \subseteq S$. Then, we have $S(\mathbf{u}_a, (1-a)) \subseteq S$.



Proof (cont'd)



Let $\mathbf{w} \in C(\mathbf{u}_a, (1-a)\epsilon)$ and let

$$\mathbf{z} = \frac{1}{1-a}(\mathbf{w} - a\mathbf{u}_1).$$

We have

$$\parallel \mathbf{z} - \mathbf{u}_0 \parallel = \frac{1}{1-a} \parallel \mathbf{w} - a\mathbf{u}_1 - (1-a)\mathbf{u}_0 \parallel = \frac{1}{1-a} \parallel \mathbf{w} - \mathbf{u}_a \parallel \leqslant \epsilon,$$

so $\mathbf{z} \in S$. Thus, $\mathbf{w} \in S$ because $\mathbf{w}$ is a convex combination of $\mathbf{z}$ and $\mathbf{w}_1$.

Preliminaries

- If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then

$$\parallel \mathbf{x} + \mathbf{y} \parallel^2 + \parallel \mathbf{x} - \mathbf{y} \parallel^2 = 2(\parallel \mathbf{x} \parallel^2 + \parallel \mathbf{y} \parallel^2)$$

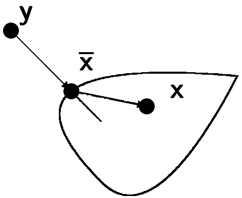
  (the Parallelogram Law).
- If $\parallel \mathbf{u} + \mathbf{v} \parallel = \parallel \mathbf{u} \parallel + \parallel \mathbf{v} \parallel$, then $\mathbf{u} = a\mathbf{v}$ for some $a \in \mathbb{R}$.

The Closest Point in a Closed and Convex Set

Theorem

Let S be a closed and convex subset of $\mathbb{R}^n$ and let $y \notin S$. There exists a unique $\bar{\mathbf{x}}$ *with minimum distance from* $\mathbf{y}$.
$\bar{\mathbf{x}}$ *is the minimizing point iff* $(\mathbf{x} - \bar{\mathbf{x}})'(\bar{\mathbf{x}} - \mathbf{y}) \geqslant 0$ *for all* $\mathbf{x} \in S$.

Note that $(\mathbf{y} - \bar{\mathbf{x}})'(\bar{\mathbf{x}} - \mathbf{x}) \geqslant 0$ for all $\mathbf{x} \in S$ is equivalent inequality.



Proof

Let $\gamma = \inf\{\| \mathbf{y} - \mathbf{x} \| \mid \mathbf{x} \in S\}$. There exists a sequence $(\mathbf{x}_k)$ such that $\| \mathbf{y} - \mathbf{x}_k \| \to \gamma$. $(\mathbf{x}_k)$ is a Cauchy sequence because, by the parallelogram law:

$$\begin{aligned} \| \mathbf{x}_k - \mathbf{x}_m \|^2 &= 2 \| \mathbf{x}_k - \mathbf{y} \|^2 + 2 \| \mathbf{x}_m - \mathbf{y} \|^2 - \| \mathbf{x}_k + \mathbf{x}_m - 2\mathbf{y} \|^2 \\ &= 2 \| \mathbf{x}_k - \mathbf{y} \|^2 + 2 \| \mathbf{x}_m - \mathbf{y} \|^2 - 4\left\| \frac{\mathbf{x}_k + \mathbf{x}_m}{2} - \mathbf{y} \right\|^2. \end{aligned}$$

Note that $\frac{\mathbf{x}_k + \mathbf{x}_m}{2} \in S$ and by the definition of γ,

$$\left\| \frac{\mathbf{x}_k + \mathbf{x}_m}{2} - \mathbf{y} \right\|^2 \geqslant \gamma^2,$$

so

$$\| \mathbf{x}_k - \mathbf{x}_m \|^2 \leq 2 \| \mathbf{x}_k - \mathbf{y} \|^2 + 2 \| \mathbf{x}_m - \mathbf{y} \|^2 - 4\gamma^2.$$

By choosing k and m, large $\| \mathbf{x}_k - \mathbf{y} \|^2$ and $\| \mathbf{x}_m - \mathbf{y} \|^2$ can be made close to γ^2, so $\| \mathbf{x}_k - \mathbf{x}_m \|^2$ can be made very small. Therefore, $(\mathbf{x}_k)$ is Cauchy and has a limit $\overline{\mathbf{x}}$.

