NEW PROOFS OF EUCLID'S AND EULER'S THEOREMS

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Abstract.

Theorem 1. *There are infinitely many prime numbers*

Proof. Suppose that $p_1 < p_2 < \ldots < p_n$ are all of the primes.

Let's declare a closed interval $[1, x]$ where $x \geq 1$ is a real number. The number of integers inside $[1, x]$ is $\lfloor x \rfloor$. Here is another way to count the number of integers inside $[1, x]$ that uses the hypothesis:

Let $i = 1, 2, 3, ..., n$, and let A_i be the set of all integers in $[1, x]$ that are divisible by p_i. Then $|A_i| = \lfloor \frac{x}{p_i} \rfloor$, the number of times p_i goes into x evenly. We will use the principle of inclusion-exclusion to determine the cardinality of the union of all of the sets, A_i.

$$|\bigcup_{i-1}^{n} A_i| = \sum_{i=1}^{n} \lfloor \frac{x}{p_i} \rfloor - \sum_{i<j} \lfloor \frac{x}{p_i p_j} \rfloor + \sum_{i<j<k} \lfloor \frac{x}{p_i p_j p_k} \rfloor - \ldots + (-1)^{n+1} \lfloor \frac{x}{p_1 p_2 \ldots p_n} \rfloor$$

Now, essentially what we are doing here is running through the sieve of Eratosthenes. We will go through the sieve crossing out all multiples of each prime in turn. When we count up all the numbers that have been crossed off, that will be every integer except 1. Since we have assumed that p_1 through p_n are all of the primes, then every integer is either prime or composite, so every integer will be crossed off at some point. We can see how this is working in the equation above. If you list out all of the integers, 1 through $\lfloor x \rfloor$, the first summation crosses off all multiples of $p_1 = 2$, then of $p_2 = 3$, and so on, where the numbers that are common multiples of any of the p_i's have been added more than once. Inclusion-exclusion principle takes care of this problem by then subtracting the number of integers that are multiples of pairs of primes. The third summation then adds back in the number of integers that are multiples of triples of primes, and so on, until the final sum accounts for the total number of integers crossed off the sieve. If we add 1, the only number that does not get crossed off, then we have another expression for the number of integers inside $[1, x]$.

Date: December 17, 2009.

$$\lfloor x\rfloor = 1 + \sum_{i=1}^{n}\lfloor\frac{x}{p_i}\rfloor - \sum_{i<j}\lfloor\frac{x}{p_ip_j}\rfloor + \sum_{i<j<k}\lfloor\frac{x}{p_ip_jp_k}\rfloor - \ldots + (-1)^{n+1}\lfloor\frac{x}{p_1p_2\ldots p_n}\rfloor$$

Our next step is to consider what happens as x gets sufficiently large. First, let's divide everything by x:

$$\frac{\lfloor x\rfloor}{x} = \frac{1}{x}+\frac{1}{x}\left(\sum_{i=1}^{n}\lfloor\frac{x}{p_i}\rfloor - \sum_{i<j}\lfloor\frac{x}{p_ip_j}\rfloor + \sum_{i<j<k}\lfloor\frac{x}{p_ip_jp_k}\rfloor - \ldots + (-1)^{n+1}\lfloor\frac{x}{p_1p_2\ldots p_n}\rfloor\right) \tag{1}$$

To take the limit of each term as $x \to \infty$, notice that,

$$\frac{\frac{x}{a}-1}{x} < \frac{\lfloor\frac{x}{a}\rfloor}{x} < \frac{\frac{x}{a}+1}{x}$$

and

$$\lim_{x\to\infty}\left(\frac{\frac{x}{a}\pm 1}{x}\right) = \frac{1}{a}$$

Therefore, by the squeeze theorem,

$$\lim_{x\to\infty}\frac{\lfloor\frac{x}{a}\rfloor}{x} = \frac{1}{a}$$

Now we can compute the limit of each term in equation (1), where $\frac{1}{x} \to 0$. We now have,

$$1 = \sum_{i=1}^{n}\frac{1}{p_i} - \sum_{i<j}\frac{1}{p_ip_j} + \sum_{i<j<k}\frac{1}{p_ip_jp_k} - \ldots + (-1)^{n+1}\frac{1}{p_1p_2\ldots p_n} \tag{2}$$

