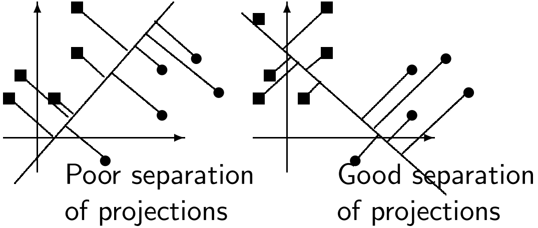
FISHER LINEAR DISCRIMINANT

Prof. Dan A. Simovici

UMB

Fisher linear discriminant (FLD) seeks to find projections on a line such that the projections of examples from different samples are well separated.



Recall that for a sequence $U = (\mathbf{u}_1, \ldots, \mathbf{u}_m)$ of vectors in $\mathbb{R}^n$ we denoted by $\tilde{U} \in \mathbb{R}^n$ the mean

$$\tilde{U} = \frac{1}{m}\sum_{i=1}^{m} \mathbf{u}_i = \frac{1}{m}(\mathbf{u}_1 \ \cdots \ \mathbf{u}_m)\mathbf{1}_m,$$

where $X = \begin{pmatrix} \mathbf{u}'_1 \\ \vdots \\ \mathbf{u}'_m \end{pmatrix} \in \mathbb{R}^{m \times n}$.

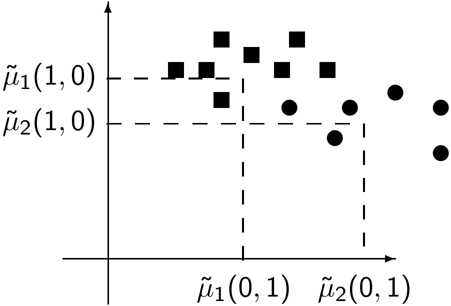
Equivalently, we have the row vector $\tilde{U}' = \frac{1}{m}\mathbf{1}'_m X$.

Let $\mathbf{d}$ be a unit vector; this defines the line used to project the vectors on.

Suppose that U is partitioned into two sequences, $U = (U_1, U_2)$ that contain m_1 and m_2 vectors, respectively. Let $\tilde{U}_1, \tilde{U}_2$ be the means of the sequences.
$\mu_1(\mathbf{d}) = \mathbf{d}'\tilde{U}_1$ and $\mu_2(\mathbf{d}) = \mathbf{d}'\tilde{U}_2$ are the projections of the means of the sequences on $\mathbf{d}$.
Is $|\mu_1(\mathbf{d}) - \mu_2(\mathbf{d})|$ a good separation measure between the set of projections?

Not necessarily!

The main problem is that $\tilde{\mu_1}(\mathbf{d})$ and $\tilde{\mu_2}(\mathbf{d})$ ignore the variances within the classes.



Even though

$$\tilde{\mu}_1(1,0) - \tilde{\mu}_2(1,0) < \tilde{\mu}_1(0,1) - \tilde{\mu}_2(0,1),$$

the separation on the vertical axis is better than the one on the horizontal axis.

Let $X \in \mathbb{R}^{m \times n}$ be a sample matrix and let $\hat{X} = H_m X$ be its centered form.
The scatter matrix of X is the matrix $S(X) = \hat{X}'\hat{X} \in \mathbb{R}^{n \times n}$.
The scatter of X in direction $\mathbf{d} \in \mathbb{R}^n$ is the number $s(X)_{\mathbf{d}}$ defined by
$s(X)^2_{\mathbf{d}} = \mathbf{d}'S(X)\mathbf{d}$.
We have

$$\begin{aligned} s(X)^2_{\mathbf{d}} &= \mathbf{d}'S(X)\mathbf{d} \\ &= \mathbf{d}'\hat{X}'\hat{X}\mathbf{d} = (\hat{X}\mathbf{d})'\hat{X}\mathbf{d}. \end{aligned}$$

