Unit Conversions

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Plan

- How much does it cost to drive to school?
- Cost of gasoline in Europe
- Unit conversions – time, length
- Metric system, metric conversion, metric prefixes
- Area and volume

Lecture notes

Unit Conversions

In real life, numbers rarely exist by themselves. They usually *measure* something like cost, distance or temperature. To express this clearly, we write each number with its *units*, which indicate what kind of measurement the number represents. For example, we say "I live 10 miles from school," or "I bought 5 pounds of potatoes." Without the units those sentences wouldn't make sense:

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I live 10 from school.
I bought 5 of potatoes.
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Because the units are not specifically listed, information is lost. We might guess which units to fill in, but our guess may be incorrect. The consequences could be significant. In Canada you may see a speed limit sign with the number 100. If you think that represents 100 *miles per hour* you risk an expensive speeding ticket, since in Canada speed is measured in *kilometers per hour*. We'll see soon how to find out what a speed of 100 kilometers/hour corresponds to when measured in miles/hour.

So we establish a policy for ourselves:

Always write down the units of the numbers.

We'll illustrate this policy by answering an easy question first.[1]

> If you drive for an hour and a half at 50 miles per hour how far do you go?

We hope you can do that one in your head – 75 miles, because one and one half times 50 is 75. You don't need the machinery we're inventing to get it right, but it's a good place to learn that machinery. We're not multiplying just the numbers, we are multiplying the numbers along with their units:

$$1.5 \text{ hours} \times 50 \ \frac{\text{miles}}{\text{hour}} = 75 \text{ miles}$$

You probably recall from elementary school or from elementary algebra the fact that when the same factor appears on the top and the bottom of a fraction you can cancel it:

$$\frac{7 \times 3}{3 \times 5} = \frac{7 \times \not{3}}{\not{3} \times 5} = \frac{7}{5}$$

You can do with the words just what you do with the numbers – cancel them when they match. So in our simple example the hours cancel, leaving just the miles for the units of the answer.

$$1.5 \ \cancel{\text{hours}} \times 50 \ \frac{\text{miles}}{\cancel{\text{hour}}} = 75 \text{ miles}$$

[1]This example was really from the next class. In writing the notes we decided to put it here, first.

Now for a problem that's a little more interesting. How much does it cost to drive to school?

As often happens, one question leads to others. The cost of driving includes the costs of gas, parking, insurance, car maintenance and depreciation. If you add up those costs the number may be surprisingly large – but not necessarily a reason not to drive. The advantages of driving are important in that decision: you save the cost of public transportation, and, perhaps more important, time and inconvenience which are hard to measure in dollars. But here we will focus just on the cost of the gasoline you need.

One practical way to find that out is to fill your tank, drive to school and back home, then go to the gas station and fill your tank again. What that second fillup costs you is the answer to the question. What this clever method highlights is that in real life it's the questions that matter, not the answer the teacher wants you to give. But we do want to take this opportunity to answer this particular question another way, in order to learn something about quantitative reasoning along the way.

We decided that we could find the cost of gasoline by thinking rather than by measuring if knew three things:

- What is the price of gas?
- How much gas does my car use?
- What is the distance from my home to school?

We then did the problem with these assumptions:

- Gas costs $2.95 \; \frac{\text{dollars}}{\text{gallon}}$
- My car's fuel efficiency is $23 \; \frac{\text{miles}}{\text{gallon}}$
- The distance from home to school is 20 miles

We started with a quick estimate:

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I drive 20 miles to school and my car gets about 23 miles per gallon,
so I use a little less than 1 gallon for the trip.  One gallon costs
$2.95, so I figure it costs me about $2.85 (a little less than the
cost of one gallon) for the gas to get me to school.
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Quick estimates are good, for several reasons. They are faster and easier than entering all the numbers in your calculator. A quick quick estimate might be all you need in some particular case. Even when you need more precision starting with a quick estimate will help you check your answer when you're done.

We can make this quick estimate even better without much more work. Since 20 miles is 15% less than 23 miles, our gas cost should be about 15% less than the cost of one gallon of gas. If we use $3 per gallon as our gas cost (to make the calculation easier), then we estimate that we pay about $2.50 for the commute, since 15% of $3 is $0.45 which we round that up to $0.50.

At this point, we have answered our original question twice. Now let's do it one more time, paying close attention to the units. In fact, let's set up the answer without even filling in any numbers.

If we can find out how many gallons of gas we need to drive to school it will easy to figure out the total cost, since we know the price of gas. So we are looking for an answer now with gallons as the units. We know about the distance, (measured in miles) and the fuel efficiency (measured in $\frac{\text{miles}}{\text{gallon}}$.) In order to "cancel" the miles and be left with gallons we would have to set up the units this way:

$$\cancel{\text{miles}} \times \frac{\text{gallons}}{\cancel{\text{miles}}} = \text{gallons}.$$

The units have told us that we should be thinking about fuel efficiency in terms of gallons/mile rather then the usual miles/gallon.

