9

Interest — Exponential Growth

In this chapter we explore how investments and populations grow and radioactivity decays — exponentially.

Chapter goals:

Goal 9.1. Understand that exponential growth (or decay) is constant relative change.

Goal 9.2. Understand how compound interest is calculated.

Goal 9.3. Work with exponential decay.

Goal 9.4. Reason using doubling times, half-lives, rule of 70.

Goal 9.5. Fit exponential models to data.

9.1 Money earns money

Imagine that you have $1,000 to invest. Would you rather earn $100 per year in interest, or 8% per year in interest? In each case the interest is added into your principal (the balance in your account) each year, and you never make any withdrawals.

The first scenario is called simple interest. You find the new balance by adding $100 each year to the previous balance. After one year the balance would be $1,100, after two years $1,200, and so on.

	Balance	
Year	Simple interest	Compound interest
now	\$1,000	\$1,000.00
1	\$1,100	\$1,080.00
2	\$1,200	\$1,166.40
3	\$1,300	\$1,259.71
4	\$1,400	\$1,360.49

Table 9.1. Simple and compound interest

The second scenario is a little more complicated. The interest each year is a fixed percentage of the balance. The one-plus trick finds the new balance in one step: after one year you would have $1.08 \times \$1,000 = \$1,080$. After two your balance would be $1.08 \times \$1,080 = \$1,166.40$. This pattern is called compound interest.

Table **??** shows your balance in each case for the first four years.

So far simple interest offers a better return on your investment. What would the numbers be in 10 years? We could continue building the table a year at a time by hand (which is tedious), we could have a spreadsheet calculate for us (we'll do that in a minute) or we could look for a pattern and find a formula for each scenario, so that we can compute for any year we like without having to do the work for all the years in between. We'll do that first.

Simple interest leads to a linear equation. Each year the balance increases by a fixed amount, \$100, so the slope is \$100/year. The intercept is the starting value, \$1,000. The linear function is

$$B = 1000 + 100 \times T$$

where B represents the balance, in dollars, and T the number of years. If you leave your money growing until $T = 10$ years, your balance will be $1000 + 100 \times 10 = 2000$ dollars.

The function describing compound interest isn't linear. The percentage increase is constant but the amount of interest changes from year to year. In the first year you earn \$80, while in the second you earn \$86.40. To see what kind of function to use, we unwind the arithmetic in the compound interest column of Table **??**.

$$\begin{aligned}
\text{Year 1:} \quad & 1080.00 = 1000.00 \times 1.08 \\
\text{Year 2:} \quad & 1166.40 = 1080.00 \times 1.08 = (1000 \times 1.08) \times 1.08 = 1000 \times 1.08^2 \\
\text{Year 3:} \quad & 1259.71 = 1166.40 \times 1.08 = (1080 \times 1.08^2) \times 1.08 = 1000 \times 1.08^3
\end{aligned}$$

It's clear that the function describing this growth is

$$B = 1000 \times (1.08)^T \tag{9.1}$$

where, as before, B represents the balance, in dollars, and T the time, in years. It's an exponential function, because the independent variable T is the exponent of 1.08. The 0.08 in $1.08 = 1 + 0.08$ is the constant relative change for each additional year. The 1000 is where we start: the value of B when $T = 0$. (You may remember but not have enjoyed the fact that $1.08^0 = 1$. If so, perhaps it makes a little more sense in this context. After 0 years you've received no interest, so your balance should be multiplied just by 1.)

Suppose you want to compare the balances at simple and compound interest after year 10. With simple interest you will have $1000 + 10 \times 100 = 2000$ dollars. But how can you compute 1000×1.08^{10} without boringly multiplying by 1.08 ten times? The calculator in Figure **??** isn't powerful enough. For that job you need a *scientific calculator*, one with a key labeled $\boxed{y^x}$ or $\boxed{x^y}$.

There are many online. Here are two: `www.math.com/students/calculators/source/scientific.html` , `web2.0calc.com/` . Each will tell you that at the end of year 10 the balance will be about $2159, so the exponential growth has caught up with the linear.