Proof (cont'd)

Since S is closed, $\overline{\mathbf{x}} \in S$. To show uniqueness suppose that $\overline{z} \in S$ is such that $\| \mathbf{y} - \mathbf{x} \| = \| \mathbf{y} - \mathbf{z} \| = \gamma$. By convexity, $\frac{\mathbf{x}+\mathbf{z}}{2} \in S$, so

$$\left\| \mathbf{y} - \frac{\mathbf{x}+\mathbf{z}}{2} \right\| \leq \frac{1}{2} \| \mathbf{y} - \mathbf{x} \| + \frac{1}{2} \| \mathbf{y} - \mathbf{z} \| = \gamma.$$

The inequality cannot be strict for this would violate the definition of γ, so equality holds, and this implies $\mathbf{y} - \mathbf{x} = a(\mathbf{y} - \mathbf{z})$ for some a. Since $\| \mathbf{y} - \mathbf{x} \| = \| \mathbf{y} - \mathbf{z} \| = \gamma$ we must have $|a| = 1$. Clearly, $a \neq -1$ because otherwise we would have $\mathbf{y} = \frac{\mathbf{x}+\mathbf{z}}{2} \in S$ contradicting the fact that $\mathbf{y} \not\in S$. So $a = 1$ and $\mathbf{x} = \mathbf{z}$, which establishes the uniqueness.

Proof (cont'd)

Finally, we prove that $(\mathbf{x} - \overline{\mathbf{x}})'(\overline{\mathbf{x}} - \mathbf{y}) \geqslant 0$ for all $\mathbf{x} \in S$ characterizes $\overline{\mathbf{x}}$.
The inequality is sufficient:
For $\mathbf{x} \in S$ we can write:

$$\| \mathbf{y} - \mathbf{x} \|^2 = \| \mathbf{y} - \overline{\mathbf{x}} + \overline{\mathbf{x}} - \mathbf{x} \|^2 = \| \mathbf{y} - \overline{\mathbf{x}} \|^2 + \| \overline{\mathbf{x}} - \mathbf{x} \|^2 + 2(\mathbf{y} - \overline{\mathbf{x}})'(\overline{\mathbf{x}} - \mathbf{x}).$$

Since $\| \overline{\mathbf{x}} - \mathbf{x} \|^2 \geqslant 0$ and $(\mathbf{y} - \overline{\mathbf{x}})'(\overline{\mathbf{x}} - \mathbf{x}) \geqslant 0$, it follows that
$\| \mathbf{y} - \mathbf{x} \|^2 \geqslant \| \mathbf{y} - \overline{\mathbf{x}} \|^2$ and $\overline{\mathbf{x}}$ the minimizing point.

Proof (cont'd)

The inequality is necessary:

Assume that $\| \mathbf{y} - \mathbf{x} \|^2 \geqslant \| \mathbf{y} - \overline{\mathbf{x}} \|^2$ for all $\mathbf{x} \in S$.

For a sufficiently small and positive a we have $\overline{\mathbf{x}} + a(\mathbf{x} - \overline{\mathbf{x}}) \in S$. Therefore,

$$\| \mathbf{y} - \overline{\mathbf{x}} - a(\mathbf{x} - \overline{\mathbf{x}}) \|^2 \geq \| \mathbf{y} - \overline{\mathbf{x}} \|^2 .$$

Also,

$$\| \mathbf{y} - \overline{\mathbf{x}} + a(\overline{\mathbf{x}} - \mathbf{x}) \|^2 = \| \mathbf{y} - \overline{\mathbf{x}} \|^2 + a^2 \| \overline{\mathbf{x}} - \mathbf{x} \|^2 + 2a(\mathbf{y} - \overline{\mathbf{x}})'(\overline{\mathbf{x}} - \mathbf{x}).$$

Thus,

$$a^2 \| \mathbf{x} - \overline{\mathbf{x}} \|^2 + 2a(\mathbf{y} - \overline{\mathbf{x}})'(\overline{\mathbf{x}} - \mathbf{x}) \geqslant 0$$

for $a > 0$ and sufficiently small. This implies $(\mathbf{y} - \overline{\mathbf{x}})'(\overline{\mathbf{x}} - \mathbf{x}) \geqslant 0$ for all $\mathbf{x} \in S$.