Now fill in the numbers and do the arithmetic:

$$20\ \cancel{\text{miles}} \times \frac{1\ \text{gallons}}{23\ \cancel{\text{miles}}} = \frac{20}{23}\ \text{gallons} = 0.86\ \text{gallons}.$$

Now we know how much gas we use for the trip. To find out what the gas costs, think first about the units:

$$\cancel{\text{gallons}} \times \frac{\text{dollars}}{\cancel{\text{gallon}}} = \text{dollars}.$$

Filling in the numbers, we can finish the problem:

$$\begin{aligned} & 0.86 \text{ \sout{gallons}} \times \frac{2.95 \text{ dollars}}{1 \text{ \sout{gallon}}} \\ = & \ 0.86 \times 2.95 \text{ dollars} \\ = & \ 2.53 \text{ dollars.} \end{aligned}$$

We can even imagine doing it all in one step:

$$\begin{aligned} & 20 \text{ \sout{miles}} \times \frac{1 \text{ \sout{gallons}}}{23 \text{ \sout{miles}}} \times \frac{2.95 \text{ dollars}}{\text{\sout{gallon}}} \\ = & \ \frac{20 \times 2.95}{23} \text{ dollars} \\ = & \ 2.53 \text{ dollars.} \end{aligned}$$

Note that this answer is very close to our estimated answer. In fact, our estimated answer of $2.50 is a better answer. The extra 3 cents is too much precision for the approximate numbers like "20 miles" we started with. The real purpose of the detailed exercise is to learn how to solve problems by manipulating the units before thinking about the numbers.

Converting Currency

Suppose you are planning a trip to France (lucky you). Purchases there are in euros rather than in dollars. You will have to know what your euros are worth (in dollars) to keep track of your spending.

The value of the euro (in dollars) varies from day to day – actually, from minute to minute. You can always find a current value on the web. When we taught this class the going rate was

$$1\$ = 0.72 \text{ euros.}$$

That means it costs 0.72 euros to buy 1 U.S. dollar. In fraction form that tells us the conversion rate is

$$\frac{0.72 \text{ euros}}{1 \ \$} = 0.72 \ \frac{\text{euros}}{\$}.$$

When we are in France and see a pair of shoes that costs 49 euroswe need to know the dollar cost. To find the conversion rate in $\frac{\$}{\text{euros}}$ we simply turn the fraction we have upside down:

$$\frac{1\$}{0.72 \text{ euros}} = \frac{1}{0.72}\frac{\$}{\text{euros}} = 1.38\frac{\$}{\text{euros}}.$$

That means it costs 1.38 U.S. dollars to buy 1 euro. When we want to estimate we can think of this as saying that the cost in dollars is 40% larger than the cost in euros. So our 49euros pair of shoes really costs about $70 – 40% more than 50euros.

To make the conversion carefully, with units carefully displayed we write

$$\cancel{\text{euros}} \times \frac{\$}{\cancel{\text{euros}}} = \$$$

Then we fill in the numbers:

$$49\cancel{\text{euros}} \times 1.38\frac{\$}{\cancel{\text{euros}}} = 68.05 \text{ dollars.}$$

Our $70 estimate was a little high. But that's probably a good thing for several reasons. First, we haven't considered the fees a bank (or credit card company) charge each time we exchange currency. That would increase the effective exchange rate. Second, when considering a purchase it's always better to overestimate than underestimate what it will cost.

In France you may be driving as well as shopping. You need to plan for the cost of gas. It's sold there in liters, not gallons. The price today is $1.30\frac{\text{euros}}{\text{liter}}$. Looks like a bargain – -let's find out. We need to know what this means in dollars per gallon.[2] As usual, think about what you need to know. We understand the relationship between euros and dollars. We need the relationship between gallons and liters. That we can look up: there are 0.264 gallons in 1 liter.

As usual, we first set up the conversion just with units:

$$\frac{\cancel{\text{euros}}}{\cancel{\text{liter}}} \times \frac{\$}{\cancel{\text{euros}}} \times \frac{\cancel{\text{liters}}}{\text{gallon}} = \frac{\$}{\text{gallon}}.$$

[2]In fact the web will answer this question, several ways. You can search for it, or ask the Google calculator to do the conversion. But we think it's worth knowing how.

And then fill in the numbers and do the arithmetic:

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\begin{aligned}
1.30\ \frac{\cancel{\text{euros}}}{\cancel{\text{liter}}} \times \frac{\$}{0.72\ \cancel{\text{euros}}} \times \frac{\cancel{\text{liters}}}{0.264\ \text{gallon}} \\
&= \frac{1.30}{0.72 \times 0.264}\ \frac{\$}{\text{gallon}} \\
&= 6.84\ \frac{\$}{\text{gallon}}.
\end{aligned}
$$

Not such a bargain after all. Gas in France costs more than twice what it costs here. When we get an answer that unexpected we should think about why. The first two answers that most people come up with are

- It's because dollars are worth less than euros.
- It's because liters are smaller than gallons.

Both of these are true, but neither explains the high cost of gas in Europe. Those facts were precisely what we took into account in converting $1.30\ \frac{\text{euros}}{\text{liter}}$ to $6.84\ \frac{\$}{\text{gallon}}$. You need to search further for the real reason: gasoline taxes in Europe are much higher than they are in the United States.