You can do the computation with the Google calculator's buttons but to use the search bar or a spreadsheet you need to know how to enter the exponent from the keyboard without a $\boxed{y^x}$ key. Both use the caret character "^" to raise a number to a power. That's meant to suggest literally "raising" the exponent. You just type

```
1000 * 1.08 ^ 10
```

into the Google search bar, or as a formula (preceded by an equal sign) in a cell in a spreadsheet to check the arithmetic in the previous paragraphs.

In *Common Sense Mathematics* we rarely put things to remember in boxes, but the moral of this discussion deserves that treatment:

In linear growth, the absolute change is constant. In exponential growth, the relative change is constant.

Interest isn't the only place exponential growth happens. In Exercise **??** we ask you to think about others.

9.2 Exploring exponential growth with a spreadsheet

You can answer "what if" questions about exponential growth by changing the initial investment and interest rate in the the spreadsheet `exponentialGrowth.xlsx` . Figure **??** shows two examples, for the equations

$$B = 1000 \times (1.08)^T$$

and

$$B = 1000 \times (1.16)^T.$$

Each swoops upward at an increasing rate. That shape is the signature for exponential growth. The graphs look similar, but the scales on the vertical axes tell a different story. The spreadsheet has a second tab (labeled "Compare two growth trajectories") that plots

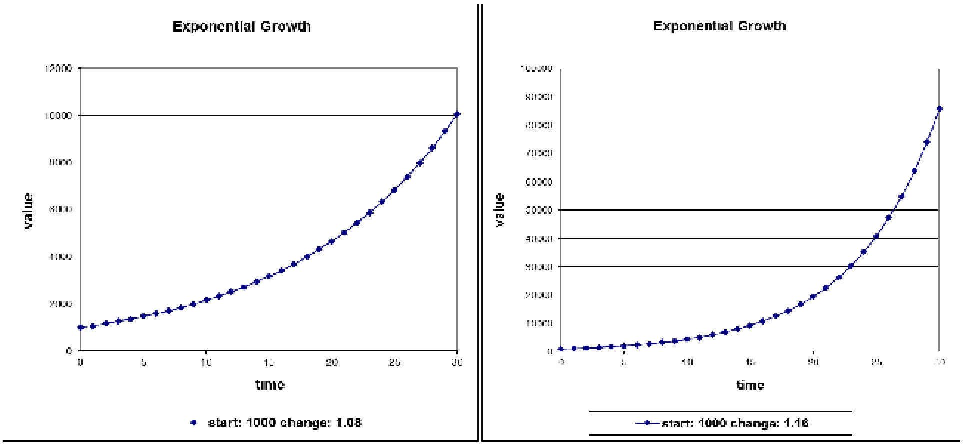


Figure 9.2. Two exponential graphs

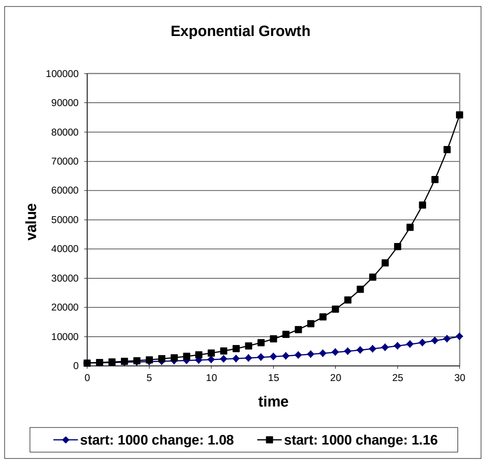


Figure 9.3. Two exponential graphs on the same set of axes

two exponential curves on the same set of axes. Figure **??** clearly shows how much faster growth is at 16% than at 8%.

Let's take some time to see how Excel updates the calculations when we change the constants in the equation. Click on one of the cells in the exponential column, say `B17`. The formula bar reads

$$= \texttt{START*RELCHANGE}^\wedge\texttt{A17}, \tag{9.2}$$

which is Excel's version of (**??**). We labeled cells `A10` and `A11` as `START` and `RELCHANGE` so that we could use (**??**) instead of

$$= \texttt{\$A\$10*\$A\$11}^\wedge\texttt{A19}.$$

The version using cell labels is much easier to understand than the one with cell references, and it doesn't need the dollar signs to tell Excel not to change those references when we copy from one row to another.