Within class scatter matrix

Let $X_1 \in \mathbb{R}^{m_1 \times n}$ and $X_2 \in \mathbb{R}^{m_2 \times n}$ be two data matrices that refer to m_1 and m_2 experiments and the same set of n variables.
The within class scatter matrix of the two data sets is the matrix $S_w(X_1, X_2) = S(X_1) + S(X_2)$. Note that this is a correct definition because we have both $S(X_1), S(X_2) \in \mathbb{R}^{n \times n}$.
The total scatter in direction **d** is

$$s(X_1)^2_{\mathbf{d}} + s(X_1)^2_{\mathbf{d}} = \mathbf{d}' S(X_1)\mathbf{d} + \mathbf{d}' S(X_2)\mathbf{d}.$$

Between class scatter matrix

The between class scatter matrix of X_1 and X_2 is

$$S_b(X_1, X_2) = (\tilde{U}_1 - \tilde{U}_2)(\tilde{U}_1 - \tilde{U}_2)' \in \mathbb{R}^{n \times n}.$$

Again, this is correctly defined because $\tilde{U}_1$ and $\tilde{U}_2$ belong to $\mathbb{R}^n$.
S_b measures the separation between the means of the two classes.

Note that

$$\begin{aligned}
\mathbf{d}' S_b \mathbf{d} &= \mathbf{d}'(\tilde{U}_1 - \tilde{U}_2)(\tilde{U}_1 - \tilde{U}_2)'\mathbf{d} \\
&= (\mathbf{d}'\tilde{U}_1 - \mathbf{d}'\tilde{U}_2)(\tilde{U}_1'\mathbf{d} - \tilde{U}_2'\mathbf{d})' \\
&= (\mu_1(\mathbf{d}) - \mu_2(\mathbf{d}))^2.
\end{aligned}$$

The objective function is

$$J(\mathbf{d}) = \frac{(\mu_1(\mathbf{d}) - \mu_2(\mathbf{d}))^2}{s(X_1)^2_{\mathbf{d}} + s(X_1)^2_{\mathbf{d}}} = \frac{(\mu_1(\mathbf{d}) - \mu_2(\mathbf{d}))^2}{\mathbf{d}'S(X_1)\mathbf{d} + \mathbf{d}'S(X_2)\mathbf{d}} = \frac{\mathbf{d}'S_b\mathbf{d}}{\mathbf{d}'S_w\mathbf{d}},$$

which should be maximized.

We have

$$J'(\mathbf{d}) = \frac{(\mathbf{d}'S_b\mathbf{d})'(\mathbf{d}'S_w\mathbf{d}) - (\mathbf{d}'S_b\mathbf{d})(\mathbf{d}'S_w\mathbf{d})'}{(\mathbf{d}'S_w\mathbf{d})^2}$$
$$= \frac{(2S_b\mathbf{d})(\mathbf{d}'S_w\mathbf{d}) - (2S_w\mathbf{d})(\mathbf{d}'S_b\mathbf{d})}{(\mathbf{d}'S_w\mathbf{d})^2} = 0$$

So, we have

$$(2S_b\mathbf{d})(\mathbf{d}'S_w\mathbf{d}) - (2S_w\mathbf{d})(\mathbf{d}'S_b\mathbf{d}) = 0,$$

or

$$S_b\mathbf{d} = S_w\mathbf{d}\frac{\mathbf{d}'S_b\mathbf{d}}{\mathbf{d}'S_w\mathbf{d}},$$

which amounts to solving a generalized eigenvalue problem $S_b\mathbf{d} = \lambda S_w\mathbf{d}$, where $\lambda = \frac{\mathbf{d}'S_b\mathbf{d}}{\mathbf{d}'S_w\mathbf{d}}$.
If S_w has full rank, this amounts to a standard eigenvalue problem

$$S_w^{-1}S_b\mathbf{d} = \lambda\mathbf{d}.$$

Fisher discriminant method consists of finding a direction **d** such that

- $\mu_1(\mathbf{d}) - \mu_2(\mathbf{d})$ is maximal, and
- $s(X_1)^2_{\mathbf{d}} + s(X_1)^2_{\mathbf{d}}$ is minimal.

This is obtained by choosing **d** to be an eigenvector of the matrix $S_w^{-1} S_b$: classes will be well separated.

