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GENERALIZED DIRICHLET TESSELLATIONS

ABSTRACT. In this paper we study how to recognize when a dissection of the plane has been constructed in one of several natural ways each of which models some phenomena in the natural or social sciences. The prototypical case is the nearest-neighbor or Dirichlet tessellation.

1. INTRODUCTION

Let $\mathbf{P}$ be a finite set of points in the plane. For each pair P, Q of points in $\mathbf{P}$ let

(1.1) $$H_{PQ} = \{X: |X - P| \leqslant |X - Q|\},$$

and

(1.2) $$K_{PQ} = H_{PQ} \cap H_{QP} = \{X: |X - P| = |X - Q|\}.$$

H_{PQ} is a half plane; K_{PQ} is its boundary. Then, for each $P \in \mathbf{P}$ let

(1.3) $$R_P = \bigcap_{Q \neq P} H_{PQ}$$

and

(1.4) $$\mathbf{R} = \{R_P | P \in \mathbf{P}\}.$$

We call $\mathbf{R} = \mathbf{R}(\mathbf{P})$ the *Dirichlet tessellation* of the plane *based on* $\mathbf{P}$. Each region $\mathbf{R}_P$ is convex and contains its *source* P. The union of the regions R_P is the whole plane. We set

(1.5) $$\partial_{PQ} = R_P \cap R_Q;$$

then $\partial_{PQ} \subset K_{PQ}$ is an edge of each of R_P and R_Q, a vertex of each, or empty.

These definitions extend with little change to n-space or indeed to any metric space. See, for example, [9] for a discussion of Dirichlet tessellations on Riemannian manifolds. Here we limit ourselves to the plane, and generalize in other ways. We shall replace $|X - P|$ by $\phi(|X - P|)$ for various smooth increasing functions $\phi: \mathbb{R} \to \mathbb{R}$, and allow the sources to be weighted by a function $w: \mathbf{P} \to \mathbf{R}$, so that (1.1) and (1.2) become

(1.6) $$H_{PQ} = \{X: \phi(|X - P|) + w(P) \leqslant \phi(|X - Q|) + w(Q)\}$$

and

(1.7) $$K_{PQ} = H_{PQ} \cap H_{QP} = \{X: \phi(|X - P|) + w(P) = \phi(|X - Q|) + w(Q)\}$$

Then we define R_P using (1.3), and call $\mathbf{R} = \mathbf{R}(\mathbf{P}) = \mathbf{R}(\mathbf{P}, \phi, w)$ a Dirichlet ϕ-

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tessellation. The standard, rectilinear case corresponds to setting $\phi(t) = t$ and w a constant function.

Sections 2, 3, and 4 discuss tessellations corresponding to particular choices for the function ϕ, introducing techniques of general use as they are encountered. In Section 5 we study how properties of ϕ influence the shapes of the regions.

When we regard the points **P** as polling places, and election law requires that each person vote at the polling place nearest his or her home; the map of election districts is the Dirichlet tessellation **R**(**P**). In [1] we solved the *recognition problem*: when is a given dissection (of the plane) a Dirichlet tessellation for some set of sources? That solves the gerrymander problem: if the legislature draws the district lines first, we can tell them whether there is a set of polling places which satisfies election law, and, if so, how to find it.

To generalize we need some definition, and a few of the results from [1], which we state here for completeness.

Let **C** be a set of plane curves — usually the level curves of the functions

$$(1.8) \qquad X \to \phi(|X - P|) - \phi(|X - Q|)$$

for all pairs P, Q of points in the plane. A set **R** of sets which covers the plane is a **C**-*dissection* when for $R \neq S \in \mathbf{R}$, $R \cap S$ is a subset of one of the curves in **C**. When the curves in **C** are given by (1.8) we may refer to **R** as a ϕ-dissection.

Now we can state the general recognition problem. Given a ϕ-dissection **R** we seek necessary and sufficient conditions for the existence of a set **P** of sources and a weight function w such that **R** is described by Equations (1.6), (1.3), and (1.4). We shall see examples in which **C** consists of circles or of hyperbolas.

In any **C**-dissection a point which belongs to three or more regions is a *vertex* of the dissection and of each of the regions to which it belongs. The number of such regions is the *valence* of the vertex. Three-valent vertices are typical: if the points of **P** are chosen at random then with probability 1 all vertices of the resulting Dirichlet ϕ-tessellation are 3-valent. That was proved for ordinary Dirichlet tessellations in [1] and will be proved for ϕ-tessellations in Section 5. When all vertices of a dissection are 3-valent we call the dissection itself 3-valent.

Recall that a regular closed set in a topological space is a set which is the closure of its interior. A **C**-dissection is *proper* when each region is regular closed and at each vertex the angles formed by the tangent rays to the boundary curves from **C** which meet there are all less than π.

Propriety is a strong local convexity assertion. For rectilinear dissections it implies the covexity of each of the regions. Ordinary Dirichlet tessellations

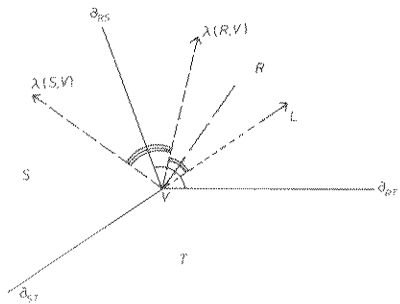

Fig. 1. Central rays at a vertex of a plane rectilinear tessellation.

with 3-valent vertices are proper. In Section 5 we shall show that ϕ-tessellations are almost always proper.

In [1] we solved the recognition problem for ordinary Dirichlet tessellations using the following construction, which we give here because it will prove useful for some generalized tessellations as well. Let R, S, and T be the regions which meet at a 3-valent vertex V of the proper rectilinear dissection $\mathbf{R}$. Let L be the extension of ∂_{ST} through V; L enters the interior of R since $\mathbf{R}$ is proper. Then let $\lambda(R, V)$, the *central ray* from V into R, be the reflection of L in the bisector of the angle formed by ∂_{RS} and ∂_{RT}. Figure 1 illustrates these definitions.

The following lemma gives a useful equivalent description of the central rays. We omit its easy proof.

LEMMA 1. *∂_{RS} makes equal angles with $\lambda(R, V)$ and $\lambda(S, V)$. Conversely, if three rays leave V so that the angles they form are bisected by the boundary arcs that meet at V then those rays are the central rays.*

Lemma 1 implies that when $\mathbf{R}$ is a Dirichlet tessellation, for each vertex V of each region R_P we have $P \in \lambda(R, V)$. One of the main theorems of [1] is the converse:

THEOREM 2. *Let $\mathbf{R}$ be a proper 3-valent rectilinear dissection of the plane. Then $\mathbf{R}$ is a Dirichlet tessellation if and only if for each region R the rays $\lambda(R, V)$ have a point in common. If there is exactly one such point it is the source in R.*

2. Hyperbolic tessellations

Imagine a factory located at each source P; suppose each factory sells the same commodity at unit price $w(P)$. Suppose that the cost of transporting one unit of the commodity to X from P is $|X - P|$. Then define H_{PQ} as the region in which it is cheaper to order from P than from Q:

$$H_{PQ} = \{X : |X - P| + w(P) \leqslant |X - Q| + w(Q)\}. \tag{2.1}$$

This is just (1.6) with $\phi(t) = t$.

If we rewrite the equation for the locus K_{PQ} as

$$|X - Q| - |X - P| = w(P) - w(Q) \tag{2.2}$$

we can see that the shape of K_{PQ} depends on the relationship between $w(P) - w(Q)$ and $|P - Q|$:

LEMMA 3. (1) *If* $w(P) - w(Q) > |P - Q|$ *then* K_{PQ} *and* H_{PQ} *are empty while* H_{QP} *is the whole plane: it is always cheaper to order from* Q.

(2) *If* $w(P) - w(Q) = |P - Q|$ *then* K_{PQ} *is the ray on line* PQ *leaving* P *in the direction away from* Q.

(3) *If* $0 < w(P) - w(Q) < |P - Q|$ *then* K_{PQ} *is one branch of a hyperbola with foci* P *and* Q, *and* H_{PQ} *is convex.*

(4) *If* $w(P) - w(Q) = 0$ *then* K_{PQ} *is the perpendicular bisector of segment* PQ.

(5) *If* $w(P) - w(Q) < 0$ *the above analysis holds with* P *and* Q *interchanged.*

Proof. Straightforward. □

In light of (3) above we let $\mathbf{C}$ be the set of branches of hyperbolas (including straight lines as special cases) and speak of hyperbolic dissections and tessellations. Note that in case (2) H_{PQ} has empty interior and cannot occur as part of a proper tessellation. Figure 2 shows an example of a hyperbolic Dirichlet tessellation. Observe that when $w(P) < w(Q)$ the region H_{PQ} is not convex. Nevertheless, it is *starshaped* with respect to P: if $X \in H_{PQ}$ then the segment $\overline{XP} \subset H_{PQ}$. The proof is an easy exercise in the geometry of hyperbolas. Since the intersection of regions starshaped with respect to a given point is starshaped with respect to that point, R_P is starshaped with respect to P and hence connected. When P is the source with maximum weight, R_P is convex. We shall prove below that hyperbolic Dirichlet tessellations with 3-valent vertices are always proper.

Next we seek to solve the recognition problem. Given a proper hyperbolic dissection $\mathbf{R}$, when can we find a set $\mathbf{P}$ of sources and a function w such that $\mathbf{R} = \mathbf{R}(\mathbf{P}, w)$? The crucial observation is that in a hyperbolic Dirichlet tessellation the hyperbolic arcs which bound any given region have a common focus which is the source in that region.

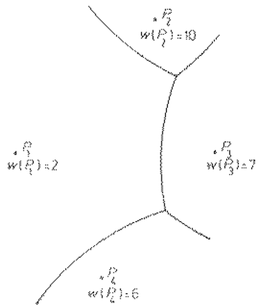

Fig. 2. A hyperbolic Dirichlet tessellation with four sources.