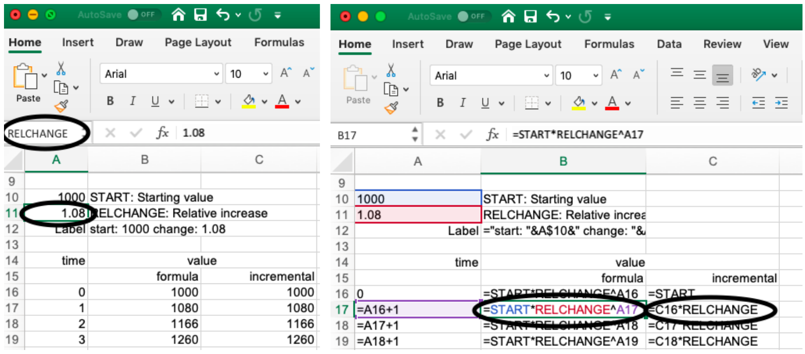


Figure 9.4. Screenshots showing Excel formulas using named cells

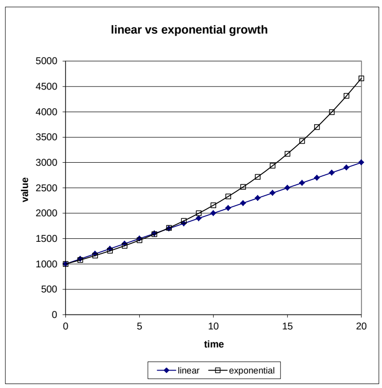


Figure 9.5. Linear vs exponential growth

To label a cell, click on it. The Name Box at the left of the Formatting Toolbar will contain the address of the cell, so if you click on cell `H4` you will see `H4` there. You can highlight the contents of the box and type in your own name.

Figure **??** shows two screen shots of our spreadsheet, the first with cell values, the second with cell formulas.

The numbers in columns `B` and `C` are the same. Excel computes them in different ways. We've seen how `B17` uses the algebra in (**??**). The value in cell `C17` comes from the previous value in `C18` instead:

```
= C16*RELCHANGE.
```

In the hypothetical investment comparison at the start of this chapter linear growth starts out better but by year 10 exponential growth leads to a higher balance. To explore what happens in more detail, use the spreadsheet `linearExponential.xlsx`. It extends Table **??** to cover 15 years. The graph in Figure **??** shows that starting at year 7, the value of the exponential function is larger than the linear.

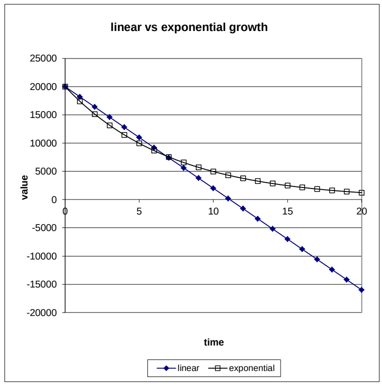


Figure 9.6. Linear vs exponential depreciation

Now we can answer "what-if" questions. Suppose, for example, our money earned 7% interest instead of 8% interest. To redo the calculations we need to change just one number: replace the 1.08 in cell `A9` with 1.07. Excel recomputes the values of the exponential function in column `C` and redraws the graph. Then you can see that with this lower interest rate, we have to wait 11 years before the exponential growth of compound interest gives us a better return.

9.3 Depreciation

It's always easier to think about increases (adding and multiplying) than decreases (subtracting and dividing) but sometimes things do decrease.

Suppose you buy a new car for $20,000. As soon as you drive it out of the dealer's lot it's worth less. In fact it's worth less each year: it *depreciates*. Its value depends on its age.

If the car is a business expense you might choose linear depreciation for tax purposes — suppose the value decreases by $1,800 each year. The equation that determines the value V as a function of the age A is

$$V = 20,000 - 1,800A.$$

But a more realistic way to model the value of the car is to assume that the percentage decrease is the same each year. Suppose it's 13%. Then each year its value is 87% of what it was the year before. The corresponding equation is

$$V = 20,000 \times 0.87^A.$$

We can use our old friend `linearExponential.xlsx` to draw Figure **??** showing what the car is worth over time in each case. Set `START` to 20,000, `ABSCHANGE` to -1,800, and `RELCHANGE` to $1 - 0.13 = 0.87$. (The relative change is still positive. It's a decrease rather than an increase because it's less than 1.)