We have

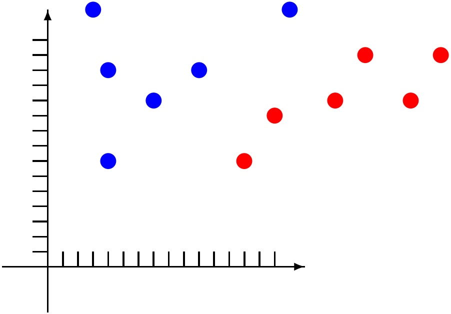
$$\begin{aligned} S_b\mathbf{d} &= (\tilde{U}_1 - \tilde{U}_2)(\tilde{U}_1 - \tilde{U}_2)'\mathbf{d} \\ &= (\tilde{U}_1 - \tilde{U}_2)\left((\tilde{U}_1 - \tilde{U}_2)'\mathbf{d}\right), \end{aligned}$$

which implies that $S_b\mathbf{d}$ is colinear with $\tilde{U}_1 - \tilde{U}_2$ for any vector $\mathbf{d}$. Therefore, the eigenvalue problem has the immediate solution $\mathbf{d} = S_w^{-1}(\tilde{U}_1 - \tilde{U}_2)$

Example

Consider the following subsets of $\mathbb{R}^2$:

$$S_1 = \left\{ \begin{pmatrix} 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 4 \\ 13 \end{pmatrix}, \begin{pmatrix} 3 \\ 17 \end{pmatrix}, \begin{pmatrix} 7 \\ 11 \end{pmatrix}, \begin{pmatrix} 10 \\ 13 \end{pmatrix}, \begin{pmatrix} 16 \\ 17 \end{pmatrix} \right\},$$
$$S_2 = \left\{ \begin{pmatrix} 13 \\ 7 \end{pmatrix}, \begin{pmatrix} 16 \\ 10 \end{pmatrix}, \begin{pmatrix} 20 \\ 11 \end{pmatrix}, \begin{pmatrix} 23 \\ 11 \end{pmatrix}, \begin{pmatrix} 22 \\ 13 \end{pmatrix}, \begin{pmatrix} 25 \\ 13 \end{pmatrix} \right\}.$$



Data matrices of the two data sets are

$$X_1 = \begin{pmatrix} 4 & 7 \\ 4 & 13 \\ 3 & 17 \\ 7 & 11 \\ 10 & 13 \\ 16 & 17 \end{pmatrix} \text{ and } X_2 = \begin{pmatrix} 13 & 7 \\ 16 & 10 \\ 20 & 11 \\ 23 & 11 \\ 22 & 13 \\ 25 & 13 \end{pmatrix}$$

The corresponding mean vectors are

$$\tilde{U}_1' = \frac{1}{6} * \mathbf{1}_6' X_1 = (7.33 \;\; 13.00) \text{ and } \tilde{U}_2' = \frac{1}{6} * \mathbf{1}_6' X_2 = (19.83 \;\; 10.83)$$

The Centered Data Matrices

We have

$$\begin{aligned}
\hat{X}_1 &= H_6 X_1 = \begin{pmatrix} 0.833 & -0.167 & -0.167 & -0.167 & -0.167 & -0.167 \\ -0.167 & 0.833 & -0.167 & -0.167 & -0.167 & -0.167 \\ -0.167 & -0.167 & 0.833 & -0.167 & -0.167 & -0.167 \\ -0.167 & -0.167 & -0.167 & 0.833 & -0.167 & -0.167 \\ -0.167 & -0.167 & -0.167 & -0.167 & 0.833 & -0.167 \\ -0.167 & -0.167 & -0.167 & -0.167 & -0.167 & 0.833 \end{pmatrix} X_1 \\
&= \begin{pmatrix} -3.333 & -6 \\ -3.333 & 0 \\ -4.333 & 4 \\ -0.333 & -2 \\ 2.666 & 0 \\ 8.666 & 4 \end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
\hat{X}_2 &= H_6 X_2 = \begin{pmatrix} 0.833 & -0.167 & -0.167 & -0.167 & -0.167 & -0.167 \\ -0.167 & 0.833 & -0.167 & -0.167 & -0.167 & -0.167 \\ -0.167 & -0.167 & 0.833 & -0.167 & -0.167 & -0.167 \\ -0.167 & -0.167 & -0.167 & 0.833 & -0.167 & -0.167 \\ -0.167 & -0.167 & -0.167 & -0.167 & 0.833 & -0.167 \\ -0.167 & -0.167 & -0.167 & -0.167 & -0.167 & 0.833 \end{pmatrix} X_2 \\
&= \begin{pmatrix} -6.833 & -3.833 \\ -3.833 & -0.833 \\ 0.167 & 0.167 \\ 3.167 & 0.167 \\ 2.167 & 2.167 \\ 5.167 & 2.167 \end{pmatrix}.
\end{aligned}
$$