Since the foci of a hyperbola are determined by any segment of that hyperbola, a necessary condition that $\mathbb{R}$ be a Dirichlet tessellation is that the hyperbolic arcs bounding each region have a common focus in that region. That condition will in fact suffice if we define the foci properly for bounding arcs which happen to be line segments. Our guide to the proper definition is statement (4) in Lemma 3. When S is a line segment then any pair P, Q of points for which S lies on the perpendicular bisector of PQ is a pair of foci for S.

THEOREM 4. *Let* $\mathbb{R}$ *be a proper hyperbolic dissection of the plane into connected regions. Suppose there is at least one boundary arc which is not a straight line. Then* $\mathbb{R}$ *is a Dirichlet tessellation if and only if the hyperbolic arcs bounding each region have a common focus in that region, and, in addition, when the boundary between two regions is part of a line L the foci determined in those regions form a pair of foci for L.*

Proof. We have already observed that the condition is satisfied for hyperbolic Dirichlet tessellations. To prove the converse we must find sources $\mathbf{P}$ and weights w such that $\mathbb{R} = \mathbb{R}(\mathbf{P}, w)$. Since there is at least one boundary arc which is not a straight line, the potential sources in the regions that the arc separates are determined. Then the hypotheses imply that by reflecting those sources over the other boundaries we can determine the potential sources in each region. To find the weights, begin by observing that when $X \in R_P$, $Y \in R_Q$ and R_P and R_Q are neighbors we know

$$(2.3) \qquad |X - P| - |X - Q| \leqslant |Y - P| - |Y - Q|,$$

with equality just when X and Y are in ∂_{PQ}. We complete the proof by invoking the following theorem, which we will find useful in solving other recognition problems later. □

Recall that R and R' are *neighbors* when $R \cap R'$ is one-dimensional.

THEOREM 5. *Let* $\mathbb{R}$ *be a* ϕ*-dissection with connected, regular closed regions and* $\mathbb{P}$ *a set of points in one-to-one correspondence with the regions of* $\mathbb{R}$. *Then there is a weight function* $w\colon \mathbb{P} \to \mathbb{R}$ *such that* $\mathbb{R} = \mathbb{R}(\mathbb{P}, \phi, w)$ *if and only if for each pair* R_P, R_Q *of neighboring regions the function*

$$(2.4) \qquad \Delta_{PQ}(X) = \phi(|X - P|) - \phi(|X - Q|)$$

satisfies

$$(2.5) \qquad \Delta_{PQ}(X) \leqslant \Delta_{PQ}(Y)$$

whenever $X \in R_P$ *and* $Y \in R_Q$,

with

$$(2.6) \qquad \Delta_{PQ}(X) = \Delta_{PQ}(Y)$$

just when $X, Y \in \partial_{PQ}$.

Proof. Suppose $\mathbb{R} = \mathbb{R}(\mathbb{P}, w)$. Conditions (1.3) and (1.6), which define the regions in a ϕ-dissection, imply (2.6), so the condition is necessary.

To prove it suffices we must construct the weight function w. Consider the abstract graph G whose vertices are the regions of R, with two vertices set adjacent just when the corresponding regions are neighbors. That graph is the dual of the dissection $\mathbb{R}$. When R_P and R_Q are neighbors (2.6) implies Δ_{PQ} is constant on the boundary ∂_{PQ}. Hence we can consider Δ_{PQ} as a real-valued function defined on the oriented edges of G.

Since the vertices of G are in one-to-one correspondence with the potential sources $\mathbb{P}$ we can think of G as a graph on the vertex set P.

We seek a weight function w defined on the vertices of G such that

$$(2.7) \qquad \Delta_{PQ} = w(Q) - w(P)$$

whenever P and Q are adjacent in G. Equation (2.7) asks that Δ be a coboundary; to prove that, it suffices to prove that Δ is a cocyle, since the 1-cohomology of any graph is trivial.

Recall that the abstract graph G is in fact represented in the plane as the dual of the original tessellation $\mathbb{R}$ (although the vertices of G in that representation may not be the hypothetical sources $\mathbb{P}$). In that planar representation we can single out a special set of cocyles – those corresponding to trips around the vertices of the original tessellation. Those cocycles generate the cocyle space of G. It follows trivially from (2.4) that at the vertex V where $\partial_{P_1P_2}, \ldots, \partial_{P_nP_1}$

meet,

$$\Delta_{P_1P_2} + \cdots + \Delta_{P_nP_1} = 0. \tag{2.8}$$

Thus the 1-cochain Δ is a cocyle, and hence a coboundary. Thus we have found the weight function w. We need only show that $\mathbb{R} = \mathbb{R}(\mathbf{P}, \phi, w)$.

Inequality (2.5) and Equation (2.6) imply that in $\mathbb{R}$, R_P is given by

$$R_P = \bigcap_{P \text{ a neighbor of } Q} H_{PQ} \tag{2.9}$$

while in $\mathbb{R}(\mathbf{P}, \phi, w)$ the corresponding region R'_P is defined by

$$R'_P = \bigcap_{Q \neq P} H_{PQ}. \tag{2.10}$$

Clearly $R'_P \subset R_P$; we must show the two intersections are equal. Suppose X is in the interior of R_P. If $X \notin R'_P$ then for some Q, $X \in R'_Q \subset R_Q$. Then X is on the boundary arc ∂_{PQ} of R_P; a contradiction. Thus R'_P lives between R_P and its interior. Since R_P is regular closed it is the closure of its interior. Since R'_P is closed as well, it must equal R_P. □

Next we discuss briefly how we might go about determining whether a hyperbolic dissection satisfies the hypothesis of Theorem 4. The central rays which were so useful for rectilinear tessellations play a role here too. At each vertex V we define them as in Section 1, using the tangents to the boundary arcs meeting at V to determine the necessary angles.

LEMMA 6. *Let V be a 3-valent vertex of the hyperbolic Dirichlet tessellation $\mathbb{R}$. Then the central rays at V pass through the foci of the three hyperbolas which meet at V.*

Proof. The line segments joining a point on a hyperbola to the foci of that hyperbola make equal angles with the tangent at the point. (That fact is the analogue for hyperbolas of the well-known optical property of the ellipse which says that rays from one focus are reflected to the other.) The lemma then follows from Lemma 1. □

COROLLARY 7. *A necessary condition that a proper hyperbolic dissection of the plane be a Dirichlet tessellation is that for each region R the central rays $\lambda(R, V)$ have a point in common. If there is exactly one such point it is the source for the region.*

Unfortunately, the necessary condition given in Corollary 7 does not suffice. Figure 3 shows a hyperbolic Dirichlet tessellation with three sources and hence one vertex V. The boundaries of its regions are the hyperbolic arcs VA, VB, and VC. If VE is a hyperbolic arc obtained by rotating VC slightly about

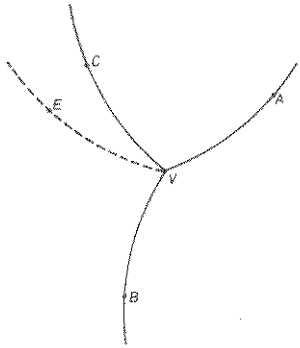

Fig. 3. A 3-valent vertex of a hyperbolic dissection which cannot be a Dirichlet tessellation.

V then the central rays in each region of this perturbed dissection have a point in common (since there is but one ray in each region) but the conditions of Theorem 4 are not satisfied. We do not have a hyperbolic Dirichlet tessellation.

3. Sectional Dirichlet tessellations

We temporarily abandon our commitment to the plane and consider a finite set $\bar{\mathbb{P}}$ of points in space. Let $\bar{\mathbb{R}}$ be the corresponding three-dimensional Dirichlet tessellation. Now let $\mathbb{R}$ be the intersection of $\bar{\mathbb{R}}$ with a plane π, we call $\mathbb{R}$ a *sectional* Dirichlet tessellation. Such tessellations occur in work of Aurenhammer [3], [4], Imai, *et al.* [12], and Sibson [15], who called them generalized Dirichlet tessellations. Since we generalize Dirichlet tessellations many ways we prefer the more descriptive name.

It is clear that the regions of $\mathbb{R}$ are (possibly unbounded) convex polygons. To fit the sectional tessellations into the framework we established in Section 1 we must discover ϕ and find sources and weights. For each $\bar{P} \in \bar{\mathbb{P}}$ whose region in $\bar{\mathbb{R}}$ meets π let P be the orthogonal projection of $\bar{P}$ on π and let $h(P) = |\bar{P} - P|$. Then $X \in \pi$ is nearer to $\bar{P}$ than to $\bar{Q}$ if and only if

$$(3.1) \qquad |X - P|^2 + h(P)^2 \leqslant |X - Q|^2 + h(Q)^2.$$

Thus sectional tessellations are generalized tessellations with $\phi(t) = t^2$ and $w(P) = h(P)^2$. One of their properties seems somewhat odd: P need not belong

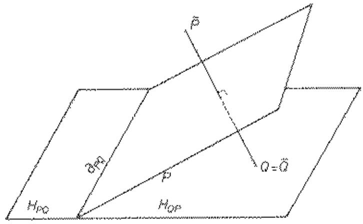

Fig. 4. A sectional Dirichlet tessellation in which $P \notin H_{PQ}$.

to R_P. Suppose, for example, that $\bar{P} = (0, 0, 3)$ and $\bar{Q} = (1, 0, 0)$ while π is t plane $z = 0$. Then $P = (0, 0)$, $Q = (1, 0)$ and H_{PQ} is defined by

$$\{(x, y): x^2 + 9 \leqslant (x - 1)^2\}$$

or

$$\{(x, y): x \leqslant -4\}.$$

Thus P and Q both belong to H_{QP} (see Figure 4).

This would seem to make sectional tessellations somewhat less than use for modelling economic and political phenomena. However, some models crystal growth lead to the development of three-dimensional Dirich tessellations; it may be necessary to know when a given plane dissection i section of such a crystal. Moreover, Sibson [15] succeeded in proving identity for Dirichlet tessellations by proving it for sectional tesselatio Finally, the recognition problem for sectional tessellations is interesting for own sake.

