Independent events

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Plan

- Independent events – probabilities multiply
- Poisson processes – runs happen
- Doubling up as a strategy for roulette
- Probability as degree of belief – who wins the game? will it rain tomorrow?

Lecture notes

The Probability of Independent Events

We saw last time that if we flip a fair coin then the probability of getting heads is equal to $\frac{1}{2}$. We know this because there are two possible outcomes (heads or tails) and heads can only occur in one way.

Now we make the question a little harder (or so it seems). Suppose we have a coin in one hand and a die in the other. What's the probability that when you flip the coin and toss the die you get a head and a 4?

As with the simple example that we started with, we could look at all of the possibilities. Remember that we want to calculate

$$\frac{\text{number of times desired outcome can occur}}{\text{number of possible outcomes}}$$

Since there is no relationship between tossing a coin and rolling a die, we call these *independent* events. We'll see a quick way to calculate the probability, but first try to work out all of the options. You'll quickly see that some of the possible outcomes are:

heads and 1
tails and 1
heads and 2
tails and 2
...
heads and 6
tails and 6

There are 12 of those combinations, so each will occur with probability 1/12.

Here's another way to look at it: if we flip a coin, the probability of getting heads is $\frac{1}{2}$. If we roll a die, the probability of rolling 4 is $\frac{1}{6}$. To compute the probability of getting heads and rolling 4, we can *multiply* the two probabilities: $\frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12}$. We won't worry too much about why this works, but it is good to remember the basic rule:

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When two events have nothing to do with one another, the probability
that both happen is the product of the probabilities for
each one separately.
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We consider another example. Suppose you flip a coin twice. What is the probability that you will get two heads in a row? This is another example of independent events. The coin does not know happened in the first flip, so when you flip it again you are looking at an independent event. We can multiply the two probabilities: $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. Or we can write out the possible outcomes: Abbreviating heads as H and tails as T, we have the following possibilities: HH, HT, TH, TT. There are four possibilities, and only one fits the outcome we want, so the probability is $\frac{1}{4}$.

You could extend this: the probability of flipping a coin 4 times and getting 4 heads in a row is $\frac{1}{16}$. The probability of flipping a coin 10 times and getting 10 heads in a row is

$$\frac{1}{2^{10}}.$$

The probability of flipping a coin 100 times and getting 100 heads in a row is awfully small, but it's not equal to zero. You can check that it should be

$$\frac{1}{2^{100}}.$$

Repeated Events - Runs Happen

Now look at a grid consisting of 64 blocks, eight across and eight down. As you go through the grid (moving left to right) mentally toss a coin in your head. With each toss, mark "H" for heads or "T" for tails in the grid. When you have finished filling it in, go back and count how many Hs and how many Ts occur in your grid.

It is unlikely that someone will have exactly 32 heads and 32 tails, but it is likely that your counts will be near that.

Now we look at the grid and look for runs of four heads in a row. That is, if your first row looks like:

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H T T H H H H T
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then this would contain one run of four Hs. If your first row looks like:

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H T T H H H H H
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then this wold contain *two* runs of four Hs. Note that there are five Hs in a row. Within this block of five Hs, we pick out two different runs: (H H H H)H and H (H H H H).

Go through and count the runs of four Hs, using this definition. Most students had a few runs of four Hs, typically fewer than five or six runs. Why is this? The mind wants to have a balanced number of Hs and Ts, so doesn't let you put too many in a row. Your vision of a fair coin is that it should alternate between H and T fairly often.

Now count down each column and look for runs of four Hs. You'll find many more such runs. What is going on here? The mind only remembers a certain amount, and basically after 8 or so flips it forgets what went before. That's why the rows are 8 blocks long, so that by the time you move to the next row you will have forgotten what you put in the previous row. Thus, we are getting a more random distribution of Hs and Ts and, as happens in nature, we will get runs of Hs and Ts.

Dependent Events

Be careful when you see these types of questions. You have to make sure that the events are independent before you try to multiply the probabilities. Think about this example. Suppose a young boy loves red socks and loves blue socks. He has the same number of each and he is equally fond of red and blue. He always wears matching socks. Suppose you look at his left foot. The probability of that sock being red is $\frac{1}{2}$. With this information, the probability of the sock on his right foot being red is close to 1. The point here is that these events are dependent. The outcome of the first event (the choice of a red sock) influences the outcome of the second event (the choice of the matching sock).

Here's another example of dependent probabilities. Suppose you have a well-shuffled deck. The probability of drawing a red card is $\frac{26}{52}$ or $\frac{1}{2}$. If you have drawn an red card, then the probability that the *second* card you draw will also be red will be: $\frac{1}{2} \cdot \frac{25}{51} = \frac{25}{102}$, or just under $\frac{1}{4}$. The key here is that the first draw has changed the deck. There are fewer cards (51 instead of 52) and fewer red cards. We have to adjust for this when we calculate the probability.

The key point here is that runs do happen. We think of the probability of getting heads on a coin toss as being $\frac{1}{2}$, and so we expect that we will not have a long run of heads, but the rules of probability say that this will happen occasionally. You need to be aware that it is a real possibility.