When the depreciation is linear the car is worthless (at least on paper) after about 11 years. Excel doesn't know that, so it continues the graph on into negative values. If we wanted to use this graph in a more formal presentation we'd have to prevent that, and change the

interest rate (%)	approximate doubling time	70/rate
2	35	35.0
3	24	23.3
5	14.5	14.0
7	10.5	10.0
8	9	8.8
10	7.5	7.0
15	5	4.7
20	4	3.5
50	1.7	1.4
100	1	0.7

Table 9.7. Double your money

labels. Leaving it this way exhibits the power of thinking abstractly in Excel — the original spreadsheet can manage shrinking just as easily as growth.

9.4 Doubling times and half-lives

How long will it take to double your money? (We'll assume you're clever enough to insist on compound interest.) The answer depends on the interest rate and the initial balance. The `exponentialGrowth.xlsx` spreadsheet shows that at 8% interest with an initial investment of $1,000 the balance is $2,000 after 9 years (the table shows $1999.004627, which is quite close enough to double).

If you change the initial investment to $100 then Excel shows a balance of $200 after the same 9 years. Experimenting with many different initial investments always shows the same doubling time. So the time it takes to double your money does not depend on the amount you start with.

What about a different interest rate? If you use 5% interest in the spreadsheet the doubling time seems to be between 14 and 15 years. We can do a calculation: $1.05^{14.5} = 2.028826162$, so 14.5 is a good guess.

At 2% interest it takes more than 30 years to double your money, so the spreadsheet doesn't give us the answer. We could find it by adding some rows, but we'll use another method instead. We'll try to guess the value of T in the equation $1.02^T = 2$ and adjust our guess until we're close enough. Perhaps the answer is $T = 40$ years:

$$1.02^{40} = 2.208039664.$$

Too big, so we need less time. Try 35:

$$1.02^{35} = 1.999889553.$$

Bingo!

We've collected these results and a few more in the second column of Table **??**.

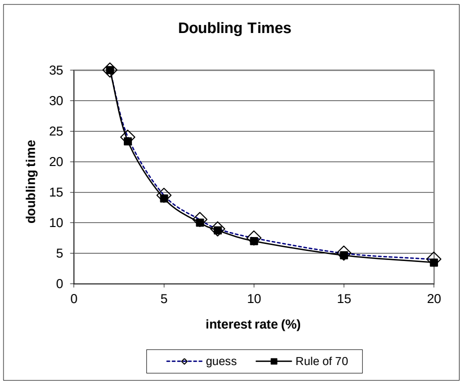


Figure 9.8. Doubling times

The third column in that table shows the results from the "Rule of 70", that says that you can estimate the compound interest doubling time by dividing the magic number 70 by the annual interest rate as a percent. The approximation is better when the interest rate isn't too large; those are just the cases that matter most in everyday investing. The most commonly quoted consequence of the Rule of 70 is that money invested at 7% will double in 10 years.

Figure **??** shows how good the Rule of 70 is for interest rates up to 20%.

When a relative increase occurs repeatedly the doubling time is independent of the initial value. So if inflation is 5% per year, all prices will double in 14 years.

Knowing the doubling time helps you make quick calculations. Since 5% inflation doubles prices in 14 years it will quadruple them in 28 years. In 42 years they will be eight times as large. The Bureau of Labor Statistics inflation calculator says that inflation in the 42 years from 1968 to 2010 increased the cost of a $100 item to $626. That's not quite eight times as much, so the average inflation rate for those years was not quite 5% per year.

The Rule of 70 applies to depreciation as well — it tells you the *half-life*. That's the equivalent for depreciation of the doubling time — the time until half the original value has disappeared. Like doubling time, the half-life depends on the depreciation rate, but not on the original value. For 13% annual depreciation it's approximately $70/13 = 5.38461538 \approx$ 5.4 years.

The term half-life comes from atomic physics, where it describes the way the quantity of a radioactive element decreases over time. The following quotation from `http://archives.nirs.us/factsheets/hlwfcst.htm` provides food for quantitative thought.