We begin our attack on the sectional Dirichlet tessellation recogniti problem by looking for properties that such a tessellation must have; we th prove that any dissection enjoying those properties must be a section o three-dimensional Dirichlet tessellation. The key observation, from which else will follow, is that the set of sources for a sectional Dirichlet tessellati forms a reciprocal figure for the tessellation.

Let **P** be a set of points in one-to-one correspondence with the regions c rectilinear plane tessellation **R**, and let **E** be the set of pairs (S, T) where S, T and S and T represent neighboring regions R_S and R_T. Then the graph (**P**, **E** a *reciprocal figure* for the tessellation when for each edge $(S, T) \in \mathbf{E}$ the lin through S and T is perpendicular to the line M containing the bounda between R_S and R_T, and, moreover, orientation is respected, in the followi sense: if you follow the ray from S through T to infinity you will end up on the side of M which contains R_T.

The existence of a reciprocal figure has implications in statics – it is equivalent to the assertion that the edges of the original tessellation can support a nonzero stress. The requirement that the reciprocal figure respects orientations implies that all the signs of that stress are the same, so that the edges of the original tessellation are the equilibrium position of a spider web. The study of reciprocal figures has a long history, stretching from Maxwell [13], [14] to the contemporary work of Crapo [8] and Whiteley [16]. We are concerned here only with its connection to our work on Dirichlet tessellations.

We observed in [1] that ordinary Dirichlet tessellations had reciprocal figures but the not all dissections with reciprocal figures were Dirichlet. The following theorem, the goal of the section, explains why.

THEOREM 8. *A dissection* $\mathbb{R}$ *of the plane into finitely many convex polygons is a sectional Dirichlet tessellation if and only if it has a reciprocal figure. Moreover, the vertices of any reciprocal figure may be taken to be the sources; suitable weights can then be found.*

Observe that we have omitted the common requirement that the vertices of $\mathbb{R}$ be 3-valent. But note that the requirement that reciprocal figures respect orientations does guarantee that, whatever the valence of a vertex, the edges meeting there do so at angles less than π.

To prove the theorem we need some definitions and some lemmas. When $P \in \pi$ we say that $\bar{P}$ is *over* P when line $P\bar{P}$ is perpendicular to π. For any points $A \neq B$ in space we write $AB^\perp$ for the plane which is the perpendicular bisector of segment AB. Our first lemma establishes Theorem 8 when the dissection consists of two regions separated by a single line.

LEMMA 9. *Suppose L is a line in π perpendicular to line PQ. Then there exist points $\bar{P}$ over P and $\bar{Q}$ over Q such that*

$$\bar{P}\bar{Q}^\perp \cap \pi = L. \tag{3.2}$$

Moreover, $\bar{P}$ may be chosen arbitrarily on the line of points over P as long as $h(P) = |\bar{P} - P|$ is large enough. Moreover, as $h(P) \to \infty$ so does $h(Q)$.

Proof. Since for any choices of $\bar{P}$ and $\bar{Q}$, $\bar{P}\bar{Q}^\perp \cap \pi$ is a line perpendicular to PQ it suffices to show that an arbitrary point X on PQ can be made to lie on $\bar{P}\bar{Q}^\perp$; that is, that $\bar{P}$ and $\bar{Q}$ can be found so that

$$|X - Q|^2 - |X - P|^2 = h(P)^2 - h(Q)^2 \tag{3.3}$$

Since $h(P)^2 - h(Q)^2$ takes on all values $\leqslant h(P)^2$ as $h(Q)$ varies, if $h(P)$ is large

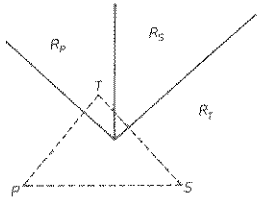

Fig. 5. A reciprocal figure which fails to respect orientations.

enough then a $\bar{Q}$ can be found so that (3.3) is satisfied. The second assertion follows trivially from (3.3). □

Now we know the theorem is true for a tessellation with just two regions. We establish it next for a proper tessellation with one vertex from which three rays emanate. To see that such a tessellation always has many reciprocal figures choose any lines perpendicular to the three rays and use their intersections as sources. Observe that we have used the restriction on the angles implied by the propriety of the tessellation. For an improper tessellation, whose regions are not all convex, this construction produces a reciprocal figure which does not respect orientations. Figure 5 shows an example.

LEMMA 10. *Let* $\mathbb{R}$ *be a proper rectilinear tessellation, in plane* π, *with one vertex* V *and three regions. Let* P, Q, S *be the vertices of a reciprocal figure for* $\mathbb{R}$. *Then there exist points* $\bar{P}, \bar{Q}$, *and* $\bar{S}$ *over* P, Q, *and* S *such that* $\mathbb{R}$ *is the intersection of* $\mathbb{R}(\bar{P}, \bar{Q}, \bar{S})$ *with* π.

Proof. Use the preceding lemma to choose $\bar{P}$ and $\bar{Q}$. Then, with $\bar{P}$ so chosen, use the lemma again to find $\bar{S}$ so that $\bar{P}\bar{S}^{\perp}$ contains the boundary ∂_{PS}. We must show that that choice of $\bar{S}$ is consistent with the known value of $\bar{Q}$. That is, we must show planes $\bar{Q}\bar{S}^{\perp}$ and π intersect in the line containing ∂_{QS}.

Since we started with a reciprocal figure, QS is perpendicular to ∂_{QS}. Hence $\bar{Q}\bar{S}^{\perp} \cap \pi$ is a line parallel to ∂_{QS}.

The planes which are the perpendicular bisectors of the sides of triangle $\bar{P}\bar{Q}\bar{S}$ meet in the line through the circumcenter of that triangle perpendicular to its plane. That line meets π at V, so $\bar{Q}\bar{S}^{\perp}$, which we have shown is parallel to ∂_{QS}, must contain V. Hence it is the line in π containing ∂_{QS}. □

Now we are ready to prove Theorem 8 for proper tessellations all of whose vertices are 3-valent. Choose any vertex P_0 of the given reciprocal figure and fix $\bar{P}_0$ over P_0 far from the plane π. Now identify the reciprocal figure with the graph G dual to $\mathbb{R}$ introduced in Theorem 5 and wander over it, starting from P_0. When you reach a source Q for the first time choose a point $\bar{Q}$ over it so that the boundary ∂_{PQ} you have just crossed lies on the appropriate perpendicular bisector, so that for $X \in R_P$ and $Y \in R_Q$

$$(3.4) \qquad |X-P|^2 - |X-Q|^2 \leqslant h(Q)^2 - h(P)^2 \leqslant |Y-P|^2 - |Y-Q|^2,$$

with equality just on the boundary. (Note that since P need not be a member of R_P you may have 'crossed' a boundary of $\mathbb{R}$ only in a combinatorial sense.) Lemma 9 guarantees that you can do that, provided that you are willing from time to time to alter your original choice of $\bar{P}$ so as to increase $h(P)$ and the heights of all the points chosen so far. In that way choose a point $\bar{Q}$ over every vertex Q of the reciprocal figure.

Lemma 10 says that the heights $h(Q)$ will be consistent when you reach a source you have already visited by crossing another of the edges of its associated region.

Inequalities (3.4) now show that Theorem 5 finishes the proof for proper tessellations. To prove it for more general tessellations, with some vertices of high valence, we exploit a suggestion of Walter Whiteley which allows us to

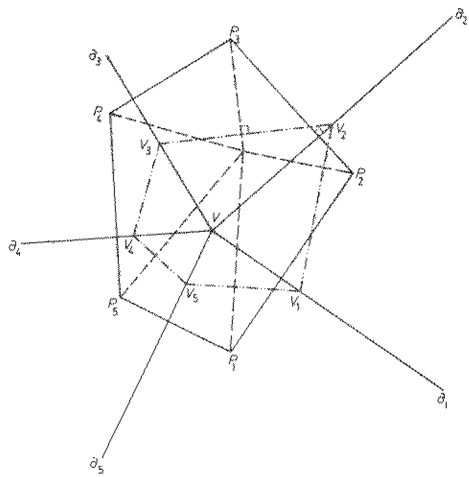

Fig. 6. Illustrating the proof of Lemma 11.

replace an n-valent vertex by n 3-valent ones. Figure 6 illustrates the following lemma and accompanying construction.

LEMMA 11. *Let V be an n-valent vertex of a tessellation $\mathbb{R}$ of the plane into convex polygons; let G be a reciprocal figure for $\mathbb{R}$. Then G contains an n-gon $P_1, \ldots, P_n$ whose edges P_iP_{i+1} are perpendicular to the edges ∂_i of $\mathbb{R}$ which meet at V. Choose a point P interior to that n-gon and add that point as a new vertex in the reciprocal figure G by joining it to each of the vertices P_i. To see that change reflected in $\mathbb{R}$ choose a point V_1 on ∂_1 near V. Draw a ray from V_1 towards ∂_2 in the direction perpendicular to PP_2; let V_2 be the point at which that ray meets ∂_2. Continue thus around V. Then the lemma asserts that polygon $V_1, V_2, \ldots$ will close. That is, the ray leaving V_n perpendicular to PP_1 meets ∂_1 at V_1.*

Proof. The following analytic argument was supplied (on 24 hours' notice) by Walter Whiteley. The cross-products it uses are motivated by the fact from mechanics that the existence of reciprocal figures corresponds to the existence of forces in equilibrium. Choose any origin in the plane, so that we can identify points with vectors. For each vector X let X^* be the rotation of X through an angle of $\pi/2$. It follows that $X^{**} = -X$ and that segments AB and XY are perpendicular just when $(Y - X)^* = \alpha(B - A)$ for some scalar α.

The choices made in the construction in this lemma then assert the existence of scalars α_i, β_i such that

$$(3.5) \qquad V_i - V = \alpha_i(P_{i+1} - P_i)^*$$

for $i = 1, \ldots, n$ (setting $P_{n+1} = P_1$) and

$$(3.6) \qquad (V_{i+1} - V_i)^* = \beta_i(P_{i+1} - P)$$

for $i = 1, \ldots, n-1$. What we must prove is that V_1V_n is perpendicular to PP_1.

