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RECOGNIZING DIRICHLET TESSELLATIONS

1. INTRODUCTION

Let Σ be a closed convex polyhedron in Euclidean N-space, (possibly unbounded, possibly all of N-space) and let $\mathbf{P}$ be a finite set of points in Σ. We call the elements of $\mathbf{P}$ *sources*, or *centers*. We shall study how $\mathbf{P}$ carves Σ into *regions*: for each $P \in \mathbf{P}$, R_P consists of those points of Σ as close to P as to any other source.

There are many applications of the resulting dissection. One is to the construction of voting precincts so that each person votes at the nearest polling place. Another is to think of the centers as business establishments and the regions as regions of economic influence. Or competing species might spread from the sources, ultimately to inhabit the regions.

These dissections are often called Dirichlet tessellations, to honor pioneering work of his in 1850 [9]. The regions have been called Dirichlet regions, Dirichlet domains, and Voronoi [36] polygons. In fact, interest in such regions predates Dirichlet by centuries. The concept appears in a 1644 work of Descartes [8] as a consequence of his theory of gravity, which asserts that a moving celestial body like a comet is attracted only by the planet or other massive body that it is closest to. Most nineteenth and early twentieth century work on Dirichlet tessellations was motivated by crystallography and thus focused on sources that form a lattice. Nowacki [25] gives a bibliography of this classical material. Grunbaum and Shephard [14] discuss more general tessellations (tilings); their paper also contains further references to Dirichlet tessellations. Recently computational geometers have become interested in Dirichlet tessellations based on finitely many irregularly placed sources. For algorithms and applications see Green and Sibson [12], work by Imai, Iri, Murota, and Ohya [1], [17], [26], [27], and Toussaint *et al.* [32]–[35]. Miles [23] considers some probabilistic properties of Dirichlet tessellations whose sources are chosen at random. Loeb [20] presents a treatment of Dirichlet tessellations in the spirit of this paper. He found our Theorem 15, although he proves only the necessity of the condition stated there which is characteristic of Dirichlet tessellations. We were not aware of his work when we did ours.

The literature on Dirichlet tessellations is surprisingly sparse given the importance of the ideas in both the physical and social sciences. It is also

scattered in the journals of many disciplines. Our bibliography is not complete; we hope it is representative of the many fields in which Dirichlet tessellations are used.

Formally, for P and $Q \in \mathbf{P}$ define the half space H_{PQ} as

(1.1) $\qquad H_{PQ} = \{X \in \Sigma : |X - P| \leqslant |X - Q|\}$

and then let

(1.2) $\qquad R_P = \bigcap_{Q \neq P} H_{PQ}.$

Here $|\cdot|$ is the Euclidean norm in N-space. The family of sets

(1.3) $\qquad \mathbf{R} = \{R_P : P \in \mathbf{P}\}$

is called the *Dirichlet tessellation* of Σ based on $\mathbf{P}$. The following properties of $\mathbf{R}$ are immediate.

    (1) Each R_P is a closed convex polyhedron, since it is a finite intersection of half spaces.

    (2) $P \in R_P$.

    (3) If $P \neq Q$ then $R_P \cap R_Q$ is a face of each of R_P and R_Q.

    (4) $\Sigma = \bigcup_{P \in \mathbf{P}} R_P.$

Figure 1 shows an example of a two-dimensional Dirichlet tessellation. Consider the boundary $R \cap R'$ between two of the regions R and R' of a tessellation. That boundary may be empty. If it is not, we say R and R' are *neighbors* when $R \cap R'$ has dimension $N - 1$ and *half-neighbors* when it has smaller dimension.

If R_i $(i = 1, \ldots, n)$ are regions of a tessellation $\mathbf{R}$ and

(1.4) $\qquad \bigcap_{i=1}^{n} R_i = \{V\}$

then the point V is a *vertex* of the tessellation (and also a vertex of each o the R_i.) If in addition $V \notin R$ for any other $R \in \mathbf{R}$ then V is an *n-valen vertex*: the intersection of exactly n regions.

Eventually, we wish to be able to decide whether a given tessellation $\mathbf{R}$ i a Dirichlet tessellation based on some set $\mathbf{P}$. To do so, we must defin 'tessellation' in general. For our purposes, a tessellation $\mathbf{R}$ of Σ will be a finite collection of closed convex polyhedra with nonempty interiors eacl
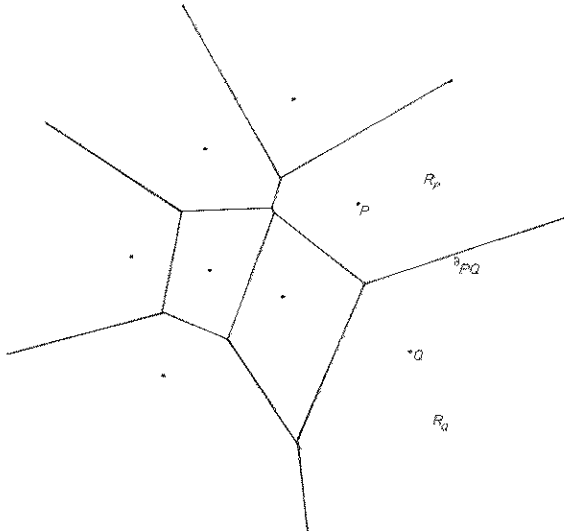
Fig. 1. A plane Dirichlet tessellation.

pair of which intersects in a face of each (so that each dihedral angle of each polyhedron is less than π) and such that

$$(1.5) \qquad \bigcup_{R \in \mathbb{R}} R = \Sigma.$$

We shall often redundantly identify tessellations as *convex* tessellations, to stress that part of the definition. A convex tessellation will be called *proper* if it contains at least one vertex. Thus a tessellation consisting of parallel slabs is improper.

LEMMA 1. *When* $\mathbb{R}$ *is a Dirichlet tessellation the boundary between regions* R_P *and* R_Q *lies on the hyperplane which is the perpendicular bisector of segment PQ.*

Proof. If the boundary is empty the lemma is obviously true, and useless. On the other hand, any $X \in R_P \cap R_Q$ is equidistant from P and Q.   □

For Dirichlet tessellations, we sometimes abbreviate the boundary $R_P \cap R_Q$ as ∂_{PQ} or $\partial(P, Q)$. The next easy theorem is a partial converse to Lemma 1.

THEOREM 2. *If R is a closed convex polyhedron in N-space and P belongs to the interior of R then there is a Dirichlet tessellation for which $R = R_P$.*

Proof. Suppose R has n facets (faces of codimension 1); suppose they lie in the hyperplanes $K_1, \ldots, K_n$. Let P_i be the mirror image of P in K_i. Then in the Dirichlet tessellation based on $\mathbf{P} = \{P, P_1, \ldots, P_n\}$ we have $R = R_P$.
□

When we try to fit convex polyhedra together and choose sources in each which match when reflected over boundaries the situation is subtler. To analyze it we first study what happens around each vertex. And to do that we must broaden our definition of Dirichlet tessellations to allow tessellations of the N-sphere as well as of N-space. Suppose $\mathbf{P}$ lies on an N-sphere Γ in $(N + 1)$-space. Then use Equations (1.1) and (1.2), with $|\cdot|$ interpreted as great circle distance on Γ, to define the Dirichlet tessellation $\mathbf{R}_\Gamma$ of Γ based on $\mathbf{P}$. If such a tessellation has at least two vertices then each of its regions lies in some closed hemisphere, and each is convex in the sense given by Grunbaum [13, p. 30]: each is the intersection of Γ with the corresponding convex region of the Dirichlet tessellation determined by $\mathbf{P}$ in the ambient $(N + 1)$-space.

Remark. Lemma 1 remains true for tessellations on spheres when we interpret 'hyperplane' as 'great $(N - 1)$-sphere'. We shall sometimes call such an $(N - 1)$-sphere a hypersphere.