Definition

Let S_1, S_2 be two non-empty subsets of $\mathbb{R}^n$.
A hyperplane $\mathbf{w}'\mathbf{x} = 0$ *separates* S_1 from S_2 if

$$\mathbf{w}'\mathbf{x} \geqslant 0 \text{ for } \mathbf{x} \in S_1 \text{ and } \mathbf{w}'\mathbf{x} \leqslant 0 \text{ for } \mathbf{x} \in S_2.$$

$\mathbf{w}'\mathbf{x} = 0$ *strictly separates* S_1 from S_2 if

$$\mathbf{w}'\mathbf{x} > 0 \text{ for } \mathbf{x} \in S_1 \text{ and } \mathbf{w}'\mathbf{x} < 0 \text{ for } \mathbf{x} \in S_2.$$

$\mathbf{w}'\mathbf{x} = 0$ *strongly separates* S_1 from S_2 if there exists ϵ such that

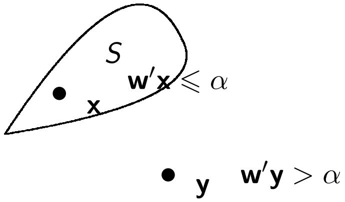
$$\mathbf{w}'\mathbf{x} \geqslant \alpha + \epsilon \text{ for } \mathbf{x} \in S_1 \text{ and } \mathbf{w}'\mathbf{x} \leqslant \alpha \text{ for } \mathbf{x} \in S_2.$$

Separation of a Convex Set and a Point

Theorem

Let $S \subseteq \mathbb{R}^n$ be a non-empty convex set and let $\mathbf{y} \notin S$. There exists $\mathbf{w} \neq \mathbf{0}$ and $\alpha \in \mathbb{R}$ such that

$$\mathbf{w}'\mathbf{y} > \alpha \text{ and } \mathbf{w}'\mathbf{x} \leqslant \alpha \text{ for } \mathbf{x} \in S.$$



Proof

There exists a unique minimizing point $\overline{\mathbf{x}} \in S$ such that

$$(\mathbf{x} - \overline{\mathbf{x}})'(\mathbf{y} - \overline{\mathbf{x}}) \leqslant 0$$

for $\mathbf{x} \in S$. Define $\mathbf{w} = \mathbf{y} - \overline{\mathbf{w}}$. Then, $(\mathbf{x} - \overline{\mathbf{x}})'\mathbf{w} \leqslant 0$, or $-\overline{\mathbf{x}})'\mathbf{w} \leqslant -\mathbf{x}'\mathbf{w}$. Therefore,

$$\mathbf{y}'\mathbf{w} - \mathbf{x}'\mathbf{w} \geqslant \mathbf{y}'\mathbf{w} - \overline{\mathbf{x}}'\mathbf{w} = \parallel \mathbf{w} \parallel^2,$$

which implies

$$\mathbf{y}'\mathbf{w} \geqslant \mathbf{x}'\mathbf{w} + \parallel \mathbf{w} \parallel^2 .$$

If $\alpha = \sup\{\mathbf{w}'\mathbf{x} \mid \mathbf{x} \in S\}$, the result follows.

Corollary

Every closed and convex subset of $\mathbb{R}^n$ equals the intersection of all half-spaces that contain S.

Proof.

Let $\mathcal{H}_S$ be the class of half-spaces that contain S. It is clear that $S \subseteq \bigcap \mathcal{H}_S$.