We start by observing that for $i = 1, \ldots, n-1$

$$(3.7) \qquad \begin{aligned} \beta_i(P_{i+1} - P) &= (V_{i+1} - V_i)^* \\ &= (V_{i+1} - V)^* - (V_i - V)^* \\ &= -\alpha_{i+1}(P_{i+2} - P_{i+1}) + \alpha_i(P_{i+1} - P_i). \end{aligned}$$

Then taking the cross-product of each member of (3.7) with $P_{i+1} - P$ and using the fact that for any three points A, B, C

$$(3.8) \qquad (A - B) \times (A - C) = (A - B) \times (B - C)$$

we deduce

$$(3.9) \qquad \begin{aligned} 0 &= \alpha_{i+1}(P_{i+2} - P_{i+1}) \times (P_{i+1} - P) - \alpha_i(P_{i+1} - P_i) \times (P_{i+1} - P) \\ &= \alpha_{i+1}(P_{i+2} - P_{i+1}) \times (P_{i+1} - P) - \alpha_i(P_{i+1} - P_i) \times (P_i - P). \end{aligned}$$

Finally, sum (3.9) for $i = 1, \ldots, n-1$ and use (3.8) again to conclude that

$$\begin{aligned} (3.10) \qquad 0 &= \alpha_n(P_1 - P_n) \times (P_n - P) - \alpha_1(P_2 - P_1) \times (P_1 - P) \\ &= -\alpha_n(P_n - P_1) \times (P_1 - P) - \alpha_1(P_2 - P_1) \times (P_1 - P) \\ &= [\alpha_n(P_{n+1} - P_n) - \alpha_1(P_2 - P_1)] \times (P_1 - P) \\ &= [-(V_n - V)^* + (V_1 - V)^*] \times (P_1 - P) \\ &= (V_1 - V_n)^* \times (P_1 - P). \end{aligned}$$

But that final zero cross-product implies $(V_1 - V_n)^*$ is a scalar multiple of $P_1 - P$, which is just what we needed to know. □

Now to finish the proof of Theorem 8, carry out the construction in Lemma 11 at each vertex of $\mathbb{R}$ of valence greater than 3. The resulting proper tessellation $\mathbb{R}'$ has for a reciprocal figure the original graph G to which some vertices have been added. The version of Theorem 8 proved so far shows that $\mathbb{R}'$ is a section of a three-dimensional Dirichlet tessellation $\bar{\mathbb{R}}'$. Finally, let $\bar{\mathbb{R}}$ be the three-dimensional Dirichlet tessellation determined by those sources for $\bar{\mathbb{R}}'$ which lie over the original vertices of the reciprocal figure G. Then $\mathbb{R}$ is the intersection of $\bar{R}$ with the plane π. □

4. Circular tessellations

Suppose the sources represent restaurants, and that a customer choosing between two is willing to travel k times as far to visit the one which is k times as desirable. Then the region H_{PQ} within which customers will choose P over Q (or will be indifferent) is given by

$$(4.1) \qquad H_{PQ} = \left\{ X : \frac{|X - P|}{\sigma(P)} \leqslant \frac{|X - Q|}{\sigma(Q)} \right\},$$

where $\sigma(P) > 0$ is the attractiveness of the restaurant at P.

The definition in (4.1) fits the pattern of (1.6) if we take the logarithm of each member of the inequality and set $\phi(t) = \log(t)$ and $w(P) = -\log \sigma(P)$. But the geometry in this section will be easier to see if we keep the algebra free of logarithms. Thus we shall refer to $\sigma(P)$ as the weight of P.

As usual, we set

$$(4.2) \qquad R_P = \bigcap_{Q \neq P} H_{PQ}.$$

If we define a *circular* tessellation or dissection as one in which the boundaries of the regions are circular arcs (allowing line segments and rays as special cases) then $\mathbb{R}(\mathbb{P}, \sigma) = \{R_P\}$ is a circular Dirichlet tessellation, because

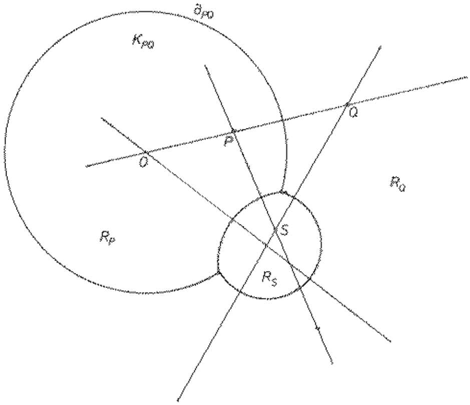

Fig. 7. A circular Dirichlet tessellation with three sources.

the boundary K_{PQ} of H_{PQ} is circle, known as the circle of Apollonius of ratio $\rho(P,Q) = \rho(P)/\sigma(Q)$.

For the facts which follow about circles of Apollonius see [7, Chap. 6]. The center of K_{PQ} lies on the line through P and Q. When $\rho(P,Q) < 1$, P is interior to K_{PQ}, but is not its center O. In fact, P is in the interior of segment OQ (see Figure 7).

When $\rho(P,Q) = 1$, K_{PQ} is a degenerate circle: the perpendicular bisector of segment PQ. Thus when σ is a constant function the circular Dirichlet tessellation based on $\langle \mathbb{P}, \sigma \rangle$ is simply the Dirichlet tessellation based on $\mathbb{P}$.

The regions of a circular Dirichlet tessellation need not be convex, or simply connected, or even connected. Indeed, in the simple case with two sources of different weights the region containing the source of greater weight is the exterior of the Apollonian circle and hence is neither convex nor simply connected. However, the region corresponding to the source of smallest weight is an intersection of disks, hence convex.

To see a disconnected region look at Figure 8 in which $\mathbb{P}$ consists of five points: the origin and the intersections of the coordinate axes with the unit circle. Let the weight be 1.1 at the origin O and 1 at the other four points. Then it is easy to see that the region R_O containing the origin contains a neighborhood of the origin and, because O has the largest weight, a neighborhood of infinity. However, if X is any point on the unit circle, its distance from one of the four sources other than O is no greater than

$$\sqrt{2-\sqrt{2}} < \frac{1}{1.1}.$$

Therefore the unit circle does not meet R_0, which is thus disconnected.

Later we shall base more complicated examples on this one, so we examine it a little more closely now. If Q is one of the four sources on the unit circle then K_{QO} is an Apollonian circle of ratio $1/1.1$, so Q is inside and O outside K_{QO}. The center of K_{QO} is on the axis containing Q, further from the origin than Q is. If Q and Q' are neighboring sources on the unit circle then $K_{QQ'}$ is the perpendicular bisector of QQ' and hence one of the lines $y = \pm x$.

The tool we use to study circular tessellations is *inversion*. If K is a circle with center C and radius r define the inversion map* by setting P^* equal to the point on the ray from C through P such that

$$|P^* - C||P - C| = r^2.$$

Inversion is an anticonformal involutory map when we compactify the plane with a point ∞ and set $C^* = \infty$. To see that, use complex arithmetic in place of geometry to define inversion: choose a complex coordinate system for which K is the unit circle. Then $P^* = 1/\bar{P}$. Adding the point at infinity to the plane

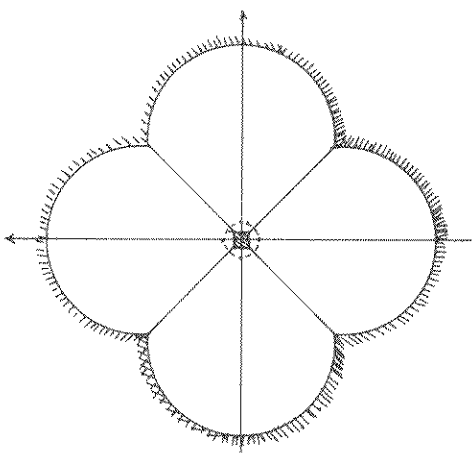

Fig. 8. A circular Dirichlet tessellation in which one region is disconnected.

means working on the Riemann sphere. Note that this restores the simple connectivity of the exterior of a circle, though not the connectivity of the disconnected region in the preceding example.

THEOREM 12. *When we invert in the circle* K_{PQ}, $P^* = Q$.

Proof. See [7, p. 89]. □

The next theorem shows that Apollonian circles and circular Dirichlet tessellations are preserved by arbitrary inversions.

THEOREM 13. *Let* $\mathbf{R}$ *be the circular Dirichlet tessellation based on* $\langle \mathbf{P}, \sigma \rangle$. *Invert in a circle K with center C and radius r. Then the image* $\mathbf{R}^*$ *is a Dirichlet tessellation based on* $\langle \mathbf{P}^*, \sigma^* \rangle$, *where the new weight* σ^* *are given by*

$$\sigma^*(P^*) = \sigma(P)\frac{r^2}{|P - C|}. \tag{4.3}$$

Proof. Let X be a point different from C and a source $P \in \mathbf{P}$. Since the triangles CXP and CP*X^* are similar [7, p. 92, Exer. 1] and since $|P^* - C|$ $|P - C| = r^2$ we have

$$\frac{|P^* - X^*|}{|P - X|} = \frac{|P^* - C|}{|X - C|} = \frac{r^2}{|P - C||X - C|}$$

or

$$\frac{|X - P|}{\sigma(P)} = \frac{|P^* - X^*||P - C||X - C|}{r^2\sigma(P)}. \tag{4.4}$$

Similarly, if $X \neq Q$

$$\frac{|X - Q|}{\sigma(Q)} = \frac{|Q^* - X^*||Q - C||X - C|}{r^2\sigma(Q)}. \tag{4.5}$$

Thus

$$\frac{|X - P|}{\sigma(P)} \leqslant \frac{|X - Q|}{\sigma(Q)}$$

just when

$$\frac{|X^* - P^*|}{r^2\sigma(P)/|P - C|} \leqslant \frac{|X^* - Q^*|}{r^2\sigma(Q)/|Q - C|}.$$

or

$$\frac{|X^* - P^*|}{\sigma^*(P^*)} \leqslant \frac{|X^* - Q^*|}{\sigma^*(Q^*)}.$$ □

Remark 1. If C, the center of inversion, happens to lie on K_{PQ} then

$$\sigma^*(P^*) = \frac{\sigma(P)r^2}{|P - C|} = \frac{\sigma(Q)r^2}{|Q - C|} = \sigma^*(Q^*)$$

and $K_{P^*Q^*}$ is the perpendicular bisector of P^*Q^*.