THEOREM 3. *Let V be an n-valent vertex of a Dirichlet tessellation* **R** *in N-space or on an N-sphere. Let $P_1, \ldots, P_n$ be the sources in the regions $R_1, \ldots, R_n$ of* **R** *with V as vertex. Then the sources P_i lie on an $(N - 1)$-sphere with center V. Moreover, if Γ is a sphere centered at V and small enough so that it meets only the regions $R_1, \ldots, R_n$ of* **R** *and $P'_i = VP_i \cap \Gamma$, then the regions of the Dirichlet tessellation of Γ based on $\mathbf{P}' = \{P'_1, \ldots, P'_n\}$ are the intersections $R_i \cap \Gamma$.*

Proof. If P_i and P_j are neighbors at V (rather than just half-neighbors) then $V \in R_i \cap R_j$ and Lemma 1 implies $|V - P_i| = |V - P_j|$. Since any pair of the regions R_i, R_j at V can be linked by a chain of regions each of which is a neighbor of its successor, the n distances $|V - P_i|$ are all equal to the same constant r, and each P_i lies on the sphere Γ with radius r and center V.

To prove the second part of the theorem, observe that if $X \in \Gamma$ then the great circle distance from X to P'_i is less than or equal to the corresponding distance to P'_j if and only if the original distances from X to P_i and to P_j are so related.
□

Let **R** be a tessellation in N-space or on an N-sphere and V a vertex of **R**. We shall say that **R** is *Dirichlet at* V if it induces a Dirichlet tessellation on any sufficiently small $(N-1)$-sphere centered at V and *locally Dirichlet* if it is Dirichlet at each of its vertices. Theorem 3 then says that a Dirichlet tessellation is locally Dirichlet.

To decide whether a given tessellation is a Dirichlet tessellation then requires two steps: a local argument at each vertex, and a pasting mechanism. Our study of plane tessellations in Sections 3 and 4 therefore begins with the local study of tessellations of the line and the circle in Section 2. In Section 5, the concluding section, we generalize what we can to tessellations of N-space.

2. TESSELLATIONS OF THE LINE AND THE CIRCLE

On the line consider the Dirichlet tessellation **R** based on the $n + 1$ sources

$$(2.1) \qquad p_0 < p_1 < \cdots < p_n.$$

For $i = 1, 2, \ldots, n$ let

$$(2.2) \qquad x_i = \tfrac{1}{2}(p_{i-1} + p_i)$$

be the midpoint of segment $[p_{i-1}, p_i]$. Then

$$(2.3) \qquad p_0 < x_1 < p_1 < x_2 < \cdots < p_{n-1} < x_n < p_n,$$

and **R** consists of the $n - 1$ bounded intervals $[x_i, x_{i+1}]$ $(1 \leqslant i \leqslant n - 1)$ and the two unbounded intervals $(-\infty, x_1]$ and $[x_n, \infty)$.

The following theorem tells us which partitions of the line into intervals are Dirichlet tessellations.

THEOREM 4. *Suppose the line is subdivided into intervals by the points* $x_1 < x_2 < \cdots < x_n$. *Let* $t_i = x_{i+1} - x_i$ *be the length of the interval* (x_i, x_{i-1}), $i = 1, \ldots, n - 1$. *Then the family of intervals forms a Dirichlet tessellation if and only if all odd alternating sums of lengths are positive, that is, if and only if*

$$(2.4) \qquad \sum_{i=r}^{s} (-1)^{i-r} t_i > 0$$

for $1 \leqslant r < s \leqslant n - 1$ *and* $s - r$ *even.*

Proof. Suppose the intervals are the Dirichlet tessellation based on $\mathbf{P} = \{p_0, \ldots, p_n\}$. Then for $i = 1, \ldots, n-1$

$$p_i \in (x_i, x_{i+1})$$

and, letting

$$(2.5) \qquad a_i = p_i - x_i = x_i - p_{i-1},$$

we have

$$(2.6) \qquad a_i > 0$$

and

$$(2.7) \qquad t_i = a_i + a_{i+1}.$$

Thus the sum on the left in (2.4) telescopes, and

$$(2.8) \qquad \sum_{i=r}^{s} (-1)^{i-r} t_i = a_r + a_{s+1} > 0.$$

Therefore the inequality in (2.4) is necessary. To prove it is sufficient we must determine scalars a_i which satisfy (2.7) subject to the constraints

$$(2.9) \qquad 0 < a_i < t_i;$$

then we can use (2.5) to define the sources p_i. Let

$$(2.10) \qquad a = \min_{k} \sum_{i=1}^{2k-1} (-1)^{i-1} t_i.$$

By hypothesis, $a > 0$. Setting $k = 1$ in (2.10) shows $a \leq t_1$. Let $a_1 = a$; then use equations (2.7) to solve for the remaining a_i:

$$(2.11) \qquad a_{i+1} = t_i - a_i,$$

or, in closed form

$$(2.12) \qquad a_{i+1} = \sum_{j=1}^{i} (-1)^{i-j} t_j + (-1)^i a.$$

To verify (2.9) it suffices to show all the a_i are positive, since the second half of that inequality will then follow from (2.11). When i is odd Equation (2.12)

and the definition of a as a minimum shows $a_{i+1} \geq 0$. Suppose i is even. Let k be the index for which the minimum in (2.10) is achieved. Then if $i > 2k - 1$,

$$(2.13) \qquad a_{i-1} = \sum_{j=2k}^{i} (-1)^j t_j > 0$$

while if $i < 2k - 1$

$$(2.14) \qquad a_{i+1} = -\sum_{j=i+1}^{2k-1} (-1)^j t_j > 0.$$

If we now replace a by $a - \varepsilon$, where ε is small enough, then the inequalities $a_{i+1} \geq 0$ for i odd become strict while the inequalities $a_{i+1} > 0$ for i even remain true.   □

Remark 1. Note that the set **P** of sources producing the given Dirichlet tessellation is not unique. It is a one parameter family: the choice of a_1 subject to some inequality constraints determines **P**.

Remark 2. The essence of the preceding theorem is that the lengths t_i of the intervals cannot vary too widely. For example, the necessary condition $t_i - t_{i+1} + t_{i+2} > 0$ may be rewritten as

$$(2.15) \qquad t_{i+1} < t_i + t_{i+2}$$

which says that a large interval may not be flanked by small intervals. Conversely, if all the intervals are the same length, say t, then every odd alternating sum also has the value t and so the family is a Dirichlet tessellation and a_1 may be chosen arbitrarily subject to $0 < a_1 < t$.

Remark 3. When the sequence $\{t_i\}$ is monotone inequalities (2.4) are always true.

Remark 4. Each inequality in (2.4) is independent of the others. To see that, we exhibit a sequence $\{t_i\}$ of arbitrary length in which exactly one alternating sum of prescribed odd length is negative. The intervals corresponding to that sum can appear at any desired place in the sequence. Suppose $k > 0$ given. Then let $\{t_i\}$ be the sequence

$$\ldots, 7, 5, 3, 1, 3, \ldots, 3, 1, 3, 5, 7, \ldots$$

where there are $2k - 1$ central 3's. It is easy to see that the only odd alternating sum which is negative is the one of length $2k + 1$ which begins and ends with a 1. (Remark 3 helps in the verification.)

A similar analysis involving odd alternating sums characterizes Dirichlet tessellations of the circle Γ; this analysis is essentially equivalent to the investigation of the finite Fourier transform Schoenberg provides in [29, Chap. 6]. Suppose $p_1, \ldots, p_n$ are n points on the circle, listed in counterclockwise order. For $i = 1, \ldots, n$ let x_i be the midpoint of the circular arc $[p_{i-1}, p_i]$; here and in what follows read subscripts modulo n. Then the Dirichlet tessellation $\mathbf{R}$ of Γ determined by the p_i consists of the n circular arcs $[x_i, x_{i+1}]$. For convenience let the circle have unit radius and let θ_i be the length of arc $[x_i, x_{i+1}]$, so that

$$(2.16) \qquad \sum_{i=1}^{n} \theta_i = 2\pi.$$

Now we wish to determine when the sequence $\theta_1, \ldots, \theta_n$ of arclengths satisfying (2.16) comes from a Dirichlet tessellation. We start with necessary conditions.