Conversely, let $\mathbf{y} \in \bigcap \mathcal{H}_S$. If $\mathbf{y} \notin S$ there exists a half-space that contains S but not $\mathbf{y}$, which is a contradiction. □

Farkas' Theorem

Theorem

Let $A \in \mathbb{R}^{m\times n}$ and let $\mathbf{c} \in \mathbb{R}^n$. Then, exactly one of the following two systems has a solution:

$$\begin{aligned}&\textit{System 1:} A\mathbf{x} \leqslant \mathbf{0} \textit{ and } \mathbf{c}'\mathbf{x} > 0, \textit{ for some } \mathbf{x} \in \mathbb{R}^n,\\ &\textit{System 2:} A'\mathbf{y} = \mathbf{c} \textit{ and } \mathbf{y} \geqslant 0, \textit{ for some } \mathbf{y} \in \mathbb{R}^m.\end{aligned}$$

Proof

Suppose that System 2 has a solution, that is, there exists $\mathbf{y}$ such that $A'\mathbf{y} = \mathbf{c}$ and $\mathbf{y} \geqslant 0$. Let $\mathbf{x}$ such that $A\mathbf{x} \leqslant \mathbf{0}$, so $\mathbf{c}'\mathbf{x} = \mathbf{y}'A\mathbf{x} \leqslant 0$, so System 1 has no solution.

Suppose now that System 2 has no solution and let $S = \{\mathbf{x} \mid \mathbf{x} = A'\mathbf{y}, \mathbf{y} \geqslant \mathbf{0}\}$. The set S is closed and convex and $\mathbf{c} \notin S$. By the previous separation theorem, there exists $\mathbf{w} \neq \mathbf{0}$ and $\alpha \in \mathbb{R}$ such that

$$\mathbf{w}'\mathbf{c} > \alpha \text{ and } \mathbf{w}'\mathbf{x} \leqslant \alpha \text{ for } \mathbf{x} \in S.$$

Since $\mathbf{0} \in S$, $\alpha \geq 0$, so $\mathbf{w}'\mathbf{c} > 0$. Also,

$$\alpha \geqslant \mathbf{w}'A'\mathbf{y} = \mathbf{y}'A\mathbf{w}$$

for all $\mathbf{w} \geqslant \mathbf{0}$. Since $\mathbf{y}$ can be made arbitrarily large, the last inequality implies $A\mathbf{w} \leq \mathbf{0}$. The vector $\mathbf{w}$ is such that $A\mathbf{w} \leqslant \mathbf{0}$ and $\mathbf{w}'\mathbf{c} > 0$ (which is the same as $\mathbf{c}'\mathbf{w} > 0$), so System 1 has a solution.

Corollary

In Farkas' Theorem replace A by

$$\begin{pmatrix} A \\ -I_n \end{pmatrix} \in \mathbb{R}^{(m+n)\times n}.$$

By Farkas' Theorem exactly one of the following two systems has a solution:

$$\text{System 1: } \begin{pmatrix} A \\ -I_n \end{pmatrix} \mathbf{x} \leqslant \mathbf{0} \text{ and } \mathbf{c}'\mathbf{x} > 0, \text{ for some } \mathbf{x} \in \mathbb{R}^n,$$
$$\text{System 2:} (A' \quad -I_n)\mathbf{z} = \mathbf{c} \text{ and } \mathbf{z} \geqslant 0, \text{ for some } \mathbf{z} \in \mathbb{R}^{m+n}.$$

This is equivalent to

Corollary

Exactly one of the following two systems has a solution:

$$\textit{System 1:} A\mathbf{x} \leqslant \mathbf{0}, \mathbf{x} \leqslant \mathbf{0} \textit{ and } \mathbf{c}'\mathbf{x} > 0, \textit{ for some } \mathbf{x} \in \mathbb{R}^n,$$
$$\textit{System 2:} A'\mathbf{y} \geqslant \mathbf{c}, \mathbf{y} \geqslant \mathbf{0}.$$

Corollary

Let $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{\ell \times n}$ and $\mathbf{c} \in \mathbb{R}^n$. In Farkas' Theorem replace A by

$$\begin{pmatrix} A \\ B \\ -B \end{pmatrix} \in \mathbb{R}^{(m+2\ell)\times n}.$$

Corollary

Exactly one of the following two systems has a solution:

$$\textit{System 1:} A\mathbf{x} \leqslant \mathbf{0}, B\mathbf{x} = \mathbf{0}, \textit{ and } \mathbf{c}'\mathbf{x} > 0, \textit{ for some } \mathbf{x} \in \mathbb{R}^n,$$
$$\textit{System 2:} A'\mathbf{y} + B'\mathbf{z} = \mathbf{c}, \mathbf{y} \geqslant \mathbf{0}.$$