> After ten half-lives, one-thousandth of the original concentration [of a radioactive substance] is left; after 20 half-lives, one millionth. Generally 10-20 half-lives is called the hazardous life of the waste. Example: plutonium-239, which is in irradiated fuel [from a nuclear power plant], has a half-life of 24,400 years. It is dangerous for a quarter million years, or 12,000 human generations. [R**??**]

This is the kind of quotation that begs to have its numbers checked.

First let's look at "ten half-lives". After one half-life the concentration is half what it was at the start. After two half-lives it's half of a half, or 1/4. After three half-lives it's 1/8. So after 10 half-lives it's

$$\frac{1}{2} \times \frac{1}{2} \times \cdots \times \frac{1}{2} = \frac{1}{2^{10}}$$

of what it was.

We saw when we studied the metric prefixes that $2^{10} = 1,024 \approx 1,000$. That's why "kilo" means "1,000" most of the time but "1,024" when computers are involved. Now we use the same fact to see why $1/2^{10}$, which is exactly 1/1,024, is approximately 1/1,000 — one one thousandth.

What about twenty half-lives? In that time the original concentration will be reduced to 1/1,000 of 1/1,000 of what it was at the start. Since a thousand thousand is a million, that's one one-millionth.

According the quotation, plutonium-239 will be dangerous for at least ten 24,000 year half-lives. That's 240,000 years, which is indeed about a quarter of a million years. Is it 12,000 generations? Yes, if you calculate with $240,000/12,000 = 20$ years per generation. That's perhaps a little low for the developed world, but good enough to highlight the danger of nuclear waste.

9.5 Exponential models

For compound interest and radioactive decay the equation for exponential change gives exact answers, just as the linear equation gives exact answers for simple interest and electricity bills.

When change is approximately linear a regression line may be useful. When it's approximately exponential, we can construct an *exponential trendline*. Most elementary texts discuss the reproduction of bacteria as a toy example of exponential growth. Bacteria reproduce by dividing, so each individual gives rise to 2, then 4, then 8 descendants, and so on. The number of bacteria grows exponentially. But that can't go on forever. Since exponential growth curves quickly become steep, it can't go on for very long. Eventually, crowding or diminishing resources cause growth to slow, perhaps even to reverse as organisms die faster than new ones are born.

Table **??** records the population of three different strains of the *Burkholderia cenocepacia* bacterium in a one day experiment conducted by Professor Vaughn Cooper at the University of New Hampshire.

The raw data points in the graph on the left in Figure **??** (built in the spreadsheet `bacteriaGrowth.xlsx`) suggest that the growth of each strain was exponential until about hour 17. To construct the graph on the right we plotted the data for each strain for that period. In the resulting chart we right-clicked on a data point for each strain, selected `Add Trendline ...`, and chose

time	population W	population R	population S
4	7.9	8.5	17.5
8	17.3	13.5	42.1
12	44.7	48.9	225.2
17	119.3	268.7	407.3
20	41.3	98.0	149.3
24	41.3	64.0	160.0

Table 9.9. Bacteria growth [R**??**]

Time: hours
Population: millions of organisms per milliliter

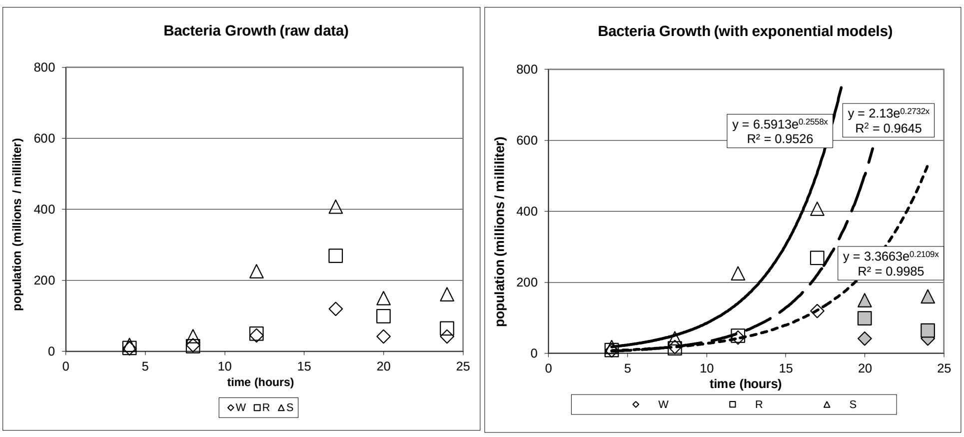


Figure 9.10. Bacteria growth

`Exponential` on the `Type` tab. As usual, we asked for the equations and the R-squared values. Those are all pretty close to 1; best for strain W, worst for strain S.