THEOREM 5. *Suppose $\theta_1, \ldots, \theta_n$ are the lengths of arcs in the Dirichlet tessellation of the unit circle based on $p_1, \ldots, p_n$. Then odd alternating sums are positive:*

$$(2.17) \qquad \sum_{i=r}^{s} (-1)^{i-r} \theta_i > 0$$

when $s - r$ is even. Moreover, if we let

$$(2.18) \qquad \alpha_i = \text{the length of } arc[x_i, p_i] > 0$$

and

$$(2.19) \qquad \beta_i = \tfrac{1}{2} \sum_{j=0}^{n-1} (-1)^j \theta_{i+j}$$

then if n is odd

$$(2.20) \qquad \beta_i = \alpha_i$$

while if n is even

$$(2.21) \qquad \beta_i = 0.$$

Proof. Since $\mathbf{R}$ is a Dirichlet tessellation, arcs $[p_i, x_{i+1}]$ and $[x_{i+1}, p_{i+1}]$ are equal in length. Hence

$$(2.22) \qquad \theta_i = \alpha_i + \alpha_{i+1}.$$

(compare (2.7)). Thus the sum on the left in (2.17) telescopes and

$$(2.23) \qquad \sum_{i=r}^{s} (-1)^{i-r}\theta_i = \alpha_r + (-1)^{s-r}\alpha_{s+1}.$$

When $s - r$ is even that is clearly positive. When $s - r = n - 1$ use the fact that $\alpha_i = \alpha_{n+i}$ to deduce from (2.19) and (2.23) that

$$(2.24) \qquad \beta_i = \tfrac{1}{2}(1 + (-1)^{n-1})\alpha_i,$$

which is α_i if n is odd and 0 if n is even.   □

COROLLARY 6. *In a Dirichlet tessellation of the circle into an odd number of parts, the sources are uniquely determined.*

Next we show that for a partition of the circle into an odd number of parts we can use Equation (2.20) to locate the sources which make it a Dirichlet tessellation.

THEOREM 7. *Let n be odd and $\theta_1, \ldots, \theta_n$ be positive real numbers whose sum is 2π. Define numbers α_i by*

$$(2.25) \qquad \alpha_i = \tfrac{1}{2} \sum_{j=0}^{n-1} (-1)^j \theta_{i+j}.$$

Then $\{\theta_1, \ldots, \theta_n\}$ corresponds to a Dirichlet tessellation of the unit circle if and only if every $\alpha_i > 0$.

Proof. Theorem 5 establishes the necessity of the condition. To prove the sufficiency let x_1 be any point on the unit circle and define x_{i+1} inductively so that arc $[x_i, x_{i+1}]$ has length θ_i. Then define p_i so that arc $[x_i, p_i]$ has length α_i. Next observe that arc $[p_i, x_{i+1}]$ has length

$$(2.26) \qquad \begin{aligned} \theta_i - \alpha_i &= \theta_i - \tfrac{1}{2}(\theta_i - \theta_{i+1} + \cdots + \theta_{i-1}) \\ &= \tfrac{1}{2}(\theta_{i+1} - \theta_{i+2} + \cdots - \theta_{i-1} + \theta_i) \\ &= \alpha_{i+1}. \end{aligned}$$

Equation 2.26 has two important consequences. First, $0 < \alpha_i$ by hypothesis, so $0 < \alpha_i < \theta_i$ and hence $p_i \in (x_i, x_{i+1})$. Second, x_i is the midpoint of the arc $[p_{i-1}, p_i]$. These facts say $\{[x_i, x_{i-1}]\}$ is the Dirichlet tessellation based on $\{p_1, \ldots, p_n\}$.

Remark. The hypothesis of Theorem 7 is not vacuous. If a circle is partitioned into five parts of lengths

$$\frac{\pi}{20}, \frac{18\pi}{20}, \frac{2\pi}{20}, \frac{18\pi}{20}, \frac{\pi}{20}$$

then

$$\alpha_1 = \frac{\pi}{40}(1 - 18 + 2 - 18 + 1) = \frac{-4\pi}{5}.$$

We shall encounter this example again later.

The study of tessellations of the circle into an even number of parts is trickier. For such a tessellation with $2k$ regions, Theorem 5 tells us

$$\sum_{j=1}^{2k} (-1)^j \theta_j = 0.$$

The arc lengths α_i are not determined by the θ_i. Indeed, we shall see that α_1 can be chosen subject only to certain inequality constraints, although the choice will force the values of the remaining α_i. The situation resembles that we encountered in studying Dirichlet tessellations on the line.

THEOREM 8. *Let n be even and $\theta_1, \ldots, \theta_n$ be positive real numbers whose sum is 2π. Then $\theta_1, \ldots, \theta_n$ corresponds to a Dirichlet tessellation of the unit circle if and only if*

$$(2.27) \quad \sum_{i=r}^{s} (-1)^{i-r} \theta_i \begin{cases} = 0 & \text{if } r = 1 \text{ and } s = n \\ > 0 & \text{if } s - r \text{ is even and } 1 \leqslant r < s \leqslant n \end{cases}.$$

Proof. Theorem 5 shows that the conditions on $\{\theta_1, \ldots, \theta_n\}$ are necessary. To prove the converse, observe first that Theorem 4 implies that the sequence $\theta_1, \ldots, \theta_{n-1}$ is the sequence of lengths for a Dirichlet tessellation of the line. Let the points p_i represent the sources and the points x_i the endpoints of the intervals, as in (2.3). Then $\theta_i = x_{i+1} - x_i$ and

$$\begin{aligned} 0 &= \sum_{i=1}^{n} (-1)^i \theta_i \\ &= \sum_{i=1}^{n-1} (-1)^i \theta_i + \theta_n \\ &= -(p_1 - x_1) - (x_n - p_{n-1}) + \theta_n. \end{aligned}$$

Therefore

$$\theta_u = (p_1 - x_1) + (x_n - p_{n-1})$$

$$= (x_1 - p_0) + (p_n - x_u).$$

Now take the interval $[p_0, p_n]$, whose length is 2π, and wrap it around the unit circle, identifying p_0 and p_n. The result is the desired Dirichlet tessellation. $\square$

3. PLANE TESSELLATIONS WITH 3-VALENT VERTICES

We start this section with a theorem showing that Dirichlet tessellations with vertices of valence 3 are in a sense typical and thus warrant special study.

THEOREM 9. *If the sources* **P** *are chosen at random in the plane then all the vertices of the resulting Dirichlet tessellation* **R** *will be 3-valent with probability* 1.

Proof. For four points chosen at random from the plane, the probability that any one of them is on the circle (or line) determined by the other three is 0. Since there are only finitely many quadruples in **P**, the probability that any four points from **P** lie on a circle is 0. But if there were an n-valent vertex with $n > 3$ then Theorem 3 would imply that n points in **P** were on a common circle. $\square$

Remark. Theorem 9 is true for tessellations in N-space when '$N + 1$' replaces '3'.