Remark 2. If the center of inversion happens to be one of the sources, then that source inverts to $\infty \in \mathbf{P}^*$. With the conventions that $\sigma^*(\infty) = \infty$ and that $|X - \infty|/\infty = 1$ for $X \neq \infty$ we see that $K_{P^*\infty}$ is a circle centered at P^* with radius $\sigma^*(P^*)$. When the results which follow are properly interpreted there is no need to exclude ∞ as a source.

Remark 3. We should not exclude ∞ as a vertex, either. Bounding arcs which hapen to be rays meet there. Thus when we consider rectilinear tessellations as special cases of circular tessellations very few will be 3-valent. Fortunately the argument in Theorem 17 below, which solves the recognition problem for circular dissections, allows a single vertex with valence greater than 3. In the example illustrated in Figure 8 the vertex at ∞ is 4-valent. Note however, that a small perturbation of the weights in that example yields one with a disconnected region but no 4-valent vertex.

Our object is to carry through as much as possible of the analysis of the rectilinear case in [1]. In particular, we shall solve the recognition problem, so that we can decide when a circular dissection $\mathbb{R}$ is the Dirichlet tessellation based on some $\langle \mathbf{P}, \sigma \rangle$ and the extent to which $\mathbb{R}$ determines $\mathbf{P}$ and σ.

The following definition and lemma, which we use here for circular Dirichlet tessellations, make sense for ϕ-tessellations, so we give their more general forms.

Let V be a vertex of the ϕ-tessellation $\mathbb{R}$, and ∂ one of the boundary arcs that meet at V. Then ∂ is part of one of the curves given by Equation (1.8). If we parametrize that curve (using arclength as parameter) so that $f(0) = V$ we call the image of f the extension of V. Suppose f_1 and f_2 extend the boundary arcs ∂_1 and ∂_2 that meet at V. If for some $s, t \neq 0$ we have $f_1(s) = f_2(t) = W$ we say that ∂_1 and ∂_2 meet at W when extended. If every other boundary arc at V also passes through W when extended we say W is a *phantom vertex* of V.

LEMMA 14. (The Phantom Vertex Lemma, or 'When shall we three meet again?'.) *Let V be a 3-valent vertex of the Dirichlet ϕ-tessellation $\mathbb{R}$. If two of the three bounding arcs which meet at V meet again when extended then that intersection is a phantom vertex of V.*

Proof. Three regions R_P, R_Q, R_S meet at V. The extension of a typical boundary curve there, ∂_{PQ}, is defined implicitly by the equation

$$\phi(|X - P|) + w(P) = \phi(|X - Q|) + w(Q).$$

›r any particular X the truth of two such equations, say for pairs P, Q and $Q,$ forces the truth of the third, involving P and S. □

ıe Phantom Vertex Lemma provides both a substantial necessary condition at a circular dissection be a Dirichlet tessellation and a tool for recovering ɛ sources and weights when it is. Suppose V is a vertex of a proper 3-valent ·cular dissection. Then any two of the circles meeting at V do in fact meet ain, at a point $\hat{V}$ at which all three circles containing the boundary arcs at V ɛet. That is, each vertex of a circular Dirichlet tessellation has a unique ıantom vertex. The phantom vertex may or may not be part of the boundary the dissection.

In Figure 7, V and $\hat{V}$ are each vertices of the dissection. In Figure 9(a) the ıantom vertex $\hat{V}$ is not part of the boundary of the dissection. Figure 9(b) ows a part of a circular dissection with a 3-valent vertex which has no ıantom vertex, thus this figure cannot be part of a Dirichlet tessellation.

ɛmark 1. We shall write $\Gamma(V, \hat{V})$ for the pencil of circles through V and $\hat{V}$, and $^{\perp}(V, \hat{V})$ for the pencil of circles orthogonal to the circles $\Gamma(V, \hat{V})$.

Remark 2. A rectilinear Dirichlet tessellation may be viewed as a circular ssellation in which all weights are equal. Then when we pass to the Riemann here for each vertex V the phantom vertex $\hat{V}$ is ∞. Conversely, if for all the ɛrtices V in a circular tessellation $\hat{V}$ exists and is a point C independent of V en inversion in a circle centered at C produces a rectilinear Dirichlet ssellation.

Remark 3. If the three boundary arcs meeting at V are tangent there then $= \hat{V}$ (see Figure 10).

In that case $\Gamma(V, \hat{V})$ consists of all circles through V tangent to the common ngent. The tangency expresses the fact that even when V and $\hat{V}$ are identical,

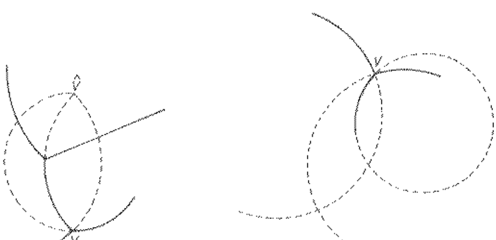

Fig. 9. (a) A phantom vertex. (b) A vertex with no phantom vertex.

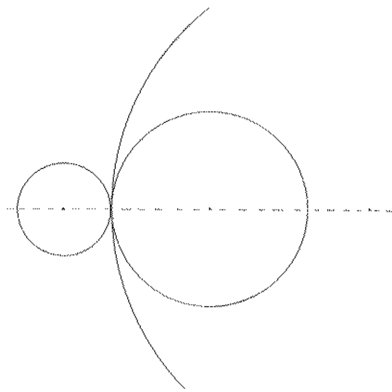

Fig. 10. A vertex at which boundary arcs are tangent.

the direction $V\hat{V}$ is known. A vertex for which $V = \hat{V}$ occurs when the sources in three neighboring regions lie on a line through V and have a weights equal to a constant multiple of their distances from V. Indeed, if K_{PQ}, K_{QS} and, hence, K_{PS} are mutually tangent at V then their centers must lie on the perpendicular to the tangent at V. Hence P, Q, and S must lie on that line too. Then, since the three circles pass through V,

$$\frac{|V-P|}{\sigma(P)} = \frac{|V-Q|}{\sigma(P)} = \frac{|V-S|}{\sigma(Q)}.$$

Such a tessellation is not proper.

Suppose we invert that configuration in a circle centered at V. Then the mutually tangent circles become mutually parallel lines, and the images of the sources lie on a line in the perpendicular pencil of lines. Thus the anomalous tessellations into parallel strips, which are essentially one-dimensional and which needed special treatment in [1], are here seen to be special cases of circular Dirichlet tessellations.

Next we use inversion to prove a lemma about orthogonal pencils, and then proceed to identify the Dirichlet tessellations among the proper circular dissections.

LEMMA 15. *$\Gamma^{\perp}(V, \hat{V})$ consists of all circles of Apollonius for the points V and $\hat{V}$.*

Proof. Let * denote inversion in a circle centered at $\hat{V}$. Under *, $\Gamma(V, \hat{V})$

becomes the pencil of all straight lines through V^* and $\Gamma^\perp(V, \hat{V})$ the pencil of circles with center V^*. Thus $\Gamma^\perp(V, \hat{V})^*$ is the family of all circles of Apollonius for the points V^* and ∞. The lemma follows when we invert back. □

Now suppose that $V \neq \infty$ is a vertex of a proper circular dissection $\mathbf{R}$ and that V has a phantom vertex $\hat{V}$. As in the hyperbolic case, use the tangents at V to the boundary arcs to define central rays $\lambda(R, V)$. But now let $\Lambda(R, V)$ be the circle through V and $\hat{V}$ which is tangent to the central ray $\lambda(R, V)$ at V.

THEOREM 16. *If V is a 3-valent vertex of the circular Dirichlet tessellation $\mathbf{R}(\mathbf{P}, \sigma)$ then the sources in the regions around V lie on a circle in the orthogonal pencil $\Gamma^\perp(V, \hat{V})$. The angle formed at V by the tangents to the boundary arcs are each less than π. Finally, the source P lies on the circle $\Lambda(R_P, V)$, where R_P is the region containing P.*

Proof. Lemma 14 guarantees the existence of the phantom vertex $\hat{V}$. Invert in a circle centered there. The three arcs meeting at $\hat{V}$ become arcs through $\hat{V}^* = \infty$ and thus are rays in the resulting Dirichlet tessellation $\mathbf{R}^*$. Since rectilinear tessellations are proper these rays meet at angles less than π. In $\mathbf{R}^*$ the sources in the regions adjacent to V^* are equally weighted, and lie on a circle centered at V^* (orthogonal to the rays from V^*). Moreover, the circles $\Lambda(R, V)$ become the central rays in $\mathbf{R}^*$. Theorem 2 shows these contain the sources P. The theorem follows when we invert back. □

THEOREM 17. *Let $\mathbf{R}$ be a proper circular tessellation each of whose regions is connected and each of whose vertices is 3-valent and has a phantom vertex. Then $\mathbf{R}$ is a Dirichlet dissection if for each region R the circles $\Lambda(R, V)$ have a unique point in R in common. Those points must be the sources; the weights, too, are uniquely determined.*

Proof. We shall use Theorem 5. Our hypothesis provides us with a potential source P in each region $R = R_P$ of $\mathbf{R}$. To prove that these will do as actual sources we must show that whenever R and R' are neighboring regions, with potential sources P and P', we have

$$\log(|X-P|) - \log(|X-P'|) \leqslant \log(|Y-P|) - \log(|Y-P'|) \tag{4.6}$$

for $X \in R$ and $Y \in R'$. That is equivalent to the assertion that $\partial(R, R')$ is part of an Apollonian circle for the pair P, P'.