Let's look at the equation for the exponential trendline for strain W, which you can see in the figure:

$$y = 3.3663\, e^{0.2109x}. \tag{9.3}$$

What is the "e" in this equation? The complete answer to that question calls for much more mathematics than you need to know to apply common sense to quantitative arguments. But since Excel uses it you may encounter it somewhere so we'll discuss it briefly.

The simplest explanation is that e is just a particular number — approximately 2.7183. Like $\pi \approx 3.1416$ it's one of those numbers whose decimal expansion "goes on forever", so the first few decimal places give just an approximation. The number e appears naturally in discussions of exponential growth just as π appears in discussions of circles. It was named by the prolific mathematician Leonhard Euler (1707-1783) who was the first to recognize its importance.

Back to our exponential growth function for the strain W. We see that

$$e^{0.2109} \approx (2.7183)^{0.2109} \approx 1.2348.$$

You don't have to remember the value of e to compute with it, since Excel and LibreCalc and the Google calculator provide the built-in function EXP to do the job. Entering `=EXP(0.2109)` in a spreadsheet cell or the same thing without the equal sign in the Google search bar will give you the same answer.

Now we can substitute 1.2348 for $e^{0.2109}$ in (**??**) and rewrite it as

$$y = 3.3663 \times 1.2348^x.$$

In that form we recognize this behavior as exponential growth at a constant rate of about 23% per hour. The Rule of 70 tells us to expect a doubling time of about $70/23 \approx 3$ hours. That matches the data, which does indeed show the strain W population doubling about every three hours for most of the day.

If you experiment with `EXP` you will find that $e^{0.7} = 2.0137527\ldots \approx 2$. It's the 0.7 in the exponent that leads to the Rule of 70. To learn just how, go on to take a course in calculus.

Using the exponential trendlines for constant growth rate to predict the future populations would fail in this experiment. About two thirds of the way through the 24 hour day a different reality appears. The gray bullets in Figure **??** show that all three populations drop dramatically. (A newspaper reporter wanting to emphasize the drama might say, incorrectly, that the populations dropped "exponentially.")

It's the biologist's job to understand why. The analysis of the exponential growth at the start has told him only how the bacteria grow when there's lots of food, lots of room and no competition.

9.6 "Exponentially"

A Google search for a definition of *exponentially* finds this first meaning:

> 1. (with reference to an increase) more and more rapidly.
> "our business has been growing exponentially"

This is from *The New York Times* in April, 2019.

> "The way ads are targeted today is radically different from the way it was done 10 or 15 years ago," said Frederike Kaltheuner, who heads the corporate exploitation program at Privacy International. "It's become exponentially more invasive, and most people are completely unaware of what kinds of data feeds into the targeting." [R**??**]

The *Daily Beast* seems particularly fond of the word.

> “[I]n 2009, the [Sketchbook Museum] project moved to New York and exponentially grew across the globe.” [R**??**]

> “Exponentially powerful technologies are transforming our sphere of possibilities.” [R**??**]

It’s sad to see “exponentially” reduced to a bland adjective when we understand its precise mathematical meaning — the kind of growth captured in the formula for compound interest:

$$\text{balance} = \text{start} \times (1 + \text{rate})^{\text{time}},$$

Even writers wno know that “exponentially” must somehow involve an exponent can get the mathematics wrong. Victoria Markovitz wrote in a *National Geographic* article that

> [t]he amount of power you can produce [with a wind turbine] is determined by the square of the blade radius. That means increasing the size of the turbine has an exponential effect on power. [R**??**]

To think about this claim we put the power output equation from the linked post at `www.nationalgeographic.com/environment/great-energy-challenge/2012/worlds-largest-wind-turbines-is-bigger-always-better/` in spreadsheet `turbinepower.xlsx`. Then we drew Figure **??** to see how power production P depends on blade length R for lengths up to 35 feet. The dotted curve predicts values using the square of the blade radius. The solid curve that grows much faster predicts values using an exponential trendline. The two equations are essentially

$$\text{quadratic}: \quad \text{power} = 0.0016\,\text{radius}^2$$

and

$$\text{exponential}: \quad \text{power} = 0.19e^{0.07\,\text{radius}} \approx 0.19\,(1.07)^{\text{radius}}.$$

When Markovitz writes that blade length has an “exponential effect on power” she is misusing the word,

9.7 Exercises

Exercise 9.7.1. [S][R][Section **??**] [Goal **??**][Goal **??**] Compound interest computations.