Let V be an n-valent vertex of a plane tessellation **R**. We study **R** near V by studying the tessellation induced by **R** on a unit circle Γ centered at V: that is, we study the angles made by various rays emanating from V. The following definitions are illustrated for $n = 5$ in Figure 2. Number the n regions which contain V as $R_1, \ldots, R_n$, so that R_i and R_{i+1} are neighbors. Read subscripts modulo n. Write ∂_i for the boundary $R_{i-1} \cap R_i$ between R_{i-1} and R_i; let x_i be the intersection of Γ with the ray from V along ∂_i, and θ_i the length of the arc $[x_i, x_{i+1}]$. When **R** is a Dirichlet tessellation let λ_i be the ray from V through P_i, the source in R_i. Let p_i be the intersection of that ray with Γ, and α_i the length of the arc (x_i, p_i). In that case all of Theorem 5 is applicable. When **R** is not known to be a Dirichlet tessellation

and n is odd, *define* the angles α_i and hence the points $p_i \in \Gamma$ using Equation (2.25). That is, let
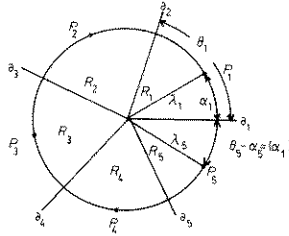
$$(3.1) \qquad \lambda_i = \lambda(R_i, V)$$

Fig. 2. A 5-valent vertex of a Dirichlet tessellation.

be the ray emanating from V which makes an angle of α_i with ∂_i, and let

$$(3.2) \qquad p_i = \lambda_i \cap \Gamma.$$

We now rewrite the equations in our discussion of tessellations of the circle when $n = 3$.

COROLLARY 10. *If V is a 3-valent vertex of a proper convex tessellation then*

$$(3.3) \qquad \alpha_1 = \pi - \theta_2 = \theta_1 + \theta_3 - \pi$$

and

$$(3.4) \qquad 0 < \alpha_1 < \theta_1$$

(and similarly for α_2 and α_3). In particular at a 3-valent vertex a proper convex tessellation is locally Dirichlet.

Proof. Equation (2.25) says

$$(3.5) \qquad 2\alpha_1 = \theta_1 - \theta_2 + \theta_3.$$

If we subtract the equation $2\pi = \theta_1 + \theta_2 + \theta_3$ we obtain the first part of (3.3) and if we add that equation we obtain the second part. Since the tessellation is proper, $\theta_2 < \pi$ so $\alpha_1 > 0$. But $\theta_3 < \pi$ also, so $\alpha_1 = \theta_1 + (\theta_3 - \pi) < \theta_1$. $\qquad\square$

We are now ready to look for conditions on a tessellation of the plane into convex polygons which will allow us to decide whether it is a Dirichlet tessellation. Our next two theorems show that not everything is possible.

Let **R** *be a plane Dirichlet tessellation: suppose the boundary $\hat{c}_{PQ}$ of two neighboring regions joins two 3-valent vertices V and W. Let θ be the angle at V which does not have $\hat{c}_{PQ}$ as a bounding ray, and ϕ the corresponding angle at W. Then*

$$\pi < \theta + \phi.$$

Proof. Corollary 10 implies that the angle at V in triangle VPW is $\pi - \theta$, while the angle at W is $\pi - \phi$. Since these angles must sum to less than π, the theorem follows. (See Figure 3.) ☐

Fig. 3 Not part of a Dirichlet tessellation.

The next theorem can be regarded as a generalization of inequality (2.15) to N-space.

THEOREM 11. *Suppose V and W are vertices of a region R. Let S be another region with vertex V and T be another region with vertex W. Then*

$$|V - W| < \text{diam}(S) + \text{diam}(T).$$

Proof. Let $C(X)$ denote the source in region X. Then

$$|V - W| \leq |V - C(R)| + |W - C(R)|$$
$$= |V - C(S)| + |W - C(T)|$$
$$< \text{diam}(S) + \text{diam}(T). \qquad \square$$

Our next task is to use Theorems 3 and 7 to produce necessary and sufficient conditions which tell whether a given tessellation of the plane is a Dirichlet tessellation. The crux of the argument is the observation that the ray $\lambda(R, V)$ defined in (3.1) must contain the hypothetical source P in R. The inequalities $0 < \alpha_i < \theta_i$, which we shall always require, say that $\lambda(R_i, V)$ meets the interior of R_i. The next lemma draws a global consequence of that fact.

LEMMA 12. *Let* **R** *be a proper convex plane tessellation and R a region of* **R** *all of whose vertices have odd valence. Suppose that for each such vertex V,* $\lambda(R, V) \cap \text{int}(R)$ *is not empty (that will always be true for the 3-valent vertices, by Corollary* 10), *and*

$$P \in \bigcap_{V \text{ a vertex of } R} \lambda(R, V).$$

Then $P \in \text{int}(R)$.

Proof. Let **H** be a minimal set of open half-planes whose intersection is the interior of R. Suppose $H \in \mathbf{H}$; we must show $P \in H$. The minimality of **H** implies that the closure of H contains some vertex V of R. Since the ray $\lambda(R, V)$ which starts at V meets the interior of R, it meets H, and thus $\lambda(R, V) - \{V\}$ is entirely contained in H. Hence $P \in H$. □

The next lemma, the last before the main theorem on tessellations with 3-valent vertices, shows that we need look only at neighboring regions to decide when a tessellation is a Dirichlet tessellation.

LEMMA 13. *Let* **R** *be a convex tessellation and suppose that for each* $R \in \mathbf{R}$ *we have a* $P \in R$. *Let* **P** *be the set of those points P, and* **R**′ *the Dirichlet tessellation based on* **P**. *If, whenever R and S are neighbors in* **R**, *with* $P \in R$ *and* $Q \in S$ *the corresponding points, the boundary* $\mathbf{R} \cap S$ *lies on the perpendicular bisector of segment PQ, then* **R** = **R**′.

Proof. Let R be a region of **R** and P the point chosen in R. For each neighbor S of R let $R \cap S^+$ be the closed half-space determined by $R \cap S$ which contains R. Then since R is convex.

$$
\begin{aligned}
(3.6) \qquad R &= \bigcap_{S \text{ a neighbor of } R} R \cap S^+ \\
&= \bigcap \{H_{PQ} : Q \in S \text{ a neighbor of } R\} \\
&\supseteq \bigcap \{H_{PQ} : Q \neq P, Q \in P\} \\
&= R_P \in \mathbf{R}'.
\end{aligned}
$$

But the interiors of the regions $R \in \mathbf{R}$ are disjoint open sets, and $\bigcup_{P \in \mathbf{P}} R_P$ is all of Σ. Consequently, $R = R_P$. □

THEOREM 14. *Let* **R** *be a proper convex plane tessellation all of whose vertices are 3-valent. Then* **R** *is a Dirichlet tessellation if and only if for each region R the rays* $\lambda(R, V)$ *have a point in common. If there is exactly one such point it is the source in R.*

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Proof. When **R** is a Dirichlet tessellation, Theorem 3 assures that $\lambda(R_P, V)$ passes through the source P. Thus P serves as a point which the rays $\lambda(R_P, V)$ share.

Conversely, suppose that for each region R the rays $\lambda(R, V)$ share a point. We wish to choose a source

$$(3.7) \qquad P \in \bigcap_{V \text{ a vertex of } R} \lambda(R, V)$$

for the region R.

Note that whenever P satisfies (3.7), Lemma 13 implies P belongs to the interior of R. Let R and R' be neighboring regions. If we can choose the sources $P \in R$ and $P' \in R'$ so that the boundary $R \cap R'$ is part of the perpendicular bisector of segment PP' then Lemma 14 will guarantee that **R** is the Dirichlet tessellation based on **P**.

(0) Usually the intersection on the right in (3.7) will contain just one point and we shall have no choice for P. That clearly happens when the region R has two vertices: if V and W are vertices joined by a line segment on the boundary of R the rays $\lambda(R, V)$ and $\lambda(R, W)$ cannot be parallel. Since they meet, they meet in a single point. In such a case let R' be the region in **R** for which $R \cap R' = VW$.

Corollary 10 and Theorem 7 imply that the rays λ_i and θ_i emanating from any vertex of **R** form a Dirichlet tessellation of a circle centered at that vertex. Thus the angles WVP and WVP' at V are equal, and the corresponding angles at W are also equal. Since the triangles share side VW, they are congruent. Hence VW is part of the perpendicular bisector of PP', as desired. (See Figure 4.)
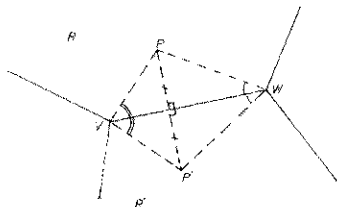

Fig. 4. Illustrating case (0), Theorem 14.