Suppose $\partial(R, R')$ joins vertices V and W. We may assume $V, W \neq \infty$ without loss of generality. Then the potential source $P \in R$ lies on $\Lambda(R, V)$ and on $\Lambda(R, W)$; the analogous assertion for $P' \in R'$ is also true.

Let K be the circle of which $\partial(R, R')$ is part. Observe that V, $\hat{V}$, W, and $\hat{W}$ are all on K. The circles $\Lambda(R, V)$ and $\Lambda(R', V)$ both pass through V and $\hat{V}$ and leave

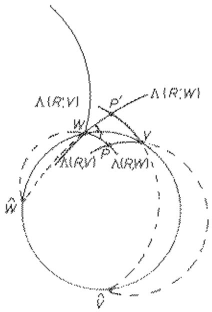

Fig. 11. Illustrating the proof of Theorem 17.

each of those points at equal angles on opposite sides of K. The same assertion is true when W replaces V. (See Figure 11.)

Now invert in a circle centered at V. In the new dissection $\mathbf{R}^*$, $\partial(R^*, R'^*)$ is part of a straight line L containing $\hat{V}^*$, W^*, and $\hat{W}^*$. Because inversion is anticonformal, $\Lambda(R^*, V^*)$ and $\Lambda(R'^*, V^*)$ are straight lines which make equal angles with L at $\hat{V}^*$. Similarly, $\Lambda(R'^*, W^*)$ is the reflection in L of $\Lambda(R^*, W^*)$. (See Figure 12.)

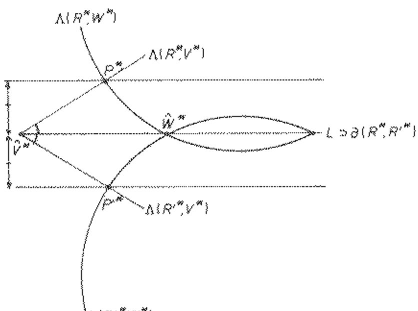

Fig. 12. Illustrating the proof of Theorem 17, after inversion.

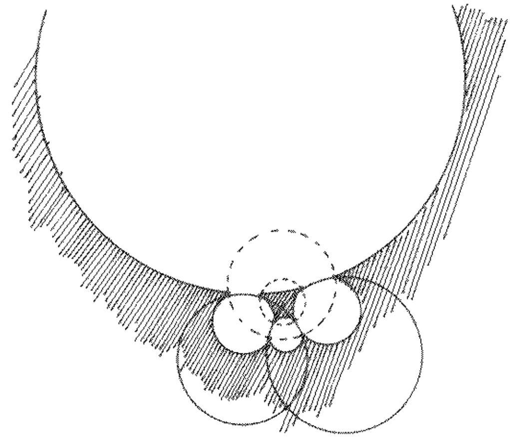

Fig. 13. A circular Dirichlet tessellation with no source at infinity in which one region is disconnected and sources are not unique.

Since

$$P^* \in \Lambda(R^*, V^*) \cap \Lambda(R^*, W^*)$$

and

$$P'^* \in \Lambda(R'^*, V^*) \cap \Lambda(R'^*, W^*)$$

the symmetry about L tells us that L is the perpendicular bisector of segment $P^*P'^*$. Thus L is the Apollonian circle for those points with ratio 1. But in Theorem 13 we showed that inversion preserves Apollonian circles, so K is indeed one for P and P'. □

Remark 1. We require connected regions in Theorem 17 because, surprisingly, the sources in a Dirichlet tessellation need not be unique. The one pictured in Figure 8 provides an example. A second set of sources yielding the same regions is $\mathbf{P}' = \{\infty, P_i'\}$, where P_i' is the center of the circle which forms part of the boundary of R_i, $i = 1, 2, 3, 4$. To obtain an example without a source at ∞ we may invert this example in any circle whose center is not one of the 10 possible sources. (See Figure 13.)

Remark 2. The nonuniqueness exhibited in Remark 1 is atypical. There can be two prospective sources for a region R only when all the circles $\Lambda(R, V)$, (V a vertex of R) happen to belong to a single pencil $\Gamma(P, P')$. Then P and P' are each candidates for the source in R. Thus it is the high degree of symmetry in Figure 8 which allows the nonuniqueness. Although the symmetry may be hidden by an inversion, we suspect that something like it must occur to cause nonuniqueness. Moreover, we do not know of a proper circular Dirichlet tessellation with connected regions and two possible sets of sources.

Remark 3. A Dirichlet tessellation may have no vertices at all. We mean by that more than is suggested by the example of parallel strips in Remark 3 following Lemma 14. We have seen how that tessellation should properly be regarded as having a vertex at ∞. A more damaging example can be constructed based on any finite set $\mathbf{P}$ of sources. It is always possible to choose weights so that there are no vertices in the resulting tessellation. Simply choose a weight for one source Q so much larger than all the other weights that the circles H_{PQ} for $P \neq Q$ are disjoint. Theorem 17 does not cover this case because for each region R the intersection of the empty set of circles $\Lambda(R, V)$ is, technically, the whole plane, and not, as required, a unique point in R.

Remark 4. In our study of rectilinear Dirichlet tessellations in [1] we saw that if for all vertices V of a region R the rays $\lambda(R, V)$ met R and were copunctual then the point of intersection was in R. We do not know whether the analogous statement (for $\Lambda(R, V)$) is necessarily true for circular dissections and have thus made it part of the hypothesis of Theorem 17.

5. ϕ-DISSECTIONS

In this section we turn from particular recognition problems to a general study of how the function ϕ used to distort distances affects the shape of the regions of $\mathbf{R}$. In Sections 2, 3, and 4 we studied the cases in which $\phi(t) = t$, $\phi(t) = t^2$ and $\phi(t) = \log(t)$, which led respectively to hyperbolic, sectional, and circular tessellations. Note that this last example forces the range of ϕ to include $-\infty$ if 0 is to be in its domain. We now place some restrictions on ϕ. These simplify theorems while still handling most cases of practical interest. We shall assume that ϕ has a continuous second derivative on $(0, \infty)$ ($\phi \in C^2$), and that $\phi'(t) > 0$, so that, in particular, ϕ is strictly increasing.

We first provide the promised proof that 3-valent vertices are in fact generic, in the sense that the set of configurations with vertices of higher valence has measure 0. The possible positions four sources $P_1, \ldots, P_4$ might occupy are parametrized by $\mathbb{R}^8$, their weights by $\mathbb{R}^4$. Then $S = \mathbb{R}^8 \times \mathbb{R}^4$ is the space of all positions and weights.

THEOREM 18. *For almost all pairs* $(\mathbf{P}, w) \in S$, $\mathbf{R}(\mathbf{P}, \phi, w)$ *has no 4-valent vertex.*

Proof. For each $X \in \mathbb{R}^2$ we test whether X is a 4-valent vertex by calculating, for $i = 1, \ldots, 4$,

$$\mu_i(X) = \phi(|X - P_i|) + w_i; \tag{5.1}$$

we have such a vertex when all four values $\mu_i(X)$ agree. Thus we are led to consider

$$\mu: \mathbb{R}^2 \times S \to \mathbb{R}^4 \tag{5.2}$$

and to see when we hit the diagonal Δ in $\mathbb{R}^4$.

We show first that μ is *transversal* to Δ in the differential geometric sense ([10, p. 28]).

In our example the differentials dw_i of the weights are independent and thus span the tangent space to the image manifold $\mathbb{R}^4$, so the differential $d\mu$ is surjective and hence μ is transversal to any submanifold of $\mathbb{R}^4$, and, in particular, to Δ. Then the Transversality Theorem [10, p. 68] asserts that for almost all $(\mathbf{P}, w) \in S$ the map

$$\mu_{(\mathbf{P},w)}: \quad \mathbb{R}^2 \to \mathbb{R}^4$$

is transversal to Δ. But the dimension of $d\mu_{(\mathbf{P},w)}$ is at most 2, while the dimension of the tangent space to Δ is 1. These spaces can never span a four-dimensional space. The Transversality Theorem thus implies that for almost all $(\mathbf{P}, w)$ the inverse image of Δ under $\mu_{(\mathbf{P},w)}$ is empty. That is, for almost all $(\mathbf{P}, w)$ we have no 4-valent vertex. In fact, the argument proves more: for all w, for almost all $\mathbf{P}$ all vertices are 3-valent, and the same is true with the roles of $\mathbf{P}$ and w reversed. □

A similar transversality argument will prove that almost all ϕ-dissections satisfy the first condition in the definition of 'proper' in Section 1. To prepare for it, for each pair of points P, Q we consider the function $\Phi: \mathbb{R}^2 \to \mathbb{R}$ given by

$$\Phi_{PQ}(X) = \Phi(X) = \phi(|X - P|) - \phi(|X - Q|). \tag{5.3}$$

If R_P and R_Q are neighboring regions in a Dirichlet ϕ-tessellation then the boundary ∂_{PQ} lies on the c-level curve Γ_c of Φ, where $c = w(Q) - w(P)$.

The best tool for studying the level curves of a function is its gradient. Using the fact that $\nabla|X| = X/|X|$ we see that

$$\nabla\Phi(X) = \frac{\phi'(|X - P|)}{|X - P|}(X - P) - \frac{\phi'(|X - Q|)}{|X - Q|}(X - Q). \tag{5.4}$$

At any point at which the gradient is not 0 it is perpendicular to the level curve. Suppose now that V is a 3-valent vertex of a Dirichlet ϕ-tessellation, where the regions corresponding to sources P, Q, and S meet. If at V none of the three relevant functions Φ has zero gradient then locally the tessellation looks rectilinear and is proper, unless two of the three boundary arcs happen to be tangent and thus meet at angle π. We saw that happen when we studied circular tessellations. Our job now is to show that it does not happen too often. We shall see below that the gradient of Φ is 0 only on a set of measure zero, so we concentrate here on showing that the tangency too is rare.