Now, our next step requires some visual creativity. The right hand side is of the form,

$$\sum a_i - \sum a_ia_j + \sum a_ia_ja_k - \ldots$$

which resembles the expansion of

$$(1-a_1)(1-a_2)\ldots(1-a_n)$$

We just need to make a slight adjustment, which we can see if we begin to expand the above term. It turns out that,

$$(1-a_1)(1-a_2)\ldots(1-a_n) = 1-\left(\sum a_i - \sum a_ia_j + \sum a_ia_ja_k - \ldots + (-1)^{n+1}a_1a_2\ldots a_n\right)$$

So then, after some rearranging and putting it in terms of $\frac{1}{p_i}$, we have,

$$(3)\quad \sum_{i=1}^{n}\frac{1}{p_i}-\sum_{i<j}\frac{1}{p_ip_j}+\sum_{i<j<k}\frac{1}{p_ip_jp_k}-\ldots+(-1)^{n+1}\frac{1}{p_1p_2\ldots p_n}=1-\prod_{i=1}^{n}\left(1-\frac{1}{p_i}\right)$$

We are almost at our contradiction. Notice that the right hand side is always less than 1, so we can turn the above expression into an inequality,

$$(4)\qquad \sum_{i=1}^{n}\frac{1}{p_i}-\sum_{i<j}\frac{1}{p_ip_j}+\sum_{i<j<k}\frac{1}{p_ip_jp_k}-\ldots+(-1)^{n+1}\frac{1}{p_1p_2\ldots p_n}<1$$

So here is our contradiction. Equation (2) tells us that the sum is equal to 1, but equation (4) tells us that the sum is less than 1. That is not possible since $1 \nless 1$. Therefore, since we followed logical reasoning based on the hypothesis that we had finitely many prime numbers and ended up with a contradiction, the hypothesis must be false. Hence, there are infinitely many primes.

□

Definition 1. *For an integer $n \in [1, x]$ and a subset $B \subseteq [1, x]$, if the probability that $n \in B$ tends to some limit as $x \to \infty$, then we call that limit the* ***Asymptotic Density*** *of B.*

We will explore this definition in the context of this paper to help set the stage for the next theorem.

Let's call B the set of integers in $[1, x]$ that are *not* divisible by $p_1, \ldots, p_n$.

From the previous proof, since

$$|\bigcup_{i=1}^{n} A_i| = \text{ number of integers divisible by } p_1, \ldots, p_n$$

then,

$$|B| = \lfloor x \rfloor - |\bigcup_{i=1}^{n} A_i|$$

So the probability that an integer in $[1, x]$ is also in B is,

$$\frac{\lfloor x \rfloor - |\bigcup_{i=1}^{n} A_i|}{\lfloor x \rfloor}$$

Now we want to see if the limit as $x \to \infty$ exists for the above probability function. We will denote the asymptotic density of B as $d(B)$:

$$d(B)=\lim_{x\to\infty}\left(\frac{\lfloor x\rfloor-|\bigcup_{i=1}^{n}A_i|}{\lfloor x\rfloor}\right)=\lim_{x\to\infty}\left(1-\frac{|\bigcup_{i=1}^{n}A_i|}{\lfloor x\rfloor}\right)=1-\lim_{x\to\infty}\left(\frac{|\bigcup A_i|}{\lfloor x\rfloor}\right)$$

$$= 1 - \lim_{x\to\infty}\left(\frac{\sum_{i=1}^{n}\lfloor\frac{x}{p_i}\rfloor - \sum_{i<j}\lfloor\frac{x}{p_ip_j}\rfloor + \sum_{i<j<k}\lfloor\frac{x}{p_ip_jp_k}\rfloor - \ldots + (-1)^{n+1}\lfloor\frac{x}{p_1p_2\ldots p_n}\rfloor}{\lfloor x\rfloor}\right) \tag{5}$$