The Scatter Matrices

The scatter matrix of X_1 is the matrix $S(X_1) = \hat{X}_1' \hat{X}_1 \in \mathbb{R}^{2\times 2}$ given by

$$S(X_1) = \hat{X}_1' \hat{X}_1 = \begin{pmatrix} 123.333 & 38 \\ 38.000 & 72 \end{pmatrix}$$

and

$$S(X_2) = \hat{X}_2' \hat{X}_2 = \begin{pmatrix} 102.833 & 45.833 \\ 45.833 & 24.833 \end{pmatrix}.$$

The within class scatter matrix is

$$S_w(X_1, X_2) = S(X_1) + S(X_2) = \begin{pmatrix} 226.167 & 83.833 \\ 83.833 & 96.833 \end{pmatrix}.$$

The between class scatter matrix

Since

$$\tilde{U}_1' = \frac{1}{6} * \mathbf{1}_6' X_1 = (7.33 \;\; 13.00) \text{ and } \tilde{U}_2' = \frac{1}{6} * \mathbf{1}_6' X_2 = (19.83 \;\; 10.83)$$

we have

$$\tilde{U}_1 - \tilde{U}_2 = \begin{pmatrix} -12.5 \\ 2.167 \end{pmatrix}$$

which implies

$$\begin{aligned} S_b &= (\tilde{U}_1 - \tilde{U}_2)(\tilde{U}_1 - \tilde{U}_2)' \\ &= \begin{pmatrix} -12.5 \\ 2.167 \end{pmatrix} (-12.5 \;\; 2.167) = \begin{pmatrix} 156.250 & -27.083 \\ -27.083 & 4.694 \end{pmatrix}. \end{aligned}$$

The inverse of the matrix

$$S_w = \begin{pmatrix} 226.167 & 83.833 \\ 83.833 & 96.833 \end{pmatrix}$$

is

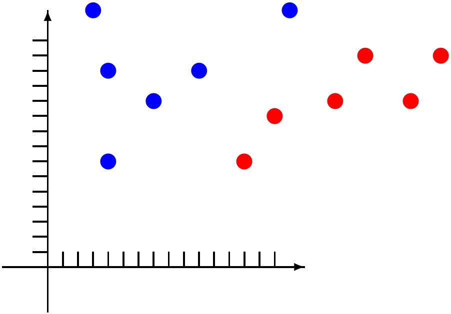
$$S_w^{-1} = \begin{pmatrix} 0.0065 & -0.0056 \\ -0.0056 & 0.0152 \end{pmatrix}$$

The optimum direction of the projection is

$$\begin{aligned} \mathbf{d} &= S_w^{-1}(\tilde{U}_1 - \tilde{U}_2) \\ &= \begin{pmatrix} -0.0936 \\ 1.034 \end{pmatrix}. \end{aligned}$$

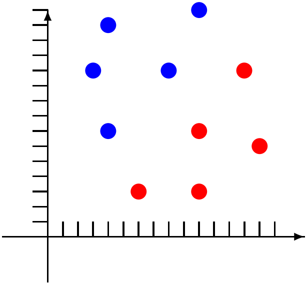
The projections of the data on the optimum direction are:

$$X_1\mathbf{d} = \begin{pmatrix} 0.3495 \\ 0.9699 \\ 1.4772 \\ 0.4823 \\ 0.4083 \\ 0.2604 \end{pmatrix} \text{ and } X_2\mathbf{d} = \begin{pmatrix} -0.4929 \\ -0.4635 \\ -0.7345 \\ -1.0153 \\ -0.7149 \\ -0.9957 \end{pmatrix}$$