As we saw above, the possible positions three sources $P_1, \ldots, P_3$ might occupy are parametrized by $\mathbb{R}^6$, their weights by $\mathbb{R}^3$. Now $S = \mathbb{R}^6 \times \mathbb{R}^3$ is the space of all positions and weights.

THEOREM 19. *For almost all pairs* $(\mathbb{P}, w) \in S$, $\mathbb{R}(\mathbb{P}, \phi, w)$ *has only proper (necessarily 3-valent) vertices.*

Proof. Theorem 18 shows that we may assume all vertices are 3-valent without loss of generality. For each $X \in \mathbb{R}^2$ we test whether X is a proper vertex first by calculating $\mu_i(X)$ for $i = 1, \ldots, 3$. We have a vertex when all three values $\mu_i(X)$ agree. That vertex is proper when the angle between any pair of boundary arcs at that vertex is nonzero. But that is equivalent to having no pair of the corresponding gradient vectors collinear. An argument like that in the previous theorem shows such collinearity occurs only on a set of measure 0 in S. □

To discover more about the shapes of the regions we study the sources two at a time. Let $\mathbb{P} = \{P, Q\}$. Then the shape of R_P depends only on the difference $c = w(Q) - w(P)$ and the distance $|P - Q|$. We have

$$(5.5) \qquad R_P = H_{PQ} = H^c = \{X; \Phi_{PQ}(X) \leqslant c\}.$$

Now note that if k is any constant, replacing $\phi(t)$ by $\phi(t) + k$ does not change Φ and thus does note change H^c. Thus we may assume without loss of generality that

$$\phi(0) = 0 \qquad \text{or} \qquad \phi(0) = -\infty.$$

THEOREM 20. $P \in H^c$ *for all* c *if and only if* $\phi(0) = -\infty$.

Proof. Suppose $\phi(0) = -\infty$. Then

$$\phi(|P - P|) - \phi(|P - Q|) = -\infty - \phi(|P - Q|) = -\infty,$$

which is $\leqslant c$ for all c. Hence $P \in H^c$ for all c. Now suppose $\phi(0) = 0$. Then

$$\phi(|P - P|) - \phi(|P - Q|) = -\phi(|P - Q|,$$

so $P \in H^c$ only for those c for which $-\phi(|P - Q|) \leqslant c$. □

We can distinguish further among the reasons for the failure of $P \in H^c$. For hyperbolic tessellations, $P \notin H^c$ only when H^c is empty, while sectional tessellations are in some sense worse: we may have $P \notin H^c$ even when H^c is nonempty. We shall see that this unpleasant phenomenon occurs just when ϕ fails to be concave. We say that ϕ is *concave* when $\phi'' < 0$, or, equivalently, the graph of ϕ lies above its chords and below its tangents. We shall need some elementary properties of concave functions.

LEMMA 21. *If $\phi(0) = 0$ and ϕ is concave then $\phi(t)/t$ is decreasing.*

Proof. Suppose $a < b$. Then the chord joining $(0, 0)$ and $(b, \phi(b))$ has height $(a/b)\phi(b)$ at a, so $(a/b)\phi(b) \leqslant \phi(a)$ and the lemma follows. □

LEMMA 22. *If for all $a, b > 0$*

$$\phi(a+b) \leqslant \phi(a) + \phi(b) \tag{5.6}$$

then ϕ is concave. Conversely, if ϕ is concave and $\phi(0) = 0$ then (5.6) holds.

Proof. Suppose (5.6) is true. Differentiating with respect to b gives

$$\phi'(a+b) \leqslant \phi'(b).$$

Since a and b are arbitrary, that says ϕ' is decreasing, so ϕ is concave.

Conversely, suppose ϕ is concave and $\phi(0) = 0$. To prove (5.6) suppose $a < b$. Then since $b < a + b$ as well, Lemma 21 applied twice gives

$$\begin{aligned}\phi(a+b) &\leqslant \frac{a+b}{b}\phi(b) \\ &= \frac{a}{b}\phi(b) + \phi(b) \\ &\leqslant \phi(a) + \phi(b).\end{aligned}$$

□

THEOREM 23. *Suppose $\phi(0) = 0$. Then*

$$H^c \neq \phi \quad \textit{implies} \quad P \in H^c \tag{5.7}$$

for all P, Q, and c if and only if ϕ is concave.

Proof. Suppose that ϕ is concave and $X \in H^c$. Then

$$\phi(|X-P|) - \phi(|X-Q|) \leqslant c. \tag{5.8}$$

But the triangle inequality and Lemma 22 imply

$$\begin{aligned}\phi(|X-Q|) &\leqslant \phi(|P-Q| + |X-P|) \\ &\leqslant \phi(|P-Q|) + \phi(|X-P|),\end{aligned}$$

which, when combined with (5.8) tells us

$$-\phi(|P-Q|) \leqslant c.$$

Since $\phi(|P-P|)=\phi(0)=0$ that implies $P\in H^c$.

Conversely, suppose (5.7) is true. We must prove (5.6). For arbitrary X, P, and Q, if we set

$$c=\phi(|X-P|)-\phi(|X-Q|)$$

then $X\in H^c$. Then by hypothesis $P\in H^c$, so

$$0-\phi(|P-Q|)\leqslant c.$$

Thus

$$\phi(|X-Q|)\leqslant\phi(|X-P|)+\phi(|P-Q|). \tag{5.9}$$

Now given a and b choose X, P, and Q collinear such that $|X-P|=a$, $|P-Q|=b$ and P is between X and Q. □

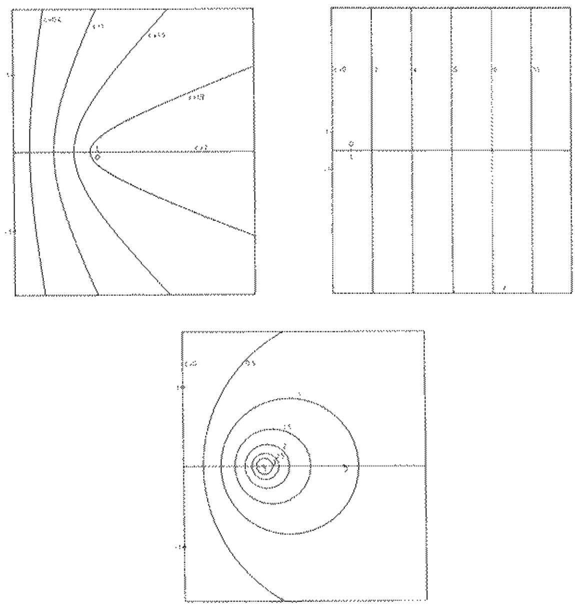

Fig. 14. The family $\{\Gamma_c\}$ for (a) $\phi(t)=t$, (b) $\phi(t)=t^2$, (c) $\phi(t)=\log(t)$.

We conclude our study of generalized Dirichlet tessellations with a study of the possible shapes for the regions H^c.

Choose a coordinate system such that $P = (-a, 0)$ and $Q = (a, 0)$. Then it is clear that Φ is symmetric about the x-axis. Since ϕ is increasing, Φ is positive in the right half plane; Γ_0 is the y-axis. Figure 14 shows the one-parameter family $\{\Gamma_c\}$ for each of the choices $\phi(t) = t$, t^2, $\log(t)$.

Using what we know about the gradient of Φ we can prove

THEOREM 24. *If ϕ is concave then H^c_{QP} is starshaped with respect to Q when $c \geqslant 0$.*

Proof. It suffices to show that $\Phi(X)$ decreases as X moves away from Q along a ray and stays in the right half plane. To prove that, take the dot product of (5.4) with $X - Q$ to calculate the directional derivative of Φ along such a ray:

$$(5.10) \qquad \nabla\Phi(X)\cdot(X-Q) = \frac{\phi'(|X-P|)|X-P||X-Q|\cos(\theta)}{|X-P|} - \frac{\phi'(|X-Q|)|X-Q|^2}{|X-Q|} = |X-Q|(\phi'(|X-P|)\cos(\theta) - \phi'(|X-Q|)),$$

where θ is the angle at X in triangle PXQ. Since X is in the right half plane, $|X-Q| \leqslant |X-P|$. Since ϕ is concave, ϕ' is decreasing, so

$$\phi'(|X-Q|) \geqslant \phi'(|X-P|) \geqslant \cos(\theta)\phi'(|X-P|).$$

Hence (5.10) is not positive, as desired. □

For hyperbolic tessellations, both H_{QP} and its complement are starshaped; the circular tessellations show that that fails for a general concave ϕ. The failure is not because $\phi(0) = -\infty$. The same qualitative behavior is observed when $\phi(t) = \log(1 + t)$, in which case the bounding curves are no longer circles. They are instead ovals of Descartes, which arise in a slightly different manner in [11]. (We have searched unsuccessfully for a function ϕ for which the level curves turn out to be ellipses.) However, we shall show below that the complementary region is connected. First we show that the region H^c_{QP} itself, while starshaped, may fail to be convex. Figure 15 shows an example.