Next consider two regions R, R' whose boundary $R \cap R'$ is a ray with a single vertex V. Let R'' be the third region with V as a vertex. We finish the proof by considering the three cases in which both, one or neither of the boundaries $R'' \cap R$ and $R'' \cap R'$ contains a vertex of R'' other than V. Note that for a region with only one vertex the intersection condition in the hypothesis is trivially true.

(1) Suppose $R'' \cap R$ contains a second vertex $W \neq V$, and $R'' \cap R'$ contains a second vertex W'. (See Figure 5(a).) Then the argument in case (0) shows that sources P, P', and P'' in R, R', and R'' respectively are all uniquely determined, that $|P - V| = |P'' - V|$ and that $|P' - V| = |P'' - V|$. Hence P and P' are equidistant from V. Since $P \in \lambda(R, V)$ and $P' \in \lambda(R', V)$, $R \cap R'$ is part of the perpendicular bisector of PP'.
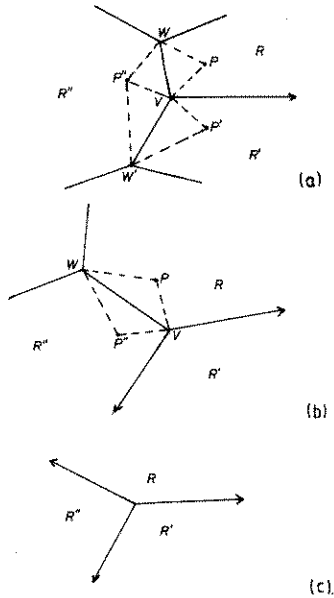

Fig. 5. Illustrating Theorem 14.

(2) Suppose only $R \cap R''$ contains a second vertex $W \neq V$. (See Figure 5(b).) Then the argument in case (0) determines P on $\lambda(R, V)$ and P'' on $\lambda(R'', V)$, equidistant from V. Since the rays leaving V form a Dirichlet tessellation of the circle, the mirror images of P in $R \cap R'$ and P'' in $R'' \cap R'$ each lie on $\lambda(R', V)$. Since those points are equidistant from V, they coincide, and the resulting point clearly serves as a source for R'.

(3) None of the three boundaries meeting at V contains a vertex other than V. Then V is the only vertex of the entire tessellation, which is then completely pictured in Figure 5(c). In that case choose a circle of arbitrary radius centered at V and let P, P', and P'' be the intersections of that circle with $\lambda(R, V)$, $\lambda(R', V)$, and $\lambda(R'', V)$. Then invoke again the fact that the rays at V are a Dirichlet tessellation of the circle to show that the original tessellation is the Dirichlet tessellation based on $\{P, P', P''\}$.   ⊐

COROLLARY 15. *If a tessellation satisfying the conditions of Theorem 15 has more than one vertex, the sources* **P** *are uniquely determined.*

Remark 1. We have excluded improper convex tessellations: those with no vertices. In the plane such a tessellation consists of parallel strips. In any dimension, recognizing when such a tessellation is a Dirichlet tessellation reduces to the one-dimensional case: it is necessary and sufficient that the thicknesses t_i of the strips satisfy inequalities (2.4). Theorem 4 can then be used to choose the sources on a line perpendicular to the hyperplanes bounding the strips.

Remark 2. Theorem 15 gives another way of looking at Theorem 11 and Figure 3. In that configuration $\lambda(R, V)$ makes an angle of $\pi - \theta$ with segment VW, while $\lambda(R, W)$ makes an angle of $\pi - \phi$. Unless $\theta + \phi > \pi$, the rays $\lambda(R, V)$ and $\lambda(R, W)$ will be disjoint and the hypothesis of Theorem 15 will not be satisfied.

Remark 3. There is a surprising connection between Dirichlet tessellations and the statics of stressed plane frameworks. A plane tessellation is said to have a *reciprocal figure* when it is possible to choose sources **P**, one for each region, such that $R \cap R'$ is perpendicular to PP' whenever R and R' are neighbors. Thus a Dirichlet tessellation has a reciprocal figure, though not conversely. Maxwell [22] showed that a bar and joint framework in the plane has a reciprocal figure just when it supports a nontrivial stress. It follows that the boundary of a Dirichlet tessellation can be realized as an equilibrium state of a spider web (though not conversely). It is easy to see that Figure 3, which is not a Dirichlet tessellation, has a reciprocal figure, and easy to see how with appropriate stresses on the edges it will be in

mechanical equilibrium. It would be interesting to know what special
mechanical properties are enjoyed by Dirichlet tessellations. In a forth-
coming paper we shall show how to characterize the tessellations with
reciprocal figures as sections of three-dimensional Dirichlet tessellations.

Maxwell showed, too, that the projection of the boundary of a convex
3-polytope has a reciprocal figure. Crapo [5] and Whiteley [37] have care-
fully proved the converse. In spite of the mention of right angles in the
definition of the reciprocal figure, the existence of such a figure depends
only on projective properties of the original tessellation. That is not so of
the existence of sources which make a given tessellation a Dirichlet tessella-
tion. (There are projections of Figure 3 which are Dirichlet tessellations.)
Nevertheless, the projective conditions which must be satisfied in order for
a tessellation to have a reciprocal figure can aid in the recognition of
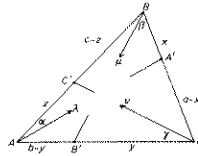Dirichlet tessellations. We give two such projective conditions.

Fig. 6. Rays entering a triangle.

The condition in Theorem 15 which requires that for each region R the
constructed rays $\lambda(R, V)$ concur is necessary but somewhat artificial. We
can substantially improve it for triangular regions. We start with some
lemmas on triangles. Let ABC be a triangle; write A, B, C too for the angles
at those vertices. Let λ, μ, v be rays emanating from A, B, C making angles
α, β, γ with sides AB, BC, CA respectively. (See Figure 6.) Let λ divide the
side BC opposite A into segments BA', $A'C$ with (directed) lengths x, $a - x$.
Similarly define y and z.

LEMMA 16. *Rays λ, μ, and v concur if and only if*

$$(3.8) \qquad \frac{\sin \alpha}{\sin(A - \alpha)} \frac{\sin \beta}{\sin(B - \beta)} \frac{\sin \gamma}{\sin(C - \gamma)} = 1.$$

Proof. Ceva's theorem [7, p. 220] says that λ, μ, and ν concur if and only if

$$(3.9) \qquad \frac{x}{a-x} \cdot \frac{y}{b-y} \cdot \frac{z}{c-z} = 1.$$

Use the law of sines to rewrite (3.9) in terms of the angles x, β, γ, A, B, and C:

$$\frac{x}{\sin x} = \frac{|AA'|}{\sin B}$$

and

$$\frac{a-x}{\sin(A-\alpha)} = \frac{|AA'|}{\sin C}$$

so

$$\frac{x \sin B}{\sin x} = \frac{(a-x)\sin C}{\sin(A-x)}$$

and

$$\frac{x}{a-x} = \frac{\sin \alpha}{\sin(A-x)} \frac{\sin C}{\sin B}.$$

Cyclic permutation of A, B, and C then shows that the product on the left in (3.9) is

$$\frac{\sin \alpha}{\sin(A-x)} \frac{\sin \beta}{\sin(B-\beta)} \frac{\sin \gamma}{\sin(C-\gamma)}.$$

The lemma follows.   □

LEMMA 17. *Let λ', μ', and ν' be the reflections of λ, μ, and ν in the angle bisectors of angles A, B, and C respectively. Then λ, μ, and ν concur if and only if λ', μ', and ν' do.*

Proof. Exchanging each ray with its primed counterpart exchanges α and $A - \alpha$, β and $B - \beta$, γ and $C - \gamma$, thus inverting the ratio in (3.9).   □

Remark. Lemma 17 defines a curious map from the interior of triangle ABC to itself. For P in that triangle, let $\lambda = AP$, $\mu = BP$, $\nu = CP$, reflect over angle bisectors, and let $f(P)$ be the intersection of the resulting rays. This map will help us discover when triangles are part of Dirichlet tessellations. We shall generalize it in Section 5, when we see how to recognize

Dirichlet tessellations in higher dimensions. Even in the plane the map f has interesting geometric properties which warrant further study. It has one fixed point, the intersection of the angle bisectors. As P approaches any point on BC, $f(P)$ approaches A. Moreover, Lemma 17 defines f even on the exterior of ABC. One can think of $f(A)$ as the whole line determined by B and C.