Consider a training data set that consists of the following sets of examples in $\mathbb{R}^2$:

$$
\begin{aligned}
S_1 &= \left\{ \begin{pmatrix} 5 \\ 4 \end{pmatrix}, \begin{pmatrix} 10 \\ 15 \end{pmatrix}, \begin{pmatrix} 4 \\ 11 \end{pmatrix}, \begin{pmatrix} 8 \\ 11 \end{pmatrix}, \begin{pmatrix} 4 \\ 7 \end{pmatrix} \right\}, \\
S_2 &= \left\{ \begin{pmatrix} 6 \\ 3 \end{pmatrix}, \begin{pmatrix} 10 \\ 3 \end{pmatrix}, \begin{pmatrix} 10 \\ 7 \end{pmatrix}, \begin{pmatrix} 14 \\ 6 \end{pmatrix}, \begin{pmatrix} 13 \\ 11 \end{pmatrix} \right\}.
\end{aligned}
$$



Data matrices of the two data sets are

$$X_1 = \begin{pmatrix} 5 & 4 \\ 10 & 15 \\ 4 & 11 \\ 8 & 11 \\ 4 & 7 \end{pmatrix} \text{ and } X_2 = \begin{pmatrix} 6 & 3 \\ 10 & 3 \\ 10 & 7 \\ 14 & 6 \\ 13 & 11 \end{pmatrix}$$

The corresponding mean vectors are

$$\tilde{U}_1' = \frac{1}{5} * \mathbf{1}_5' X_1 = (6.2 \ \ 9.6) \text{ and } \tilde{U}_2' = \frac{1}{5} * \mathbf{1}_5' X_2 = (10.6 \ \ 6).$$

The Centered Data Matrices

We have

$$\hat{X}_1 = H_5 X_1 = \begin{pmatrix} 0.8 & -0.2 & -0.2 & -0.2 & -0.2 \\ -0.2 & 0.8 & -0.2 & -0.2 & -0.2 \\ -0.2 & -0.2 & 0.8 & -0.2 & -0.2 \\ -0.2 & -0.2 & -0.2 & 0.8 & -0.2 \\ -0.2 & -0.2 & -0.2 & -0.2 & 0.8 \end{pmatrix} X_1 = \begin{pmatrix} -1.2 & -5.6 \\ 3.8 & 5.4 \\ -2.2 & 1.4 \\ 1.8 & 1.4 \\ -2.2 & -2.6 \end{pmatrix}$$

and

$$\hat{X}_2 = H_5 X_2 = \begin{pmatrix} 0.8 & -0.2 & -0.2 & -0.2 & -0.2 \\ -0.2 & 0.8 & -0.2 & -0.2 & -0.2 \\ -0.2 & -0.2 & 0.8 & -0.2 & -0.2 \\ -0.2 & -0.2 & -0.2 & 0.8 & -0.2 \\ -0.2 & -0.2 & -0.2 & -0.2 & 0.8 \end{pmatrix} X_2 = \begin{pmatrix} -4.6 & -3.0 \\ -0.6 & -3.0 \\ -0.6 & 1.0 \\ 3.4 & 0 \\ 2.4 & 5.0 \end{pmatrix}.$$

The Scatter Matrices

The scatter matrix of X_1 is the matrix $S(X_1) = \hat{X}_1'\hat{X}_1 \in \mathbb{R}^{2\times 2}$ given by

$$S(X_1) = \hat{X}_1'\hat{X}_1 = \begin{pmatrix} 28.8 & 32.4 \\ 32.4 & 71.2 \end{pmatrix}$$

and

$$S(X_2) = \hat{X}_2'\hat{X}_2 = \begin{pmatrix} 39.2 & 27 \\ 27 & 44 \end{pmatrix}.$$

The within class scatter matrix is

$$S_w(X_1, X_2) = S(X_1) + S(X_2) = \begin{pmatrix} 68 & 59.4 \\ 59.4 & 115.2 \end{pmatrix}.$$

