History of Mathematics
Homework 3

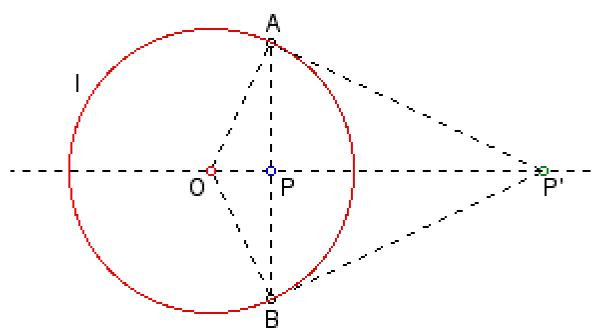
Ethan Bolker

March 31, 2014

This homework is due Thursday, February 20.

1. Let P and Q be two points in the unit disk. Find a Euclidean construction for the (hyperbolic/pseudo) line joining them. That is, find a straightedge-and-compass construction for the circle through those points that meets the unit circle at a right angle. (This is not easy. After trying it yourself (or with a classmate) feel free to try to find a solution on the web. If you do, you should write it out in your own words (and pictures) and tell me where and how you found it.)

   The clue to this problem is the idea of *inversion in a circle*. In this figure (from `http://jwilson.coe.uga.edu/emt668/emat6680.f99/challen/inversion/index.html`)



   P is the inverse of P'. $|OP'| = 1/|OP|$. It's not hard to prove that every circle that goes through P and P' is perpendicular to the circle in the figure. Now suppose Q is some other point in the circle. Draw the circle through P, Q and P'. The arc of that circle connecting P to Q is the hyperbolic line joining them.

2. Calculate the (hyperbolic/pseudo) distance from the center of the unit disk to a point at (Euclidean) distance $r < 1$ from the center.

   Recall that distances at a point P at distance $r < 1$ from the center of the unit disk are stretched by a factor $1/(1-r^2)$.

   Hint. Your argument should lead to an integral that you can evaluate using what you learned in Calculus. If you don't remember what you need you should be able to look it up.

   There's no loss of generality assuming that P is on the x-axis. The distance from x to $x + dx$ is $dx/(1 - x^2)$ so the distance to r is

$$\begin{aligned}\int_0^r \frac{dx}{1-x^2} &= \frac{1}{2}\int_0^r \left(\frac{1}{1+x} + \frac{1}{1-x}\right) dx \\ &= \frac{1}{2}(\log(1+x) - \log(1-x))\Big|_0^r \\ &= \frac{1}{2}\frac{\log(1+r)}{\log(1-r)}.\end{aligned}$$

3. Use your answer to the previous problem to show that lines through the center are infinitely long - in other words, that the distance from the center approaches ∞ as $r \to 1$.

   As $r \to 1$ the logarithm in the denominator of the distance formula above approaches 0 so the distance approaches ∞.

4. At http://math.stackexchange.com/questions/675522/whats-the-intuition-behind-pythagoras-theorem you will find a question and several answers about the Pythagorean Theorem. I read through them rapidly, but found no mention of the fact that the theorem depends on the parallel postulate.

   I think we should provide another answer to point that out. (I have enough reputation to do that.) For this homework, write something that I might post.

   Here's a hint. What's a square in spherical or hyperbolic geometry? Can you compute the area of a square by squaring the length of its side?

   I did post a short answer, with not much detail: http://math.stackexchange.com/questions/675522/whats-the-intuition-734277#734277

5. Build a paper model of the hyperbolic plane. There are instructions at http://euler.slu.edu/escher/index.php/Hyperbolic_Paper_Exploration.

   - Answer the two questions there.
   - Let ABC be one of the equilateral triangles you taped together to make your hyperbolic plane. Let D be the foot of the perpendicular from C to AB. Try to compute the angle of parallelism $\Pi(CD)$. (I do not know the answer. I don't even know whether the question makes sense. You should be able to find some upper and lower bounds for that angle by experiment.)

   I haven't built my model yet, but I will, with my seventh graders.

Here is the LaTeX source for this document. You can cut it from the pdf and use it to start your answers. I used the `\jobname` macro for the source file name, so you can call your file by any name you like.

```
% Math 370 hw3 Spring 2014
%
\documentclass{article}
\pagestyle{empty}
\usepackage[textheight=10in, textwidth=7.5in]{geometry}

\usepackage{amsmath}
\usepackage{amsthm}
\usepackage{hyperref}
\usepackage{graphicx}
\usepackage{verbatim}

%% create an environment for theorems
\newtheorem*{thm}{Theorem}

\newcommand{\coursehome}
{http://www.cs.umb.edu/~eb/370}

\title{History of Mathematics \\
Homework 3
}
\author{Ethan Bolker}

\begin{document}

\maketitle

This homework is due Thursday, February 20.

\begin{enumerate}

\item Let $P$ and $Q$ be two points in the unit disk. Find a Euclidean
  construction for the (hyperbolic/pseudo) line joining them. That is,
  find a straightedge-and-compass construction for the circle through
  those points that meets the unit circle at a right angle. (This is
  not easy. After trying it yourself (or with a classmate) feel free
  to try to find a solution on the web. If you do, you should write it
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The clue to this problem is the idea of \emph{inversion in a circle}. In this figure (from
\url{http://jwilson.coe.uga.edu/emt668/emat6680.f99/challen/inversion/index.html})
%
\begin{center}
\includegraphics{inversion}
\end{center}

$P$ is the inverse of $P'$. $|OP'| = 1/|OP|$. It's not hard to prove
  that every circle that goes through $P$ and $P'$ is perpendicular to
  the circle in the figure. Now suppose $Q$ is some other point in the
  circle. Draw the circle through $P$, $Q$ and $P'$. The arc of that
  circle connecting $P$ to $Q$ is the hyperbolic line joining them.

\item
Calculate the (hyperbolic/pseudo) distance from the center of the unit
disk to a point at (Euclidean) distance $r<1$ from the
center.
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Recall that distances at a point $P$ at distance $r < 1$ from the
center of the unit disk are stretched by a factor $1/(1-r^2)$.

\emph{Hint.} Your argument should lead to an integral that you can
evaluate using what you learned in Calculus. If you don't remember
what you need you should be able to look it up.

There's no loss of generality assuming that $P$ is on the
$x$-axis. The distance from $x$ to $x+dx$ is $dx/(1-x^2)$ so the
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%
\begin{align*}
\int_0^r \frac{dx}{1-x^2} & =
\frac{1}{2}\int_0^r \left( \frac{1}{1+x} + \frac{1}{1-x}\right) dx \\
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\end{align*}

\item
Use your answer to the previous problem to show that lines through the
center are infinitely long - in other words, that the distance from
the center approaches $\infty$ as $r \rightarrow 1$.

As $r \rightarrow 1$ the logarithm in the denominator of the distance
formula above approaches $0$ so the distance approaches $\infty$.

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  \url{http://math.stackexchange.com/questions/675522/whats-the-intuition-behind-pythagoras-theorem}
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  \url{http://euler.slu.edu/escher/index.php/Hyperbolic_Paper_Exploration}.

\begin{itemize}
\item  Answer the two questions there.
\item Let $ABC$ be one of the equilateral triangles you taped together
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\newpage

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\verbatiminput{\jobname}

\end{document}
```