A first look at exponential models

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Lecture notes

A comparison between linear growth and exponential growth

If you had \$1,000 to invest, would it be better to get a return of \$100 per year, or to earn 8% interest on your money each year? We'll assume that the interest is added into your account each year and that we never make any withdrawals.

Because this is a very concrete example, it makes sense to start to explore it by calculating what happens for the first few years. We can do this by setting up a table and making some calculations by hand. First the first scenario, we simply add \$100 each year to get the new balance. After one year the balance would be \$1,100, after two years the balance would be \$1,200, and so on. Under the second scenario, we have to calculate a percentage each time. After the first year, the new balance would be $\$1,080 = \$(1.08)(1000)$. After the second year, the new balance would be \$1166 = \$(1.08)(1080). Here is what the table would look like:

Year	Scenario 1	Scenario 2
now	\$1,000	\$1,000
1	\$1,100	\$1,080
2	\$1,200	\$1,166.40
3	\$1,300	\$1,259.71
4	\$1,400	\$ 1,360.49

So far the first scenario is giving us a better return on the investment. If we wanted to carry these calculations out to 20 years to see what happens, we have some choices for how to do it. We could continue to do each year's calculation by hand (which is tedious); we could use Excel to calculate this for us (we'll do that in a minute); or we could look for a pattern and find functions that fit each scenario. Let's find the functions that fit each scenario.

The first scenario is a *linear* one, since a one year increase results in a \$100 increase in the balance. The starting value is \$1,000 and therefore the linear function is

$$B = 1000 + 100 * T$$

where T represents the number of years and B represents the balance. If $T = 20$, the balance would be $\$3000 = 1000 + 100 * 20$.

The second scenario does not follow a linear pattern. Why not? One reason is that you cannot find a slope. For each additional year, you are not getting a fixed increase in the balance. From now to the first year, you earn \$80, but from the first year to the second year you earn \$86.40. To see what kind of function to use, we can think about how each year's calculation is done. Here is one way to look at it:

Year 1	$1080.00 = 1000 * 1.08$
Year 2	$1166.40 = 1080.00 * 1.08 = (1000 * 1.08) * 1.08 = 1000 * 1.08^2$
Year 3	$1259.71 = 1166.40 * 1.08 = (1080 * 1.08) * 1.08 = 1000 * 1.08^3$

You can verify that for the fourth year the balance is $1000*(1.08)^4$ and for year 20 the balance will be $1000(1.08)^{20} = \$4,660.96$. If you are using a calculator to figure this out, you will have to look for the button that lets you do an exponent. On many calculators it looks like y^x. You can also evaluate this using Excel, typing in

```
=1000*(1.08)^20.
```

We found out in class that you can even put it into the search bar in Google.

If we let T represent the number of years, then a function that models the second scenario is the function $Y = 1000(1.08)^T$. This is called an *exponential* function. Notice that the variable T is the exponent of 1.08 (hence the name exponential for this type of function). If you look carefully at this function, you can see some important information about our scenario. The 0.08 in the number 1.08 in the function represents the relative change for each additional year. This relative change is always the same for this type of function. The number 1000 in the function represents our starting value.

Putting all of this together, we make a very important observation about linear and exponential functions.

```
With linear growth, the absolute change is always the same
With exponential growth, the relative change is always the same.
```

Using Excel to explore exponential growth

Now we look again at our original question of which scenario is better. From the work above, we see that initially the linear growth is better but that by year 20 the exponential growth gives us a higher balance. We can explore this with Excel to see what happens each year and to make a graph.

Open the Excel file `http://www.cs.umb.edu/~eb/m114/lectureNotes/1127/linearExponential.xls`. This spreadsheet includes a table giving the balance from now through 15 years. The data are graphed on the scatter plot and you can see that at year 7, the balance in the exponential model begins to outpace the balance in the linear model. Therefore, the first scenario (the linear model) is better for the first 7 years; after that, the exponential model is better. We will see that exponential growth will always (eventually) beat linear growth.

Now we use some of the power of Excel to change the questions. Suppose, for example, our money earned 7% interest instead of 8% interest. Rather than repeating all of the calculations, we can change one number in our spreadsheet and see how the rest of the numbers change. Go to cell A6 in the spreadsheet and change 1.08 to 1.07. When you hit return, you'll see that the numbers in the exponential column change, and the exponential curve on the graph changes. We see that with this lower interest rate, we have to wait 11 years before the exponential scenario gives us a better return on the money.

To see how Excel updated the calculations, click on one of the new numbers in the exponential column, say C19. The formula bar will read

```
=START*RELCHANGE^A19.
```

We changed the number in the cell labeled RELCHANGE when we changed cell A6 from 1.08 to 1.07. By labeling this cell RELCHANGE, we can refer to it in an Excel formula and the reference is fixed. When we update the value of this cell, Excel immediately updates the calculations in the table (and the graph) based on this change. [1]

You can experiment with other changes with this spreadsheet. It's good practice to think about what you are doing and make a guess first about the outcome. This may help you see whether the change that you made is what you actually intended to do. Remember that you can always click on the "undo" button in the standard toolbar (or go to the Edit menu and click on "Undo") if you want to get back to a previous spreadsheet.

Here are a few more examples to try with this spreadsheet. If you change the STEP to 2, notice how every column in the table changes. Now we are looking at the balance every 2 years, instead of every year. Our graph then goes out to 30 years. You can extend the graph to 45 years by choosing a STEP of 3. The STEP does not have to be a whole number - you could put any number in there and Excel will incorporate it into the formula.

[1] To give a cell a name, click on the cell and then go to the Name Box in the Formatting Toolbar. The Name Box is a small window on this toolbar that usually contains the name of the cell (so if you click on cell H4 you will see H4 in that box). You can highlight the contents of the box and type in your own name. If you click on 1.08 in the spreadsheet and look at the Name Box, you'll see that we named this cell RELCHANGE.