

OPTIMIZATION - 0

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1 Unconstrained Optimization

2 Optimization with Inequality Constraints

Theorem

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a function differentiable in $\bar{\mathbf{x}}$. If there exists $\mathbf{d} \in \mathbb{R}^n$ such that $(\nabla f)(\bar{\mathbf{x}})' \mathbf{d} < 0$, then there exists $\delta > 0$ such that

$$f(\bar{\mathbf{x}} + \lambda \mathbf{d}) < f(\bar{\mathbf{x}})$$

for each $\lambda \in (0, \delta)$.

We say that \mathbf{d} is a **descent direction** of f at $\bar{\mathbf{x}}$.

Proof

By differentiability, we have

$$f(\bar{\mathbf{x}} + \lambda \mathbf{d}) = f(\bar{\mathbf{x}}) + \lambda(\nabla f)(\bar{\mathbf{x}})' \mathbf{d} + \lambda \|\mathbf{d}\| \alpha_{\bar{\mathbf{x}}}(\lambda \mathbf{d}),$$

where $\lim_{\lambda \rightarrow 0} \alpha_{\bar{\mathbf{x}}}(\lambda \mathbf{d}) = 0$.

This implies

$$\frac{f(\bar{\mathbf{x}} + \lambda \mathbf{d}) - f(\bar{\mathbf{x}})}{\lambda} = (\nabla f)(\bar{\mathbf{x}})' \mathbf{d} + \|\mathbf{d}\| \alpha_{\bar{\mathbf{x}}}(\lambda \mathbf{d}).$$

Since $(\nabla f)(\bar{\mathbf{x}})' \mathbf{d} < 0$ and $\lim_{\lambda \rightarrow 0} \alpha_{\bar{\mathbf{x}}}(\lambda \mathbf{d}) = 0$ it follows that there exists $\delta > 0$ such that $(\nabla f)(\bar{\mathbf{x}})' \mathbf{d} + \|\mathbf{d}\| \alpha_{\bar{\mathbf{x}}}(\lambda \mathbf{d}) < 0$ for all $\lambda \in (0, \delta)$. This gives the result.

A first-order condition for minimum

Corollary

If $\bar{\mathbf{x}}$ is a local minimum for the function f that is differentiable in $\bar{\mathbf{x}}$, then $(\nabla f)(\bar{\mathbf{x}}) = \mathbf{0}$.

Proof: Suppose that $(\nabla f)(\bar{\mathbf{x}}) \neq \mathbf{0}$. Define $\mathbf{d} = -(\nabla f)(\bar{\mathbf{x}})$, so $(\nabla f)(\bar{\mathbf{x}})' \mathbf{d} < 0$. By the previous theorem, there exists $\delta > 0$ such that $f(\bar{\mathbf{x}} + \lambda \mathbf{d}) < f(\bar{\mathbf{x}})$ for $\lambda \in (0, \delta)$, which contradicts the fact that $\bar{\mathbf{x}}$ is a local minimum.

A second-order condition for minimum

Theorem

Let $f : \mathbb{R}^n \rightarrow \mathbb{N}$ be a twice differentiable function. If $\bar{\mathbf{x}}$ is a local minimum for f , then $(\nabla f)(\bar{\mathbf{x}}) = 0$ and the Hessian matrix $H(\bar{\mathbf{x}})$ is positive semidefinite.

Proof

Since f is differentiable in $\bar{\mathbf{x}}$ we have

$$f(\bar{\mathbf{x}} + \lambda \mathbf{d}) = f(\bar{\mathbf{x}}) + \lambda(\nabla f)(\bar{\mathbf{x}})' \mathbf{d} + \frac{1}{2} \lambda^2 \mathbf{d}' H(\bar{\mathbf{x}}) \mathbf{d} + \lambda^2 \|\mathbf{d}\|^2 \alpha_{\bar{\mathbf{x}}}(\lambda \mathbf{d}),$$

where $\lim_{\lambda \rightarrow 0} \alpha_{\bar{\mathbf{x}}}(\lambda \mathbf{d}) = 0$. By the previous corollary,

$$\frac{f(\bar{\mathbf{x}} + \lambda \mathbf{d}) - f(\bar{\mathbf{x}})}{\lambda^2} = \frac{1}{2} \mathbf{d}' H(\bar{\mathbf{x}}) \mathbf{d} + \|\mathbf{d}\|^2 \alpha_{\bar{\mathbf{x}}}(\lambda \mathbf{d}),$$

Since $\bar{\mathbf{x}}$ is a local minimum, $\frac{f(\bar{\mathbf{x}} + \lambda \mathbf{d}) - f(\bar{\mathbf{x}})}{\lambda} \geq 0$, so $\frac{1}{2} \mathbf{d}' H(\bar{\mathbf{x}}) \mathbf{d} + \|\mathbf{d}\|^2 \alpha_{\bar{\mathbf{x}}}(\lambda \mathbf{d}) \geq 0$ for λ small. Taking $\lambda \rightarrow 0$, it follows that $\mathbf{d}' H(\bar{\mathbf{x}}) \mathbf{d} \geq 0$, so $H(\bar{\mathbf{x}})$ is positively semidefinite.