THEOREM 18. *Three 3-valent vertices forming a triangle are part of a Dirichlet tessellation if and only if the rays exterior to the triangle concur when extended. The point of concurrence is in the interior of the triangle.*

   Proof. Call the triangular region R and its vertices U, V, and W. Call the three boundary lines exterior to the triangle at U, V, and W, ∂_U, ∂_V, and ∂_W respectively. (See Figure 7.) Then by Theorem 15 the construction is part of a Dirichlet tessellation if and only if the rays $\lambda(R, U)$, $\lambda(R, V)$, and $\lambda(R, W)$ concur. But by Lemma 18, that happens just when their reflections in the angle bisectors concur. And by Corollary 10, these reflections are precisely the extensions of ∂_U, ∂_V, and ∂_W. Since the reflection map f preserves the interior of R, the point of concurrence is in R.   □
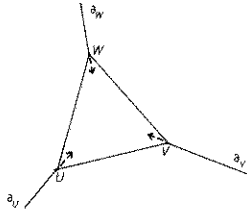

Fig. 7. A triangular candidate.

   Remark. The implication in one direction could have been proved by simply letting P_i ($i = 1, 2, 3$) be the sources in the regions exterior to R and observing that ∂_U, ∂_V, and ∂_W are portions of the perpendicular bisectors of the sides of triangle $P_1 P_2 P_3$ and are therefore concurrent. However, proving the converse — that concurrence implies a Dirichlet tessellation — seems to require the machinery we have developed.

   It is sometimes useful to think of the configuration in Figure 7 as the result of creating a new source P and hence a new region R which covers the intersection T which, before the addition of P, was the vertex at which R_1, R_2, and R_3 met. (P must be interior to the circle Γ containing P_1, P_2, and P_3; if it were on Γ the vertex T would become 4-valent instead of

disappearing.) Adding a second new source and obliterating another vertex creates a quadrangular region with 3-valent vertices and leads to the following theorem.

THEOREM 19. *Suppose ABCD is a convex quadrilateral with 3-valent vertices. Let ∂_A, ∂_B, ∂_C, and ∂_D be the boundary rays at A, B, C, and D respectively that are exterior to ABCD. Let X be the intersection of ∂_A and ∂_B and let Y be the intersection of ∂_C and ∂_D when the rays are extended. If this configuration is part of a Dirichlet tessellation, then the lines through AD, BC, and XY concur in the sense of projective geometry, e.g. if AD is parallel to BC then so must XY be, if ∂_A is parallel to ∂_B then XY is the line through Y parallel to ∂_A. (See Figure 8.)*
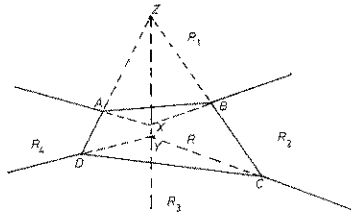

Fig. 8. A quadrangular candidate.

Proof. Call the interior of *ABCD* region *R*, and label the regions surrounding *R* by R_1, R_2, R_3, R_4 as in the figure. Let P_i be the source in R_i and let *P* be the source in *R*. Then *X* is on the perpendicular bisectors of P_1P_4 and of P_1P_2 and therefore on the perpendicular bisector of P_2P_4. Similarly, *Y* is on the perpendicular bisectors of P_2P_3 and P_3P_4 and therefore on the perpendicular bisector of P_2P_4. Now *AD* is on the perpendicular bisector of PP_4 and *BC* is on the perpendicular bisector of PP_2 so that the intersection of *AD* and *BC*, which we call *Z*, is on the perpendicular bisector of P_2P_4. But *X* and *Y* are also on the perpendicular bisector of P_2P_4. Therefore the lines *AD*, *BC*, and *XY* concur at *Z*. Note that if *AD* and *BC* are parallel the triangle PP_2P_4 degenerates to a straight line, and the point of concurrence of the lines is a point at infinity.   ⊡

Remark. The converse of the theorem is false. The projective conditions in the hypothesis suffice to prove the existence of a reciprocal figure, or equivalently, that Figure 8 is the projection of a convex polyhedron, but, as

we have seen, that does not guarantee that we started with a Dirichlet
tessellation. If in this example the sum of the angle formed by ∂_A and AD
and the angle formed by ∂_B and BC does not exceed π then Theorem 11
says the configuration can not be part of a Dirichlet tessellation.

4. PLANE TESSELLATIONS WITH VERTICES OF HIGH VALENCE

We next generalize Theorem 15 to tessellations with vertices of odd valence.

THEOREM 20. *Let* **R** *be a proper convex plane tessellation all of whose
vertices have odd valence. Suppose that*

(1) **R** *is a Dirichlet tessellation at each vertex* V;
(2) *if any boundary ray from a vertex* V *is unbounded, then all such
unbounded rays are adjacent, in the sense that it is possible to traverse a
small circle around* V *meeting first all the bounded rays (if any) and
then all the unbounded ones.*

Then **R** *is a Dirichlet tessellation if and only if for each region* R *the rays*
$\lambda(R, V)$ *concur.*

Remarks. Theorem 3 shows that (1) is satisfied by Dirichlet tessellations.
Corollary 10 shows that (1) is always true for 3-valent vertices. Note too
that (2) is trivially true for 3-valent vertices. We shall investigate the
meaning of (2) further after we prove the theorem.

Proof. The proof of Theorem 21 is modelled on that of Theorem 15. If **R** is
a Dirichlet tessellation then when $R = R_P$ the discussion preceding Corol-
lary 10 shows that $\lambda(R, V)$ passes through the source P, which thus lies on
the intersection

$$\bigcap_{V \text{ a vertex of } R} \lambda(R, V).$$

Hence that intersection is nonempty.

Conversely, suppose that for each R that intersection is nonempty.
Lemma 13 guarantees that whenever P lies in that intersection, $P \in \text{int}(R)$;
note that we have included the hypotheses for Lemma 13 in assumption (1)
about **R**. Thus, as before, it suffices to choose the sources $P \in R$, $P' \in R'$ so
that for neighboring regions, $R \cap R'$ is a part of the perpendicular bisector
of PP'.

(0) When $R \cap R'$ contains two vertices the argument is identical to that
in Theorem 15; the sources P and P' are uniquely determined.

(1) Next consider two regions R, R' whose boundary $R \cap R'$ is a ray
with a single vertex V. If none of the boundary rays emanating from V is

bounded (that is, if none contains a vertex other than V) then the argument proceeds as in case (3) of Theorem 15; $\mathbf{R}$ is easily seen to be a Dirichlet tessellation.

