# History of Mathematics
# Homework 4

Ethan Bolker

March 31, 2014

This homework is due on Thursday, March 6 .

Note: I've used some new (to you) LaTeX idioms in this document. Learn them. Make sure to submit LaTeX source with your assignment – preferably as part of the pdf, as here.

1. Write a short essay synthesizing what you learned about mathematics and the history of mathematics from our work on the parallel postulate. Please make this interesting for me to read. Don't just write a summary with names, dates and theorems. Consider telling me what was hard, what was easy, what was fun, what you would have liked more or less of, how the material connects to what you knew and to what you hope to know.

2. Positional notation

   In class this week we stumbled into a discussion of positional notation in our exploration of the history of the search for big primes.

3. Write a little bit about the history of logarithms. Were they *invented* or *discovered*? Write down some arguments for *both sides* of that question.

4. The algebraic identity

$$\begin{aligned} a^n - b^n &= (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}) \qquad (1) \\ &= \sum_{k=0}^{n-1} a^{n-1-k}b^k \end{aligned}$$

   is one of the most important in mathematics. When $n = 2$ it's the well known identity for the difference of two squares. We'll see soon how Euclid proved that case – with geometry, of course, not algebra.

   (a) Prove identity 1.

   When you multiply the terms in the second parenthesis first by $a$ and then by $-b$ and add, the middle terms cancel

$$\begin{aligned} a^n + a^{n-1}b + \cdots + a^2b^{n-2} + ab^{n-1} \\ - a^{n-1}b - a^{n-2}b^2 - \cdots - ab^{n-1} - b^n \end{aligned}$$

   leaving just $a^n - b^n$.

   Note: I think best when I write sums like this with an ellipsis ($\ldots$) rather than trying to manipulate the indices for the summation sign ($\Sigma$).

   (b) Use identity 1 to prove that $2^N - 1$ can be prime (hence a Mersenne prime) only if $N$ is prime. (This is Exercise 4.3 in the text.)

   If $N = rs$ is *not* prime then I can factor $2^N - 1$ this way:

$$2^{rs} - 1 = (2^r)^s - 1 = (2^r - 1)(\text{other stuff}).$$

   For example both $2^3 - 1 = 7$ and $2^5 - 1 = 31$ are factors of $2^{15} - 1 = 32,767 = 7 \times 31 \times 151$.

   (c) Use identity 1 to prove that $2^N + 1$ can be prime (hence a Fermat prime) only if $N$ is a power of 2.

   Hint: A number is a power of 2 if and only if it has no odd factors.

   If $n$ is odd then replacing $b$ by its negative in Equation 1 we get

$$a^n + b^n = (a+b)(a^{n-1} - a^{n-2}b + \cdots - ab^{n-2} + b^{n-1}). \qquad (2)$$

   Now suppose $N$ is not a power of 2. Then (using the hint) $N = rs$ for some odd number $r > 1$. Then Equation 2 with $n = r$, $a = 2^s$ and $b = -1$ tells me that $2^s + 1$ is a factor of $(2^s)^r + 1$.

   For example $2^2 + 1 = 5$ is a factor of $2^{10} + 1 = 1025$.

5. On page 172 our text mentions the Pythagorean triple $(12709, 13500, 18541)$ known to the Babylonians. Show how to construct it using the characterization of primitive Pythagorean triples as $(m^2 - n^2, 2mn, m^2 + n^2)$.

   From Josh:

   Using $(m^2 - n^2, 2mn, m^2 + n^2)$, we have $13500 = 2mn$ to give $mn = 6750$. We can now use the factors of 6750 to determine $m$ and $n$. Since we know $a$ to be 12709, we can use these factors to determine which with work for $a = m^2 - n^2$. These turn out to be $m = 125$ and $n = 54$. We can check the numbers by using $c = m^2 + n^2$) to get 18541 as well. We now have our $a^2 + b^2 = c^2$ also written as $12709^2 + 13500^2 = 18541^2$.

6. Modify Euler's argument from our text to prove that there are no nontrivial integral solutions to the equation

$$x^4 - y^4 = z^2.$$

   *Note: The question as I originally asked it had an error. I wrote $z^4$ when I meant $z^2$. But that made the problem trivial. Just move the $-y^2$ to the other side of the equal sign and you have exactly what Euler proved to be impossible.*

   This is an exercise in the book. You can use modern notation, or mimic Euler's style.

   I will catch up on this one later.

Here is the LaTeX source for this document. You can cut it from the pdf and use it to start your answers. I used the `\jobname` macro for the source file name, so you can call your file by any name you like.

```
% Math 370 hw4 Spring 2014
%
\documentclass{article}
\pagestyle{empty}
\usepackage[textheight=10in, textwidth=7.5in]{geometry}

\usepackage{amsmath}
\usepackage{amsthm}
\usepackage{hyperref}
\usepackage{graphicx}
\usepackage{verbatim}

%% create an environment for theorems
\newtheorem*{thm}{Theorem}

\newcommand{\coursehome}
{http://www.cs.umb.edu/~eb/370}

\title{History of Mathematics \\
Homework 4
}
\author{Ethan Bolker}

\begin{document}

\maketitle

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\begin{enumerate}

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primes.

\item Write a little bit about the history of logarithms. Were
 they \emph{invented} or \emph{discovered}? Write down some arguments
  for \emph{both sides} of that question.

\item The algebraic identity
% Leaving an empty line here makes the TeX easier to read. Making it
% a comment tells TeX not to start a new paragraph.
```

```latex
%
\begin{align}\label{eq:anbn}
  a^n - b^n & = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}) \\
         & = \sum_{k=0}^{n-1} a^{n-1-k} b^k \notag
\end{align}
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is one of the most important in mathematics. When $n=2$ it's the well
known identity for the difference of two squares. We'll see soon how
Euclid proved that case -- with geometry, of course, not algebra.

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\item Prove identity~\ref{eq:anbn}.

When you multiply the terms in the second parenthesis first by $a$ and
then by $-b$ and add, the middle terms cancel
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a^{n}  & + a^{n-1}b + \cdots + a^{2}b^{n-2} + ab^{n-1} \\
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leaving just $a^n - b^n$.

Note: I think best when I write sums like this with an ellipsis (\ldots)
rather than trying to manipulate the indices for the summation sign
($\Sigma$).

\item Use identity~\ref{eq:anbn} to prove that $2^N -1$ can be prime (hence a
  Mersenne prime) only if $N$ is prime. (This is Exercise 4.3 in the text.)

If $N = rs$ is \emph{not} prime then I can factor $2^N
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\begin{equation}\label{eq:anplusbn}
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\end{equation}

Now suppose $N$ is not a power of $2$. Then (using the hint)
$N=rs$
for some odd number $r > 1$. Then Equation~\ref{eq:anplusbn} with
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For example $2^2 + 1 = 5$ is a factor of
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\verbatiminput{\jobname}

\end{document}
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