Definition

Let $S \subseteq \mathbb{R}^n$ and let $\overline{\mathbf{x}} \in \partial S$. A *supporting hyperplane of S at $\overline{\mathbf{x}}$* is a hyperplane $\mathbf{p}'(\mathbf{x} - \overline{\mathbf{x}}) = 0$ if

- either $S \subset H^+$, that is, $\mathbf{p}'(\mathbf{x} - \overline{\mathbf{x}}) \geqslant 0$ for $\mathbf{x} \in S$, or
- $S \subset H^-$, that is, $\mathbf{p}'(\mathbf{x} - \overline{\mathbf{x}}) \leqslant 0$ for $\mathbf{x} \in S$.

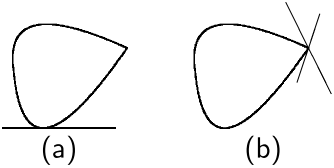
If $S \not\subset H$, then H is a *proper supporting hyperplane* of S.

Equivalent definition of supporting hyperplane

Definition

The hyperplane $\mathbf{p}'(\mathbf{x} - \overline{\mathbf{x}}) = 0$ is a supporting hyperplane of S at $\overline{\mathbf{x}} \in \partial S$ if $\mathbf{p}'\overline{\mathbf{x}} = \inf\{\mathbf{p}'\mathbf{x} \mid \mathbf{x} \in S\}$, or $\mathbf{p}'\overline{\mathbf{x}} = \sup\{\mathbf{p}'\mathbf{x} \mid \mathbf{x} \in S\}$.

Examples



(a): unique supporting hyperplane
(b): multiple supporting hyperplanes

Theorem

Let S be a convex subset of $\mathbb{R}^n$ and let $\overline{\mathbf{x}} \in \partial S$. There exists a hyperplane that supports S at $\overline{\mathbf{x}}$. In other words, there exists $\mathbf{p} \neq \mathbf{0}$ such that $\mathbf{p}'(\mathbf{x} - \overline{\mathbf{x}}) \leqslant 0$ for each $\mathbf{x} \in \mathbf{K}(S)$.

Proof

Since $\bar{\mathbf{x}} \in \partial S$, there exists a sequence $(\mathbf{y}_k)$ in S such that

$$\lim_{k \to \infty} \mathbf{y}_k = \bar{\mathbf{x}}.$$

By Theorem on slide 20, there exists $\mathbf{p}_k$ with $\parallel \mathbf{p}_k \parallel = 1$ such that $\mathbf{p}'\mathbf{y}_k > \mathbf{p}'\mathbf{x}$ for each $\mathbf{x} \in \mathbf{K}(S)$.
Since the sequence $(\mathbf{p}_k)$ is bounded, it contains a convergent subsequence with limit $\mathbf{p}$ with $\parallel \mathbf{p} \parallel = 1$. Taking the limit $\mathbf{p}$ of this subsequence we get $\mathbf{p}'(\mathbf{x} - \bar{\mathbf{x}}) \leqslant 0$ for each $\mathbf{x} \in \mathbf{K}(S)$.

Separation of Two Convex Sets

Theorem

Let S_1, S_2 be two subsets of $\mathbb{R}^n$ such that $S_1 \cap S_2 = \emptyset$. There exists a hyperplane that separates S_1 and S_2, that is, there exists $\mathbf{p} \neq \mathbf{0}$ such that

$$\inf\{\mathbf{p}'\mathbf{x} \mid \mathbf{x} \in S_1\} \geqslant \sup\{\mathbf{p}'\mathbf{x} \mid \mathbf{x} \in S_2\}.$$

Proof

The set $S = \{\mathbf{x} - \mathbf{y} \mid \mathbf{x} \in S_1 \text{ and } \mathbf{y} \in S_2\}$ is convex and $\mathbf{0} \notin S$.
There exists $\mathbf{p} \neq \mathbf{0}$ such that $\mathbf{p}'\mathbf{x} \geqslant 0$ for all $\mathbf{x} \in S$, so $\mathbf{p}'\mathbf{x}_1 \geqslant \mathbf{p}'\mathbf{x}_2$ for all $\mathbf{x}_1 \in S_1$ and $\mathbf{x}_2 \in S_2$, which implies the result.