(2) Thus we may suppose there are $k > 0$ bounded rays emanating from V. Since they are adjacent, they determine $k + 1$ regions $R_0, \ldots, R_k$ for which the argument in (0) applies and the source is determined. Repeated application of the argument in case (1) of Theorem 15 shows that the sources $P_0, \ldots, P_k$ are equidistant from V. Then reflecting P_k successively across the unbounded rays determines the rest of the sources (if any) and the fact that the tessellation is a Dirichlet tessellation at the vertex V shows that the unbounded rays are the perpendicular bisectors of the required segments. Finally, the argument in case (2) of Theorem 15 shows that the last reflection just gives the source P_0.   □

Remark. The following example shows that in Theorem 21 some condition like (2) is necessary. It shows why the regions around V in which the sources are determined must be adjacent. In Figure 9 $|VW| = 2$, $|VU| = 1$, all angles θ_i about V measure $2\pi/5$ and the angles P_1VW, P_1WV, P_2UV, and P_2VU are each $\pi/5$. Thus triangles P_1VW and P_2UV are similar and $|VP_1| = 2|VP_2|$. Since $|VP_1| \neq |VP_2|$, this cannot be Dirichlet tessellation. Note that the boundary lines $R_2 \cap R_7$, $R_2 \cap R_1$, and $R_1 \cap R_6$ are parallel, as are the pairs $R_7 \cap R_3$ and $R_3 \cap R_4$, and $R_4 \cap R_5$ and $R_5 \cap R_6$. Thus there are no hidden vertices. A slight perturbation of this counterexample gives one without parallel lines.
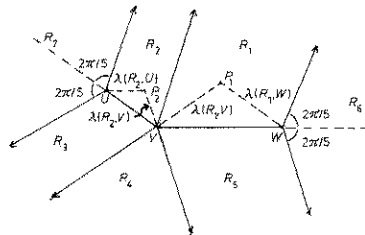

Fig. 9. An example with unbounded rays.

Another modification of this example shows that condition (2) itself need not always be satisfied by a Dirichlet tessellation: choose five sources at the vertices of a regular pentagon and two more as in Figure 10. In the

resulting Dirichlet tessellation the unbounded rays at V are not adjacent in the sense of (2).
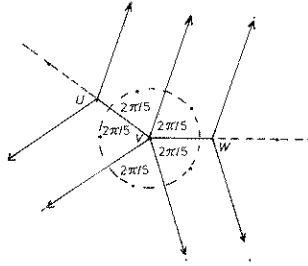

Fig. 10. A Dirichlet tessellation.

If a plane tessellation has vertices of even valence there are additional problems we have not yet solved. Of course, we assume that the tessellation is a Dirichlet tessellation at such vertices V too. But Theorem 8 tells us that the angles α_i, and hence the rays $\lambda(R, V)$, are not uniquely determined, so a condition like that given in Theorem 15 cannot be hoped for. When there are not too many even vertices there are enough rays $\lambda(R, V)$ to determine the positions of the sources, which must be consistent with some *ad hoc* conditions at each even vertex. But we do not see what the general theorem is.

But one case in which all the vertices are 4-valent is easy to handle. Consider a tessellation formed by m vertical lines and n horizontal lines ($n, m \geqslant 2$). Since the lines cross at right angles, the tessellation is locally Dirichlet. At each vertex one angle, α_1, say, is arbitrary subject to an inequality constraint. If the tessellation is a Dirichlet tessellation, once we know the source in any one rectangular region we can locate all the sources by a sequence of reflections over the horizontal and vertical boundaries. Thus, the Dirichlet tessellation is a Cartesian product of two one-dimensional Dirichlet tessellations. (See Figure 11.) Therefore, it is the spacing of the parallel lines which determines whether a tessellation of this type is a Dirichlet tessellation; that is, such a tessellation is a Dirichlet tessellation if

$$\frac{s}{\cdot\cdot}$$

and

$$\sum_{j=r}^{t}(-1)^{j-r}k_j > 0 \quad (t-r \text{ even}, \ 1 \leqslant r \leqslant t \leqslant n),$$

where h_i is the distance between the ith and $(i+1)$st vertical lines and k_j is the distance between the jth and $(j+1)$st horizontal line.
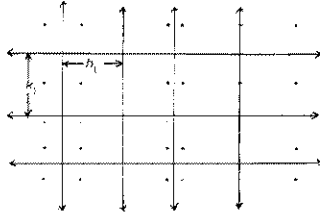

Fig. 11. The product of two one-dimensional tessellations.

5. HIGHER DIMENSIONAL TESSELLATIONS

In this section we indicate how to generalize several of the theorems we have proved for plane tessellations. Since much of the argument depends on localizing at vertices, the exposition is streamlined by considering only tessellations on N-spheres. That limitation also rules out the bizarre behavior of unbounded rays we took such care with in the previous two sections. Moreover, we shall restrict attention to tessellations of the N-sphere all of whose vertices have valance $N+1$. The remark following Theorem 9 shows that behavior is typical. An interested reader may work out for herself what our theorems say when properly generalized to the customary rectilinear tessellations of N-space.

A tessellation $\mathbf{R}$ of an N-sphere Γ induces a tessellation $\mathbf{R}^*$ of the ambient $N+1$-space: if $R \in \mathbf{R}$ let R^* be the cone over R with vertex the center of Γ. Recall that, following Grunbaum [13, p. 30] we shall say $\mathbf{R}$ is *convex* when $\mathbf{R}^*$ is. $\mathbf{R}$ is *proper* if each region contains at least one pair of nonantipodal vertices.

We now generalize Theorem 15 in two stages. We begin by stating, without proof, a lemma about geometry on the N-sphere which generalizes

Euclid's angle-side-angle criterion for the congruence of triangles. Let the *base* of the nondegenerate N-simplex $\{P, V_1, \ldots, V_N\}$ on an N-sphere be the $N-1$-simplex $\{V_1, \ldots, V_N\}$ and the *base angles* the angles between the circular rays PV_i and the hypersphere containing the base.

LEMMA 21. *Two simplices with congruent bases and equal base angles are congruent. Hence two simplices with the same base and equal base angles are either identical or are mirror images with respect to the hypersphere containing the base.*

Now suppose **R** is a proper convex tessellation of an N-sphere Γ and V is a vertex at which **R** is locally Dirichlet. Let Γ_V be a small $(N-1)$-sphere centered at V. Then for each region $R \in \mathbf{R}$ which contains V we can find a source $p \in R \cap \Gamma_V$. Let $\lambda(R, V)$ be the great circular ray which starts at V and proceeds through p.

THEOREM 22. *Let **R** be a proper convex tessellation of an N-sphere. Then **R** is Dirichlet tessellation if and only if*

(1) *it is locally Dirichlet, and*

(2) *for each region R, the rays $\lambda(R, V)$ have a point in R in common.*

Proof. We copy the proof of Theorem 15. When **R** is a Dirichlet tessellation, Theorem 3 assures that $\lambda(R_P, V)$ passes through the source P. Thus P serves as a point which the rays $\lambda(R_P, V)$ share.

Conversely, suppose that for each region R the rays $\lambda(R, V)$ share a point in R. Since R contains at least one pair of nonantipodal vertices, those rays have a unique intersection in R:

$$(5.1) \qquad P = \bigcap_{V \text{ a vertex of } R} \lambda(R, V) \cap R.$$

We shall show that P can serve as the source in the region R.

Let R and R' be neighboring regions and $P \in R$ and $P' \in R'$ the corresponding hypothetical sources. Since the N-dimensional spherical analogue of Lemma 14 is true, once we show that the boundary $R \cap R'$ is part of the perpendicular bisector of segment PP' we can conclude that **R** is the Dirichlet tessellation based on **P**.

Since $R \cap R'$ is $(N-1)$-dimensional, we can find a nondegenerate $(N-1)$-simplex $\sigma = \{V_1, \ldots, V_N\}$ of vertices shared by R and R'. Since **R** is Dirichlet at each V_i, the rays $\lambda(R, V_i)$ and $\lambda(R', V_i)$ make equal angles at V_i with the hypersphere H containing σ, and lie on opposite sides of H. Then Lemma 21, applied to the N-simplices P, σ and P', σ, shows that P is the reflection in H of P'.

$\square$

The second stage of our development is to eliminate hypothesis (1) from Theorem 22 when all the vertices are $(N + 1)$-valent by proving that such vertices are always Dirichlet. To do so we must generalize Theorem 18. And that we accomplish by following Andrew Gleason's suggestion that the heart of that theorem was the inversion of barycentric coordinates.

