Continued Fractions and their application to solving Pell's equations

Number Theory: Fall 2009

Peter Khoury
University of Massachusetts-Boston, MA

Gerard D.Koffi
University of Massachusetts-Boston, MA

Outline

Pell's Equation

The quadratic Diophantine equation of the form

$$x^2 - dy^2 = \pm 1$$

where d is a positive square free integer is called a Pell's equation.

Pell's Equation and History

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The name Pell's equation comes from Euler who in a letter to Goldbach confused the name of William Brouncker, the first mathematician who gave an algorithm to solve the equation, with that of the English mathematician John Pell(1 March 1611-12 December 1685).

Exercises, Part I

Find(if any) a solution to the following equations:

1. $x^2 - 8y^2 = 1$
2. $x^2 - 13y^2 = 1$
3. $x^2 - 13y^2 = -1$
4. $x^2 - 58y^2 = 1$(To do for homework)
5. $x^2 - 58y^2 = -1$(To do for homework)
6. $x^2 - 58y^2 \pm 1$(To do for homework: Here find a solution that is different from those found in (4) and (5))

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A solution to equation (1) is given by (3,1) and a solution to (2) is given by (649,180). Equation (3) on the other does not have any solution: it is not solvable.

Lemma

If (a, b) is a solution to $x^2 - dy^2 = 1$ where $a > 1$ and $b \geq 1$, then (x, y) such that

$$x + y\sqrt{d} = (a + b\sqrt{d})^n$$

for $n = 1, 2, 3, 4, \ldots,$ is also a solution.

Similarly, if (c, d) is a solution to $x^2 - dy^2 = -1$ where $c > 1$ and $d \geq 1$, then (x, y) such that

$$x + y\sqrt{d} = (c + d\sqrt{d})^n$$

for $n = 1, 3, 5, 7, \ldots,$ is also a solution.

Proof: The proof is done by induction.

Continued Fractions

Definition

The expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

where the a_i's are integers, is called the continued fraction expansion of a real number.

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Example: The continued fraction of $\frac{987}{610} = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\cdots}}}$.

We use the notation $[a_0, a_1, a_2, a_3, \cdots]$ to denote the continued fraction expression of a real number. Hence, in the above example, we have

$$\frac{987}{610} = [1; 1, 1, 1, 1, 1, 1, 1, \cdots].$$

Continued Fractions(Cont'd)

In the definition of the continued fraction of a real number R, we call the k-convergent of R, the truncated continued fraction of R at the kth term. We denote this convergent as C_k. Hence,

$$C_0 = a_0, \quad C_1 = a_0 + \frac{1}{a_1}, \quad C_2 = a_0 + \frac{1}{a_1 + \frac{1}{a_2}}, \quad C_3 = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}},$$

and so on.
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Exercise(to do for homework)

Show that $A_k B_{k-1} - A_{k-1} B_k = (-1)^k$ for $k \geq 0$.
Hint: Write two recurrence relations involving the a_i's and A_k, B_k

Algorithm to find continued fraction expansion

Algorithm

Let R be a real number.

Step 1: Write $R = \lfloor R \rfloor + \frac{1}{x_1}$

Step 2: Solve for x_1

Step 3: Write $x_1 = \lfloor x_1 \rfloor + \frac{1}{x_2}$ and replace the new expression of x_1 into step 1.

Step 3: Repeat steps 2 and 3 for x_2 and so on.

Using the algorithm above, we obtained that the continued fraction of $\sqrt{8}$ and $\sqrt{13}$ are given respectively, by $[2; \overline{1, 4}]$ and $[3; \overline{1, 1, 1, 1, 6}]$.

Exercise(to do for homework)

Let $P, Q \in \mathbb{Z}$ with $Q \neq 0$. Show that the continued fraction of $\frac{P}{Q}$ is obtained by performing the Euclidean algorithm, and deduce that its continued fraction eventually stops.

