Discrete Mathematics
Exam 1 Solutions

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October 16, 2014

The first question was worth 16 points. Each part of each succeeding question was worth 12 points, for a total of 100. (I didn't count the last optional hard question.)

I tried to be reasonably generous with part credit.

Here's the grade distribution, with approximate letter grade equivalents (you can't take those literally).

range	90-99	80-89	70-79	60-69	50-59	40-49	30-39	20-29
number	1	1	3	5	3	8	4	5
grade?	A	A	A-	B	C+	C	C-/D	D/F

1. Consider the word `MISSISSIPPI`. (It's a favorite for this kind of problem.) How many ways are there to permute the letters if all four `I`s or all four `S`s are together? (`IIII`, `SSSS`) .

   I should not have to remind you that in mathematics and in computer science, "or" means "and/or".

   Solution

   If I treat `IIII` as a single letter there are 8!/4!2! permutations. The factorials in the denominator take into account the fact that there are four `S`s and two `P`s.

   There are the same number of permutations containing `SSSS`.

   I can add those two answers to find the number of permutations containing one or the other – as long as I correct because I've double counted the permutations that contain both (inclusion-exclusion principle). There are 5!/2! of those, so my answer is

   $$2 \times \frac{8!}{4!2!} - \frac{5!}{2!} = 1680 - 60 = 1620.$$

   That's an interesting historical number: it's the year the pilgrims landed in Plymouth on the Mayflower.

   I gave partial credit for knowing how to calculate the number of permutations when some letters are repeated. I was surprised at how many students didn't see the need to worry about double counting the strings in which both blocks appeared. I'd hoped my hint would point you in that direction.

2. 18 people have gathered for dinner. Tables at the restaurant seat 6.

   (a) How many ways are there to divide the 18 people into three groups of 6?

   Solution

   I can choose the first group of 6 in 18!/6!12! ways, then the second group in 12!/6!6! ways. The last 6 people make the last group. Multiplying these independent choices gives me 18!/6!6!6! ways to choose the groups *in order*.

   Now in the problem statement the tables aren't mentioned. They are clearly indistinguishable. There is no way to assign a particular group to a particular table. That means choosing the same three groups in any other order would lead to the same division, so I need to divide by 3!. The answer is

   $$\frac{18!}{6!6!6!3!} = 21,237,216$$

(if I did the arithmetic right). On the exam you didn't have a calculator, so I didn't expect you to do the arithmetic at all.

I was a little concerned about whether I really needed to do that last division by 3! so I checked my answer by thinking about how I would divide three people into three groups of one person each. Clearly there's only one way!

Very few students divided by that last 3!. I gave partial credit for the answer $\binom{18}{6,6,6}$.

This is similar to the poker hands homework problem, easier since we're "dealing out the whole deck" but harder since the order in which the groups occur doesn't matter.

(b) If half the 18 people are women and half men, how many ways are there to divide them so that each table has the same number of women and men?

Solution

Using the same logic, I can divide up the women and men each in 9!/3!3!3!3! ways. I need to multiply those together and then multiply by 3! to take into account the different ways to pair triples of women with triples of men. The answer is

$$\frac{9!}{3!3!3!3!} \times \frac{9!}{3!3!3!3!} \times 3! = 235,200 \text{ ways.}$$

That's just about one percent of the previous answer.

Here's another way that comes to the same conclusion. After you choose the three groups of men in $9!/(3!)^4$ ways you choose the groups of women in $9!/(3!)^3$. You *don't* divide by the extra 3! since it matters which groups of women join which groups of men.

(c) How many ways are there to seat six people at a round table if all that matters is who each person's neighbors are?

Solution

If the six people were in a line there would be 6! permutations. Arranging that line in a circle I can no longer tell where the line started, so there are only 5! circular arrangements. But each of those arrangements can go around the table either clockwise or counterclockwise without changing who sits next to whom, so I have to divide by two. The answer is $5!/2 = 60$ ways.

3. Imagine a test with 20 questions where each question has two possible answers. They are not true/false questions – one or both or neither answer might be correct. Here is an example:

> Is discrete mathematics fun
>
> - for mathematics students?
> - for computer science students?

