October 9, 2007

Ethan D. Bolker
Maura B. Mast

October 9, 2007

Plan

- Frequentist probabilities: coins, cards, dice
- The fair price of a lottery ticket
- The house advantage. Insurance.

Lecture notes

0.1 Probability

What is probability? Several classmates offered definitions that amounted just to synonyms: “likelihood”, “chance”.

In our examples, we looked at the probability of an outcome occurring when all outcomes are equally likely. Here are some outcomes that we looked at: getting heads or tails on a coin toss; rolling a certain number with a fair die; choosing a particular card from a shuffled deck.

What does it mean to say that the probability of a coin coming up heads is $\frac{1}{2}$? One way to think about this is that if we flip the coin again and again, it will come up heads roughly one half the time. This assumes that our coin is fair. How do we know if a coin is fair? If we flip it many times and get heads about half the time, we can conclude that it is fair. Does this seem circular? Sure. We can make it clearer by giving a more formal way to calculating a probability. We do this by looking at all of the possible outcomes and constructing a fraction as follows:

$$\frac{\text{number of times desired outcome can occur}}{\text{number of possible outcomes}}$$

In our coin flipping example, there are two possible outcomes: heads or tails. It is only possible get heads one way, so the probability of flipping a coin and getting heads is $\frac{1}{2}$. If you roll a die, the probability that it comes up 6 is $\frac{1}{6}$. If you have a deck of cards, the probability of choosing an ace is $\frac{4}{52}$, or $\frac{1}{13}$. It may help to say this a different way: If I choose a card randomly (from a full deck) many times, then 1 out of every 13 times I should choose an ace. Of course, this doesn't mean that you *will* choose an ace every 13 draws, just that over the long run it will average out to one ace every 13 draws.

We'll look at more complicated versions of this question next time (for example, the probability of getting 4 heads in a row in four coin tosses); for now make sure you understand the basic approach.

Why do we write these probabilities as fractions? We usually use decimals in our class (and in life). Certainly we could in this case. We would get:

- The probability of getting heads in a coin toss is 0.5
- The probability of rolling a 6 with a fair die is 0.1666... = 0.17.
- The probability of drawing an ace from a well shuffled deck is about 0.769.

The decimals themselves are small (notice that they are all less than 1) and so not too easy to understand. But if we turn them into percentages, the meaning may be clearer. It is also more aesthetically pleasing. We could re-write each of the above statements:

- The probability of getting heads in a coin toss is 50%.
- The probability of rolling a 6 with a fair die is about 17%.
- The probability of drawing an ace from a well shuffled deck is about 77%.

Keep in mind that probabilities (when written as fractions or decimals) are always numbers between 0 and 1. If an outcome can never occur, its probability is 0. If the outcome will always occur, its probability is 1. You can always rephrase these statements using percentages (and so your percentages will be between 0% and 100%); if an outcome has a 100% probability, then it will always occur.

Now that we have the basic idea of how to find a probability, let's look at raffles and wagers.

A simple raffle

Imagine a raffle with a \$1000 prize. Suppose 500 tickets are sold. What's the fair price for a ticket?

We know the probability of a *particular* ticket winning is $\frac{1}{500}$, since there are 500 tickets and each has an equal chance of being the winner. This may not be the same as the probability of a particular person winning. If I buy 10 tickets then my probability of winning is $\frac{10}{500}$. If I don't play, the probability that I will win is 0. If I buy all of the tickets, the probability that I will win is 1, or 100%.

How much should you have to pay for a ticket? In other words, what is a fair price? If there are 500 tickets and a \$1000 prize, a fair price for a ticket is \$2, or $\frac{\$1000}{500}$.

Would you buy a ticket for \$1.50? Think about the possibilities: 499 times out of 500, you will lose your \$1.50. But 1 time out of 500 you will win the prize.

Would you buy a ticket for \$2.50? Here you are losing more money if you lose. But perhaps you really wanted to win and you could afford to spend \$2.50. Or perhaps the thrill you get from gambling is worth the extra money.

Certainly there are common situations where you are willing to pay more than the fair price for a ticket. If your local school's PTO has a raffle with a \$1000 prize, then they may charge \$3.00 for a ticket. The probabilities that we worked out above still hold, so why would we pay \$3 rather than the fair price of \$2? Perhaps because it is for a good cause, or we like the thrill that we may win. In this case, part of your decision about buying a ticket should be: "how much more than the fair price am I paying, and is it worth it?" In other words, you should have some knowledge about the fair price so that you can make a decision with all of the information.

In reality, you may not have all of the information. You may not know whether all of the tickets are sold, for example. You will have to make an estimate to then make your decision.

The house advantage

We can take the same principle and apply it to casino gambling. Many casinos feature a roulette wheel. For now let's simplify this type of wheel and imagine that it is divided equally into 36 wedges. The wedges are numbers consecutively 1 to 36, and colored alternately red and black. What kind of bets could we place, and what would be the probabilities of winning?

- You could bet on red, with a probability of $\frac{1}{2}$ of winning.
- You could bet on the number 17, with a probability of $\frac{1}{36}$ of winning.
- You could bet on odd, with a probability of $\frac{1}{2}$ of winning.
- You could bet on hitting one of the numbers 1 through 9, with a probability of $\frac{9}{36} = \frac{1}{4}$ of winning.

Now suppose you bet \$1 in each of the above situations. What should the payoff be?

- If you bet on red, the payoff should be \$2.
- If you bet on the number 17, the payoff should be \$36.
- If you bet on odd, the payoff should be \$2.
- If you bet on hitting one of the numbers 1 through 9, the payoff should be \$9.

Why is this? Think of the raffle again. If you bet on red, it is as if there are 2 raffle tickets sold (one red and one black), each at a price of \$1. If you win, you win the total amount bet, or \$2. If you bet on the number 17, it is as if there were 36 raffle tickets sold (each numbered 1 through 36). The total payoff is the total prize, or \$36. Finally, if you bet on 1 through 9, it is as if there are 4 raffle tickets sold (numbered 1 - 9; 10 - 18; 19 - 27; 28 - 36) and so \$4 is paid in as the total prize.

Of course, we would not see this at a casino, as the casino needs to make money. They need to collect *more* money than they give back in prizes. Thus, the prices are never fair. The cost of a bet is always bigger than the payoff. This is called the house advantage, and it is always present. Does this mean you shouldn't bet? As we said above, you need to realize that you are not paying a fair price, and you need to have a good sense of how far away your price is from fair. Then you need to take into account anything else you are getting (the pleasure of gambling, the entertainment, and so on) and use that to make your decision.

To return to our roulette example. The casino will pay \$36 for a \$1 bet, but they put at least one extra wedge into the wheel (numbered 0; often there's another one numbered 00 - often colored green). This changes the probabilities. For example, if you bet on hitting 17 your probability of winning is now $\frac{1}{37}$. The fair payoff should be \$37, but they will pay \$36 instead. If there are two extra wedges on the wheel, your probability of hitting 17 is now $\frac{1}{38}$ and the house will pay \$36 instead of \$38. If you bet on red, the probability is now $\frac{18}{38}$ and the payoff should be $\$\frac{38}{18}$, or \$2.11. Instead, it's still \$2.00.

Insurance