Let $\sigma = \{V_0, \ldots, V_N\}$ be a nondegenerate simplex in N-space. When we come to use such simplices later they will lie on the N-sphere, but now we wish to study rectilinear, not spherical geometry. Each point P in the hyperplane H which is the affine span of σ can be uniquely expressed in the form

$$P = b_0 V_0 + \cdots + b_N V_N$$

if we demand that

$$b_0 + \cdots + b_N = 1.$$

We call the sequence

$$b(P) = \langle b_0, \ldots, b_N \rangle$$

the *barycentric coordinates* of P. P is in the interior of σ just when all its barycentric coordinates are positive. Suppose P is such a point. Let $r(P)$ be the point in σ whose barycentric coordinates are the scalars $1/b_i$, normalized to sum to 1.

Remark 1. When $N = 1$, σ is a line segment $V_0 V_1$ and $r(P)$ is just the reflection of P in the midpoint of the segment.

Next we study what happens when we project P and $r(P)$ onto each facet of σ. Let σ_j be the $(N - 1)$-simplex spanned by all the vertices of σ except V_j, and P^j the intersection of line $V_j P$ with σ_j. For $j = 0, \ldots, N$ call the barycentric coordinates of P^j the jth local barycentric coordinates of P in σ. They are the scalars

$$b(P^j) = \langle b_i(P) \rangle, \quad i \neq j,$$

normalized to sum of 1.

Remark 2. The local barycentric coordinates of $r(P)$ are the inverses of the local barycentric coordinates of P:

$$b(r(P^j)) = b(r(P)^j).$$

Suppose now that we are given $P^j \in \sigma_j, j = 0, \ldots, N$. Then the $N + 1$ lines $V_j P^j$ will concur in a point in σ if and only if the barycentric coordinates

$b(P^j)$ satisfy a set $\mathbf{C}_N$ of consistency conditions. When $N = 2$ Ceva's theorem asserts that $\mathbf{C}_2$ contains the single condition

$$\frac{b_1(P^3)}{b_2(P^3)} \frac{b_2(P^1)}{b_3(P^1)} \frac{b_3(P^2)}{b_1(P^2)} = 1.$$

A precise statement of the contents of $\mathbf{C}_N$ would constitute a generalization of Ceva's theorem. We do not need that much precision. The following remark suffices.

Remark 3. If a set of local barycentric coordinates, one sequence for each facet, satisfy the identities in $\mathbf{C}_N$ then so do the inverted local coordinates.

The proof is straightforward. Since the original local coordinates satisfy $\mathbf{C}_N$, they correspond to a point P in σ. Then Remark 2 shows the inverted local coordinates correspond to $r(P)$, and hence satisfy $\mathbf{C}_N$.

Let σ be a spherical N-simplex on the N-sphere Γ centered at C. Define barycentric coordinates and the map r on σ by identifying each point P in σ on Γ with the intersection $\bar{P}$ of the radius CP and the hyperplane in $(N + 1)$-space spanned by the vertices of σ. Thus $r(P)$ is the point $Q \in \sigma$ such that $\bar{Q} = r(\bar{P})$.

When $N = 1$, σ is a circular arc V_0V_1 and $r(P)$ is just the reflection of P in the midpoint of that arc.

The map r on simplices is related to but not identical with the curious map f discussed in Section 3. That map depended on symmetry about angle bisectors while this one is based on symmetry about midpoints. In Section 3 we connected the two with the law of sines. Here we shall use a duality argument on the N-sphere.

Each simplex σ on the N-sphere has a polar simplex σ^0 whose vertices V_j^0 are poles corresponding to the hyperspheres containing the facets σ_j and whose facets σ_j^0 lie on the hyperspheres for which the original vertices V_j are the poles. Points on σ_j^0 correspond to great circular rays emanating from V_j, and vice versa. Moreover, when $N = 2$, so that each facet is a circular arc, reflection of rays over angle bisectors at a vertex corresponds to reflection about the midpoint of the corresponding arc in the polar triangle.

Now we are ready to generalize Lemma 17 and Theorem 18. Let V be an $(N + 1)$-valent vertex of a tessellation $\mathbf{R}$ of the N-sphere. For each region R of $\mathbf{R}$ containing V there is exactly one ray at V which is not a bounding ray of R. Let $\tau(R, V)$ be the extension of that ray through V. Since $\mathbf{R}$ is proper $\tau(R, V)$ meets the interior of R.

THEOREM 23. *An* $(N + 1)$-*valent vertex of a proper convex tessellation* **R** *of the N-sphere is a Dirichlet vertex, and for each region R containing V,*

(5.2) $\lambda(R, V) = r(\tau(R, V))$.

Proof. In Section 3 we proved the theorem in the plane. The proof on the 2-sphere is essentially the same, so we can now proceed by induction. Let Γ be a small $(N - 1)$-sphere centered at V, and examine the tessellation $\mathbf{R}_\Gamma$ which **R** induces on Γ. It has $N + 1$ vertices $V_0, \ldots, V_N$ and $N + 1$ simplicial regions, each of which has vertices of valence N. Figure 12 illustrates the case $N = 3$. We show first that $\mathbf{R}_\Gamma$ is a Dirichlet tessellation, thus showing that **R** is Dirichlet at V.
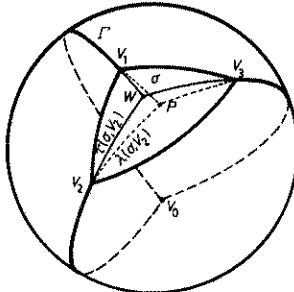

Fig. 12. A tessellation of the 2-sphere induced by a 4-valent vertex of a three-dimensional tessellation.

Let $\sigma = \{V_1, \ldots, V_N\}$ be one of the regions of $\mathbf{R}_\Gamma$. The ray $\tau(\sigma, V_j)$ entering σ from vertex V_j is the extension of arc $V_0 V_j$ and hence lies on a great circle through V_0. The N great circles so constructed meet again at the point W antipodal to V_0, so the N arcs $\tau(\sigma, V_j)$ concur. Since, by induction,

$\lambda(\sigma, V_j) = r(\tau(\sigma, V_j))$

it follows from Remark 3 and the definition of the map r that the rays $\lambda(\sigma, V_j)$ concur at a point P in σ. Theorem 22 then implies $\mathbf{R}_\Gamma$ is a Dirichlet tessellation, so the original tessellation **R** is Dirichlet at V.

To finish the proof we must verify (5.2). To do so, suppose that R is a region of **R** with V as a vertex and that $R \cap \Gamma$ is the simplex σ labelled as above. Then $\tau(R, V)$ lies on the great circle joining V_0 to V in the original tessellation. That great circle is a diameter of Γ, and hence passes through W. (Figure 13 illustrates the case $N = 3$.)
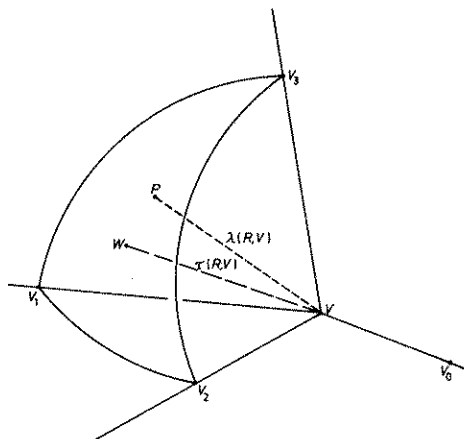
Fig. 13. Illustrating Theorem 23.

Since, inductively, $P = r(W)$, and $\lambda(R, V)$ passes through P while $\tau(R, V)$ passes through W, it follows that $\lambda(R, V) = r(\tau(R, V))$.   □

Thus, coupling Theorems 22 and 23, we have proved

THEOREM 24. *On an N-sphere a proper convex tessellation all of whose vertices have valence $N + 1$ is a Dirichlet tessellation if and only if for each region R the rays*

$$\lambda(R, V) = r(\tau(R, V)) \quad (V \text{ a vertex of } R)$$

concur at a point in R. For simplicial regions that is equivalent to the concurrence of the rays $\tau(R, V)$.

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