We will again use the squeeze theorem to show that $\lim_{x\to\infty}\frac{\lfloor\frac{x}{a}\rfloor}{\lfloor x\rfloor} = \frac{1}{a}$. Notice,

$$\frac{\frac{x}{a}-1}{x+1} < \frac{\lfloor\frac{x}{a}\rfloor}{\lfloor x\rfloor} < \frac{\frac{x}{a}+1}{x-1}$$

$$\lim_{x\to\infty}\frac{\frac{x}{a}\pm 1}{x\pm 1} = \frac{1}{a}$$

Now we can solve (5) using (3)

$$\begin{aligned} d(B) = 1 - \sum_{i=1}^{n}\frac{1}{p_i} - \sum_{i<j}\frac{1}{p_ip_j} + \sum_{i<j<k}\frac{1}{p_ip_jp_k} - \ldots + (-1)^{n+1}\frac{1}{p_1p_2\ldots p_n} \\ = 1 - \left[1 - \prod_{i=1}^{n}\left(1 - \frac{1}{p_i}\right)\right] \\ = \prod_{i=1}^{n}\left(1 - \frac{1}{p_i}\right) \end{aligned}$$

Let's define $D = \lim_{n\to\infty} d(B)$. Take the natural log of both sides,

$$\ln D = \ln\left|\prod_{i=1}^{\infty}\left(1 - \frac{1}{p_i}\right)\right|$$

$$\ln D = \sum_{i=1}^{\infty}\ln\left|1 - \frac{1}{p_i}\right| \tag{6}$$

Let's compare $\sum_{p}^{\infty}\ln\left|1-\frac{1}{p}\right|$ with $= \sum_p \frac{1}{p}$. Notice that we can rewrite (6) without the absolute value, since the term is always positive. By the limit comparison test,

$$\lim_{p\to\infty}\left|\frac{\ln(1-\frac{1}{p})}{\frac{1}{p}}\right|, \text{ apply L'hospital's Rule:}$$

$$\lim_{p\to\infty}\left|\frac{\left(\frac{1}{1-\frac{1}{p}}\right)\left(\frac{1}{p^2}\right)}{-\frac{1}{p^2}}\right| = \lim_{p\to\infty}\left|-\left(\frac{1}{1-\frac{1}{p}}\right)\right| = 1$$

both series diverge or both converge.

Our goal is to show that $D = 0$. If $D = 0$, then $\sum_{i=1}^{\infty}\ln\left|1-\frac{1}{p_i}\right|$ diverges, and that would imply that $\sum_p \frac{1}{p}$ diverges. Now we are ready to prove the next theorem.

Theorem 2. *The series $\sum_p \frac{1}{p}$ diverges.*

Proof. Let us assume that $D > 0$ and the convergence of $\sum_p \frac{1}{p}$ happen simultaneously. We will choose $\varepsilon > 0$ and n big enough such that,

$$\varepsilon < D \quad \text{and} \quad \sum_{p>p_n} \frac{1}{p} < \varepsilon$$

Let $S(p_n)$ be the set of all integers which have prime factors greater than p_n. Recall that B is the set of integers divisible by none of $p_1, \dots, p_n$. Then,

$$B \subseteq S(p_n)$$

Furthermore, since D is the asymptotic density of B, let's call $D(S_n)$ the asymptotic density of $S(p_n)$. Then,

$$D \subseteq D(p_n)$$

So D, the asymptotic density of the integers not divisible by $p_1, \dots, p_n$, is bounded below by ε because of the condition we established. However, the asymptotic density of the integers which have prime factors greater than p_n is bounded above by ε: If we calculate the asymptotic density of the integers that have prime factors greater than p_n using the methods we used above for B and the principle of inclusion-exclusion, we see that the density is certainly less than $\sum_{p>p_n} \frac{1}{p}$, and we picked an n big enough so that that number is less than ε. Now we have,

$$\varepsilon < D < D(p_n) < \sum_{p>p_n} \frac{1}{p} < \varepsilon$$

We have a contradiction since ε cannot be greater than and bigger than itself. Therefore, it cannot be true that both $D > 0$ and $\sum_p \frac{1}{p}$ converges. So we can conclude that if $D > 0$, then $\sum_p \frac{1}{p}$ diverges, but if $\sum_p \frac{1}{p}$ diverges, then so does $\sum_{i=1}^{\infty} \ln\left(1 - \frac{1}{p_i}\right)$, but that only happens when $D = 0$. So it must only be true that $D = 0$ and therefore $\sum_p \frac{1}{p}$ diverges.

$\square$