Fact about continued fractions

A continued fraction is purely periodic with period m if the initial block of partial quotients $a_0, a_1, \ldots, a_{m-1}$ repeats infinitely and no block of length less than m is repeated and is periodic with period m if it consists of an initial block of length n followed by a repeating block of length m.

Purely periodic continued fraction $\mapsto [\overline{a_0; a_1, \ldots, a_{m-1}}]$
Periodic continued fraction $\mapsto [a_0; a_1 \ldots, a_{n-1}, \overline{a_n, \ldots, a_{n+m-1}}]$.
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Theorem

Let $d > 1$ be a rational number that is not the square of another rational number. Then

$$\sqrt{d} = [a_0; \overline{a_1, a_2, \ldots, a_2, a_1, 2a_0}].$$

Solving Pell's Equations

The solutions to both Pell's equations $x^2 - dy^2 = 1$ and $x^2 - dy^2 = -1$ are related to the continued fraction expansion of $\sqrt{d}$. In fact,

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Theorem

The equation $x^2 - dy^2 = 1$ is always solvable and the fundamental solution is (A_k, B_k) where $k = r$ or $2r$ and A_k/B_k is a convergent of $\sqrt{d}$. The equation $x^2 - dy^2$ is solvable if and only if the length of the period of the continued expansion of $\sqrt{d}$ is odd. The fundamental solution is (A_k, B_k) where $k = r$ or $r + 1$.

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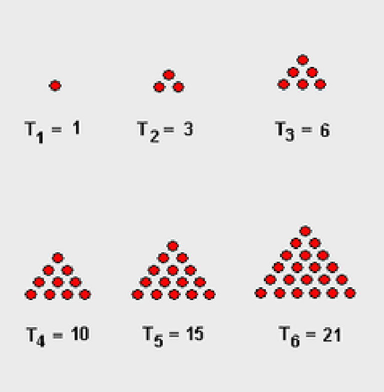
Theorem

The positive solutions to the Pell'equation $x^2 - dy^2 = \pm 1$ are given by the convergent A_k/B_k with $k = r, 2r, 3r, \cdots$

The numbers $1, 3, 6, 10, 15, 21, 28, 36, , 45, \ldots, t_n = \frac{1}{2}n(n+1), \ldots$ are called triangular numbers, since the nth number counts the number of dots in an equilateral triangular array with n dots to the side. It happens that individual triangular numbers are square. We want to find them or at least generate them.



Solution

The condition that the nth triangular number t_n is equal to the mth square is $\frac{1}{2}n(n+1) = m^2$. Rewriting that expression, we can put it in the form $(2n+1)^2 - 8m^2 = 1$. Now setting $x = 2n+1$ and $y = m$, we are can solve the equation $x^2 - 8y^2 = 1$.

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$$A_0/B_0 = 2$$
$$A_1/B_1\text{: } 3 = 3$$
$$A_2/B_2 = 14/5$$
$$A_3/B_3 = 17/6$$
$$A_4/B_4 = 82/29$$
$$A_5/B_5 = 99/35$$
$$A_6/B_6 = 478/169$$
$$A_7/B_7 = 577/204$$
$$A_8/B_8 = 2786/985$$

Exercises, Part II(Cont'd)

Determine integers n for which there exists an integer m for which

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Solution:

The condition is that $n(n+1) = 2m(m+1)$ or $(2n+1)^2 - 2(2m+1)^2 = -1$. The continued fraction expansion of $\sqrt{2}$ give us infinitely many solutions.

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$$A_0/B_0 = 1$$
$$A_1/B_1 = 3/2$$
$$A_2/B_2 = 7/5$$
$$A_3/B_3 = 17/12$$
$$A_4/B_4 = 41/29$$
$$A_5/B_5 = 99/70$$
$$A_6/B_6 = 239/169$$
$$A_7/B_7 = 577/408$$

Last Homework Problem

The root-mean-square of a set of $\{a_1, \ldots, a_n\}$ of positive integers is equal to

$$\sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_k^2}{k}}.$$

Is the root-mean-square of the first n positive integers ever an integer?.

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QUESTIONS?