(a) Count the number of possible ways to answer all the questions on that test.

Solution

Each question has four possible answers so there are 4^{20} possible tests.

Many students thought there were just three possible answers. If you read the sample question in the box carefully you can see four:

- Everyone finds discrete math fun.
- Noone finds discrete math fun.
- Only math students like it.
- Only cs students like it.

(b) Which of the following is a good estimate of your count?

10^6 10^9 10^{12} 10^{15} 10^{18} none of these

Solution

$$4^{20} = 2^{40} = (2^{10})^4 \approx (10^3)^4 = 10^{12}.$$

Some of you did this with $\log_{10}(2) \approx 0.30103$, which is correct but overkill.

If you found 3^{20} as the (incorrect) answer to the previous question you could still get full credit here for 10^9.

If you got a ridiculously small or large wrong answer to the previous question you could earn free points here for "none of these".

4. A parking lot has n spaces in a row. Cars arrive and fill spaces at random. Then Auntie Em drives up in her SUV, which needs two adjacent spaces.

   (a) If exactly two spaces are free, what is the probability that she is able to park?

   Solution

   I need to compute

   $$\frac{\text{number of configurations with adjacent empty spaces}}{\text{number of ways to pick two empty spaces}}.$$

   To find the numerator, treat the two empty spaces as one item among $n-1$. There are $n-1$ places to put that pair. The denominator is just $\binom{n}{2}$ so the answer is

   $$\frac{n-1}{\frac{n(n-1)}{2}} = \frac{2}{n}.$$

   (b) Suppose f spaces are free. (In the previous question $f = 2$.) What is the minimum value of f that guarantees that Auntie Em can park? (Explain how you know you've answered the question correctly.)

   Solution

   Think of the parking lot as a bit string, with 0 for empty spaces.

   If Auntie Em can't park, then every 0 must be followed by a 1, except possibly at the end.

   If we want as many 0's as possible, we don't want two 1's together. That says that the bitstrings $0101\cdots01$ and $101\cdots10$ (for n even) and $0101\cdots010$ (for n odd) have the fewest 1s that can prevent Auntie Em. Therefore $f = 1 + n/2$ for n even and $f = 1 + (n+1)/2$ for n odd.

   (c) (Optional, hard, take home). Answer the first question if there are f empty spaces. (You've just done $f = 2$, and found values of f that make the probability 1.) Check that your answer agrees with your answer to the first question. Evaluate your expression when $n = 16$ and $f = 4$.

   Solution

   To answer this question I counted the configurations where she couldn't park. I considered two cases.

   - If the bitstring doesn't end with a 0 then every 0 must be followed by a 1. So think of building the string from f copies of (01) and the remaining $n - 2f$ copies of 1. That can be done $\binom{n-f}{f}$ ways.
   - For the bitstrings ending with 0 I need $f - 1$ copies of (01) and $n - 2f + 1$ more copies of 1. I can do that in $\binom{n-f}{f-1}$ ways.

   So Auntie Em can part with probability

   $$1 - \frac{\binom{n-f}{f} + \binom{n-f}{f-1}}{\binom{n}{2}} = 1 - \frac{\binom{n-f+1}{f}}{\binom{n}{2}}.$$

   Checking when $n = 2$:

$$1 - \frac{\binom{n-1}{2}}{\binom{n}{2}} = 1 - \frac{(n-1)(n-2)}{n(n-1)} = 1 - \frac{n-2}{n} = \frac{2}{n}.$$

Sho Inaba counts the ways Em can't park much more elegantly. First he observes that there must be at least one filled space between each of the f free spaces. That uses up $f-1$ filled spaces, so there are $n-2f+1$ more to insert somewhere. There are $f+1$ possible places to put them – before the first free space, between two, or after the last one. Then he sees that this is just the apples/bananas/cherries problem in disguise, so there are

$$\binom{(n-2f+1)+f}{f} = \binom{n-f+1}{f}$$

ways she can't park. That's all he needs to finish the problem, pretty much the way I did from that point.