The between class scatter matrix

Since

$$\tilde{U}_1' = \frac{1}{5} * \mathbf{1}_5' X_1 = (6.2 \;\; 9.6) \text{ and } \tilde{U}_2' = \frac{1}{5} * \mathbf{1}_5' X_2 = (10.6 \;\; 6),$$

we have

$$\tilde{U}_1 - \tilde{U}_2 = \begin{pmatrix} 6.2 \\ 9.6 \end{pmatrix} - \begin{pmatrix} 10.6 \\ 6 \end{pmatrix} = \begin{pmatrix} -4.4 \\ 3.6 \end{pmatrix},$$

which implies

$$\begin{aligned} S_b &= (\tilde{U}_1 - \tilde{U}_2)(\tilde{U}_1 - \tilde{U}_2)' \\ &= \begin{pmatrix} -4.4 \\ 3.6 \end{pmatrix} (-4.4 \;\; 3.6) = \begin{pmatrix} 19.36 & -15.84 \\ -15.84 & 12.96 \end{pmatrix}. \end{aligned}$$

The inverse of the matrix

$$S_w = \begin{pmatrix} 68 & 59.4 \\ 59.4 & 115.2 \end{pmatrix}$$

is

$$S_w^{-1} = \begin{pmatrix} 0.0268 & -0.0138 \\ -0.0138 & 0.0158 \end{pmatrix}$$

The optimum direction of the projection is

$$\begin{aligned} \mathbf{d} &= S_w^{-1}(\tilde{U}_1 - \tilde{U}_2) \\ &= \begin{pmatrix} 0.0268 & -0.0138 \\ -0.0138 & 0.0158 \end{pmatrix} \begin{pmatrix} 6.2 \\ 9.6 \end{pmatrix} = \begin{pmatrix} 0.0334 \\ 0.0661 \end{pmatrix}. \end{aligned}$$

The projections of the data on the optimum directions are:

$$X_1\mathbf{d} = \begin{pmatrix} 0.4316 \\ 1.3258 \\ 0.8607 \\ 0.9945 \\ 0.5964 \end{pmatrix} \text{ and } X_2\mathbf{d} = \begin{pmatrix} 0.3989 \\ 0.5327 \\ 0.7971 \\ 0.8648 \\ 1.1618 \end{pmatrix}.$$

Example

Consider the following subsets of $\mathbb{R}^2$:

$$\begin{aligned} S_1 &= \left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \end{pmatrix}, \begin{pmatrix} 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 4 \\ 9 \end{pmatrix} \right\}, \\ S_2 &= \left\{ \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 2.5 \\ 0.5 \end{pmatrix}, \begin{pmatrix} 6 \\ 6 \end{pmatrix}, \begin{pmatrix} 7.5 \\ 4.5 \end{pmatrix} \right\}. \end{aligned}$$

Data matrices of the two data sets are

$$X_1 = \begin{pmatrix} 1 & 3 \\ 1 & 5 \\ 4 & 7 \\ 4 & 9 \end{pmatrix} \text{ and } X_2 = \begin{pmatrix} 3 & 2 \\ 2.5 & 0.5 \\ 6 & 6 \\ 7.5 & 4.5 \end{pmatrix}$$

The corresponding mean vectors are

$$\tilde{U}_1' = \frac{1}{4} * \mathbf{1}_4' X_1 = (2.5 \;\; 6) \text{ and } \tilde{U}_2' = \frac{1}{4} * \mathbf{1}_4' X_2 = (4.75 \;\; 3.25)$$

The Centered Data Matrices

We have

$$
\begin{aligned}
\hat{X}_1 &= H_4 X_1 = \begin{pmatrix} 0.75 & -0.25 & -0.25 & -0.25 \\ -0.25 & 0.75 & -0.25 & -0.25 \\ -0.25 & -0.25 & 0.75 & -0.25 \\ -0.25 & -0.25 & -0.25 & 0.75 \end{pmatrix} X_1 \\
&= \begin{pmatrix} -1.50 & -3 \\ -1.50 & -1 \\ 1.50 & 1 \\ 1.50 & 3 \end{pmatrix}
\end{aligned}
$$

The Centered Data Matrices

We have

$$\begin{aligned}\hat{X}_2 &= H_4 X_1 = \begin{pmatrix} 0.75 & -0.25 & -0.25 & -0.25 \\ -0.25 & 0.75 & -0.25 & -0.25 \\ -0.25 & -0.25 & 0.75 & -0.25 \\ -0.25 & -0.25 & -0.25 & 0.75 \end{pmatrix} X_2 \\ &= \begin{pmatrix} -1.75 & -1.25 \\ -2.25 & -2.75 \\ 1.25 & 2.75 \\ 2.75 & 1.25 \end{pmatrix}.\end{aligned}$$