A Sufficient Condition for a Local Minimum

Theorem

*If f is a function that is twice differentiable at $\bar{\mathbf{x}}$, $(\nabla f)(\bar{\mathbf{x}}) = 0$ and $H(\bar{\mathbf{x}})$ is positive *definite*, then $\bar{\mathbf{x}}$ is a local minimum.*

Proof

Since f is twice differentiable in $\bar{\mathbf{x}}$,

$$f(\mathbf{x}) = f(\bar{\mathbf{x}}) + (\nabla f)\bar{\mathbf{x}}'(\mathbf{x} - \bar{\mathbf{x}}) + \frac{1}{2}(\mathbf{x} - \bar{\mathbf{x}})'H(\bar{\mathbf{x}})(\mathbf{x} - \bar{\mathbf{x}}) + \|\mathbf{x} - \bar{\mathbf{x}}\|^2 \alpha_{\bar{\mathbf{x}}}(\mathbf{x} - \bar{\mathbf{x}})$$

for $\mathbf{x} \in S$ and $\lim_{\mathbf{x} \rightarrow \bar{\mathbf{x}}} \alpha_{\bar{\mathbf{x}}}(\mathbf{x} - \bar{\mathbf{x}}) = 0$.

Suppose $\bar{\mathbf{x}}$ is not a local minimum, that is, there exists sequence $\mathbf{x}_n \rightarrow \bar{\mathbf{x}}$ such that $f(\mathbf{x}_k) < f(\bar{\mathbf{x}})$ for each \mathbf{x}_k . This implies that

$$(\mathbf{x}_k - \bar{\mathbf{x}})'H(\bar{\mathbf{x}})(\mathbf{x}_k - \bar{\mathbf{x}}) + \|\mathbf{x}_k - \bar{\mathbf{x}}\|^2 \alpha_{\bar{\mathbf{x}}}(\mathbf{x}_k - \bar{\mathbf{x}}) < 0$$

for each \mathbf{x}_k .

Proof (cont'd)

Let \mathbf{d}_k be the unit vector $\mathbf{d}_k = \frac{1}{\|\mathbf{x}_k - \bar{\mathbf{x}}\|}(\mathbf{x}_k - \bar{\mathbf{x}})$. We have

$$\frac{1}{2}\mathbf{d}'_k H(\bar{\mathbf{x}})\mathbf{d}_k + \alpha_{\bar{\mathbf{x}}}(\mathbf{x} - \bar{\mathbf{x}}) < 0$$

Since (\mathbf{d}_k) is a bounded sequence it contains a convergent subsequence (\mathbf{d}_{k_i}) such that $\lim_{i \rightarrow \infty} \mathbf{d}_{k_i} = \mathbf{d}$. This implies $\mathbf{d}' H(\bar{\mathbf{x}})\mathbf{d} \leq 0$, which contradicts the fact that $H(\bar{\mathbf{x}})$ is positive definite.

Let $f = (x^2 - 1)^3$. We have

$$(\nabla f)(x) = f'(x) = 6x(x^2 - 1)^2$$

and $H(x) = f''(x) = 6(x^2 - 1)^2 + 24x^2(x^2 - 1) = 6(x^2 - 1)(x^2 - 1 + 4x^2) = 6(x^2 - 1)(5x^2 - 1)$.

We have $f'(x) = 0$ for $x_1 = 0$, $x_2 = 1$ and $x_3 = -1$. Furthermore, $f''(0) = 6$, $f''(1) = f''(-1) = 0$.

A local minimum exists at $x = 0$; -1 and 1 fail to be minima because $f''(x) = 0$ in both cases.

Feasible Directions

Definition

Let $S \subseteq \mathbb{R}^n$ and let $\bar{\mathbf{x}} \in \mathbf{K}(S)$. The vector $\mathbf{d} \neq \mathbf{0}$ is a **feasible direction** in $\bar{\mathbf{x}}$ if there exists $\delta > 0$ such that $\bar{\mathbf{x}} + \lambda \mathbf{d} \in S$ for $0 < \lambda < \delta$.

Let

$$D = \{\mathbf{d} \mid \mathbf{d} \text{ is a feasible direction in } \bar{\mathbf{x}}\}$$
$$F_0 = \{\mathbf{d} \mid (\nabla f)(\bar{\mathbf{x}})' \mathbf{d} < 0\}.$$

- if $\mathbf{d} \in F_0$, \mathbf{d} is an improving direction: a small movement in the direction of \mathbf{d} away from $\bar{\mathbf{x}}$ reduces the value of f ;
- if $\bar{\mathbf{x}}$ is a local minimum, $\mathbf{d} \in F_0$, then $\mathbf{d} \notin D$.

Theorem

If $f : S \rightarrow \mathbb{R}$ is differentiable at \bar{x} and \bar{x} is a local optimum, then $D \cap F_0 = \emptyset$.

Feasible Region

Let $g_i : \mathbb{R}^n \rightarrow \mathbb{R}$, where $1 \leq i \leq m$. The *feasible region* is the set

$$S = \{\mathbf{x} \in \mathbb{R}^n \mid g_i(\mathbf{x}) \leq 0 \text{ for } 1 \leq i \leq m\}.$$

Optimization Problem:

minimize $f(\mathbf{x})$;

subject to $g_i(\mathbf{x}) \leq 0$ for $1 \leq i \leq m$.