The function ϕ studied there is

$$\phi(t) = \begin{cases} Mt, & t \leqslant T \\ mt - (m - M)T, & t \geqslant T \end{cases}$$

with $P = (-1, 0)$, $Q = (1, 0)$, $M = 1$, $m = 1/5$, and $T = 2.5$. This ϕ is piecewise

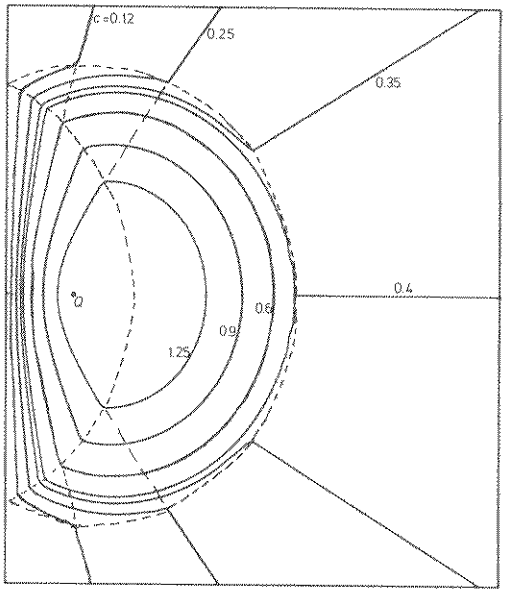

Fig. 15. Even a concave ϕ may yield nonconvex regions.

linear and hence not twice differentiable, but the same pathology rem: when it is smoothed. Let S be the lune in the right half plane between the cir of radius T centered at P and Q. To see the nonconvexity, observe that complement of S in the right half plane has two components in each of wh the level curves Γ_c of Φ are the familiar hyperbolas of Figure 14(a). But th labels on those curves change by the factor m/M as you move through S fr one component to the other. In S, in fact, each Γ_c is a Cartesian oval.

But even when ϕ is not concave we can say more about the geometry of level curves. We first find all the critical points of Φ: the places where $\nabla\Phi(X)$ or undefined.

THEOREM 25. *Φ has no critical points off the x-axis. If ϕ is strictly conc ($\phi'' < 0$) then P and Q are the only critcal points on the x-axis. When $\phi(0)$ P is a minimum and Q a maximum of Φ. When $\phi(0) = -\infty$ each is a pole, $\Phi(P) = -\infty$ while $\Phi(Q) = \infty$.*

Proof. Let $X = (x, y)$ and separate the components of $\nabla\Phi$. To simplify the resulting expression, set

$$\alpha(X) = \frac{\phi'(|(X - P)|)}{|X - P|}, \quad X \neq P,$$

and

$$\beta(X) = \frac{\phi'(|(X - Q)|)}{|X - Q|}, \quad X \neq Q.$$

On the y-axis $\alpha(X) = \beta(X)$ while, since $\phi' > 0$, $\alpha(X)$ and $\beta(X)$ are strictly positive. Then

$$(5.11) \qquad \nabla\Phi(X) = ([\alpha(X) - \beta(X)]x + [\alpha(X) + \beta(X)]a, [\alpha(X) - \beta(X)]y).$$

Off the x-axis $y \neq 0$ so we can have a critical point only when $\alpha(X) = \beta(X)$. But then $[\alpha(X) + \beta(X)]a \neq 0$, so the first component of $\nabla\Phi(X)$ is not 0.

If ϕ is concave, Theorem 24 shows that Φ decreases as you leave Q in either direction along the x-axis, which shows there are no critical points in the right half plane except the maximum (or pole) at Q. By considering $\Phi_{QP}(X) = -\Phi_{PQ}(X)$ we see that Φ_{PQ} has only one critical point in the left half plane, namely a minimum (or pole) at P. □

Note that we need the strict concavity of ϕ to show that Φ strictly decreases to the right of Q. When $\phi(t) = t$ the strict concavity fails, and in fact, $\Phi(x, 0) \equiv 2a$ for $x > a$. Then the entire x-axis to the right of Q consists of critical points, a fact visible in Figure 14(a).

COROLLARY 26. *If ϕ is concave then H^c_{PQ} is path-connected.*

Proof. Recall that we know that this region is starshaped with respect to P and hence connected when $c \leqslant 0$. The problem is proving the connectedness of the complement. The key is the absence of critical points, which shows that each level curve of Φ is an arc or a simple closed curve. Suppose $c \geqslant 0$; let X be a point at which $\Phi(X) = d \leqslant c$. If $d \leqslant 0$ then X is in the left half plane and is connected to P by the segment XP which lies entirely in H^c_{PQ}. That argument and the symmetry about the x-axis then alows us to assume $d \geqslant 0$ and that X is in the first quadrant, as shown in Figure 16.

Theorem 25 implies that as you travel along the x-axis from Q towards the origin you will cross the level curve Γ_d just once, say at Z. If we first follow the x-axis from P to Z and then Γ_d into the upper half plane we must encounter X, which thus lies on a curve in H^c_{PQ} which connects it to P. Thus H^c_{PQ} is path-connected. □

Note that when there are more than two sources this argument fails to show

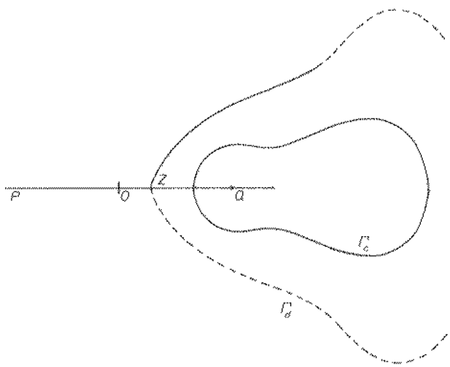

Fig. 16. Illustrating Corollary 26.

that the regions R_p are connected. We saw a counterexample when studying circular tessellations.

Whether or not ϕ is concave we can use $\nabla\Phi$ to explore what happens on the x-axis in more detail. Let $X = (x, 0)$, $x \geqslant 0$, and write

$$(5.12) \qquad \Phi(x) = \Phi(x,0) = \phi(x+a) - \phi(|x-a|) = \begin{cases} \phi(a+x) - \phi(a-x), & 0 \leqslant x \leqslant a \\ \phi(x+a) - \phi(x-a), & a \leqslant x. \end{cases}$$

Then

$$(5.13) \qquad \Phi'(x) = \begin{cases} \phi'(a+x) + \phi'(a-x), & 0 \leqslant x \leqslant a \\ \phi'(x+a) - \phi'(x-a), & a \leqslant x \end{cases}.$$

Since ϕ is increasing $\Phi(x)$ increases on the interval $[0, a]$ and takes on every value between $\Phi(0)$, which is 0, and $\Phi(a) = \phi(2a) - \phi(0)$. The latter value is either $\phi(2a)$ or ∞.

What happens betwen a and ∞ depends on whether ϕ is concave. If it is, then since Φ assumes its maximum at Q, every level curve meets the x-axis between 0 and a. Moreover, Theorem 24 shows Φ is decreasing on $[a, \infty)$. Then the slope of ϕ has a limiting value as $x \to \infty$; write $\phi'(\infty) = b \geqslant 0$. Then the mean value theorem shows that as $x \to \infty$

$$\Phi(x) = \phi(x+a) - \phi(x-a) \to 2a\phi'(\infty) = 2ab.$$

Thus just the level curves Γ_c with $2ab < c < \phi(2a) - \phi(0)$ will meet the x-axis a

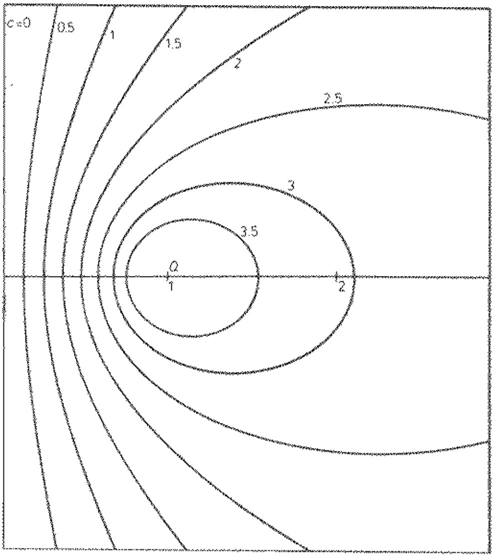

Fig. 17. The family $\{\Gamma_c\}$ for $\phi(t) = t + \log(t)$.

second time. When $\phi(t) = t$ we have $b = 1$; no hyperbola meets the x-axis twice (Figure 14(a)). When $\phi(t) = \log(t)$ we have $b = 0$; every Apollonian circle meets the x-axis twice (Figure 14(c)). Intermediate cases are possible: Figure 17 shows what happens when $\phi(t) = t + \log(t)$. Then $b = 1$.

We conclude with two examples illustrating what can happen when ϕ is not concave. Theorem 25 tells us we need search for critical points only on the x-axis, and (5.13) tells us they will be found where

$$\phi'(x + a) = \phi'(x - a), \quad x > a. \tag{5.14}$$

When $\phi(t) = t^4$ (5.14) has no solutions, so there are no critical points. The level curves of Φ all approach the y-axis asymptotically.

When ϕ' oscillates even more bizarre behavior is possible. For example, let

$$\phi(t) = t + k\sin(t) \quad (k < 1).$$

Then ϕ is increasing but not concave. Equation (5.14) is true when $\sin(x) = 0$. Analysis shows that the even multiples of π correspond to local maxima at

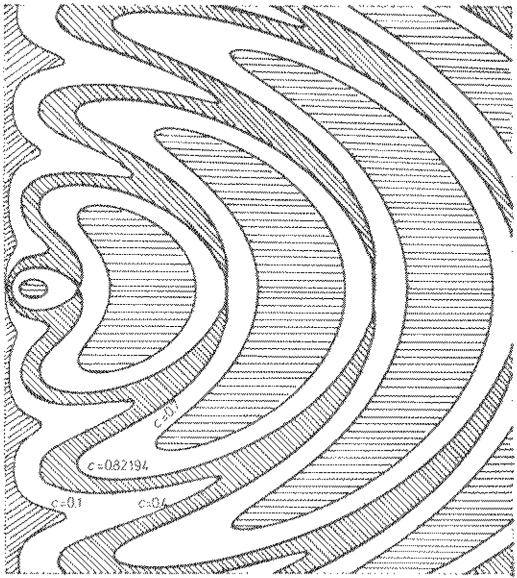

Fig. 18. The family $\{\Gamma_c\}$ for $\phi(t) = t + 0.7\sin(t)$.

each of which Φ has value $2(a + k\sin(a))$ while the odd multiples of π correspond to saddle points for Φ, all of which lie on the connected level curve with value $2(a - k\sin(a))$. For c between these values the level curve has infinitely many components. Figure 18 shows some of the level curves when $a = 1$ and $k = 0.7$.

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