The Scatter Matrices

The scatter matrix of X_1 is the matrix $S(X_1) = \hat{X}_1' \hat{X}_1 \in \mathbb{R}^{2\times 2}$ given by

$$S(X_1) = \hat{X}_1' \hat{X}_1 = \begin{pmatrix} 9 & 12 \\ 12 & 20 \end{pmatrix}$$

and

$$S(X_2) = \hat{X}_2' \hat{X}_2 = \begin{pmatrix} 17.25 & 15.25 \\ 15.25 & 18.25 \end{pmatrix}.$$

The within class scatter matrix is

$$S_w(X_1, X_2) = S(X_1) + S(X_2) = \begin{pmatrix} 26.25 & 27.25 \\ 27.25 & 38.25 \end{pmatrix}.$$

The between class scatter matrix

Recall:

$$\tilde{U}_1' = \frac{1}{4} * \mathbf{1}_4' X_1 = (2.5 \;\; 6) \text{ and } \tilde{U}_2' = \frac{1}{4} * \mathbf{1}_4' X_2 = (4.75 \;\; 3.25)$$

Thus, we have

$$\tilde{U}_1 - \tilde{U}_2 = \begin{pmatrix} -2.25 \\ 2.75 \end{pmatrix}$$

which implies

$$\begin{aligned} S_b &= (\tilde{U}_1 - \tilde{U}_2)(\tilde{U}_1 - \tilde{U}_2)' \\ &= \begin{pmatrix} -2.25 \\ 2.75 \end{pmatrix} (-2.25 \;\; 2.75) = \begin{pmatrix} 5.0625 & -6.1875 \\ -6.1875 & 7.5625 \end{pmatrix}. \end{aligned}$$

The inverse of the matrix

$$S_w = \begin{pmatrix} 26.25 & 27.25 \\ 27.25 & 38.25 \end{pmatrix}.$$

is

$$S_w^{-1} = \begin{pmatrix} 0.1463 & -0.1042 \\ -0.1042 & 0.1004 \end{pmatrix}$$

The optimum direction of the projection is

$$\begin{aligned} \mathbf{d} &= S_w^{-1}(\tilde{U}_1 - \tilde{U}_2) \\ &= \begin{pmatrix} -0.6157 \\ 0.5195 \end{pmatrix}. \end{aligned}$$

The projections of the data on the optimum direction are:

$$X_1\mathbf{d} = \begin{pmatrix} -0.6080 \\ 0.4130 \\ -0.4130 \\ 0.6080 \end{pmatrix} \text{ and } X_2\mathbf{d} = \begin{pmatrix} 0.4393 \\ -0.0186 \\ 0.6343 \\ -1.0550 \end{pmatrix}$$

Multiple Discriminant Analysis

- this is a generalization of Fisher's discriminant to c classes;
- reduces the dimensionality of the data to $c - 1$.

Let V be the projection matrix on a $(c-1)$-dimensional space.
Notations:

- $\tilde{U}_i$ the mean of the n_i examples of class C_i,

$$\tilde{U}_i = \frac{1}{n_i} \sum \{\mathbf{x} \mid \mathbf{x} \in C_i\}.$$

  for $1 \leqslant i \leqslant c$;
- $\tilde{U}$: the mean of the total mean of samples;

Scatter matrices

- within class scatter matrix: $S_w = \sum_{1 \leqslant i \leqslant c} \hat{X}_i' \hat{X}_i$;
- between class scatter matrix: $S_b = \sum_{1 \leqslant i \leqslant c} n_i (\tilde{U}_i \tilde{U})(\tilde{U}_i \tilde{U})'$ (maximum rank is $c - 1$);
- objective function is

$$J(V) = \frac{V' S_b V}{V' S_w V}.$$

$$J(V) = \frac{V' S_b V}{V' S_w V}.$$

- solve the generalized eigenvalue problem $S_b \mathbf{v} = \lambda S_w \mathbf{v}$, which has at most $c - 1$ distinct eigenvalues;
- let $\mathbf{v}_1, \ldots, \mathbf{v}_{c-1}$ are the respective eigenvectors;
- the optimal projection matrix to a subspace of dimension k is given by the eigenvectors that correspond to the largest k eigenvalues.