If you invest \$1500 at 7% interest compounded every year, how much will you have at the end of 10 years? 15 years? 20 years? Use the formula for exponential growth; then check your answers with the `exponentialGrowth.xlsx` spreadsheet.

The Google calculator tells me that

$$\boxed{1\,500 * (1.07^{\wedge}10) = 2\,950.72704},$$

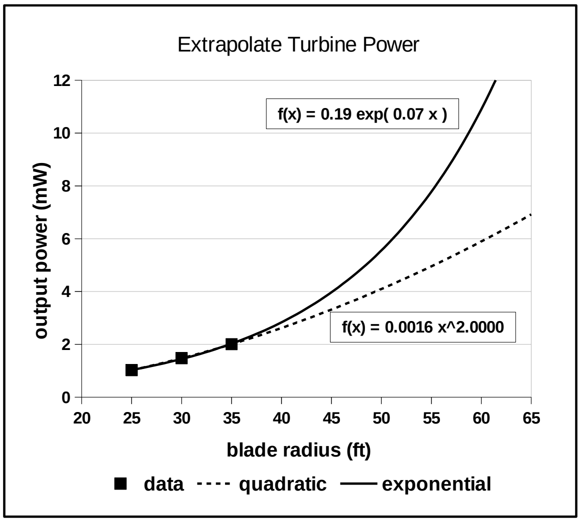


Figure 9.11. Turbine power

which is about \$2950.

Similarly,

$$\$1500 \times (1.07)^{15} = \$4138.54731 \approx \$4140,$$
$$\$1500 \times (1.07)^{20} = \$5804.52669 \approx \$5800.$$

I've rounded the results to the nearest \$10. The spreadsheet gives the same answers.

Exercise 9.7.2. [S][Goal **??**] [Section **??**][Goal **??**] When do you expect exponential growth?

In each of the following situations, explain why you would expect linear or exponential growth.

Think about whether the change is best described as an absolute rate (like dollars per hour or gallons per mile) or a percentage (like percent per year or percent per washing).

Write the units for the kind of rate you decide on.

(a) Price increases from year to year due to inflation.

(b) How the amount of gas you use depends on how far you drive.

(c) The amount of money left on your public transportation debit card as the days go by and you commute to school or work.

(d) The amount of sales tax you pay, depending on how much you buy.

(e) The amount of dirt left in your kid's filthy jeans when you wash them over and over again.

(f) The population of the world as the years go by.

(g) Your credit card balance if you stop making payments. (We will study credit cards in the next chapter.)

(h) The height of the snow as it accumulates in a big storm.

(i) The number of people sick in the first weeks of the flu season.

(j) The number of subscribers to a hot new social network in its first days.

Think about your answers before you look at the hints.

[See the back of the book for a hint.]

(a) Inflation is usually reported as a percent increase.

(b) No hint needed.

(c) No hint needed.

(d) Read this one carefully to think about what depends on what. Don't just jump at the word "percent".

(e) The washing happens over and over again on the same day — that's how dirty they were.

(f) No hint needed.

(g) Interest on unpaid balances accumulates.

(h) What units would the weatherman use to report the rate at which snow was accumulating?

(i) Think about how the number of people exposed to germs depends on the number of people sick.

(j) (Electronic) word of mouth generates new subscribers from old ones.

(a) Price increases from year to year due to inflation.

This should be (approximately) exponential since annual inflation is measured as a percentage increase, not an absolute amount.

(b) How the amount of gas you use depends on how far you drive.

This is a linear relationship. The equation that computes gas used depending on miles driven has slope gallons/mile.

(c) The amount of money left on your public transportation debit card as the days go by and you commute to school or work.

This is linear, with a negative slope: the rate of decrease is the number of dollars per day you spend.

(d) The amount of sales tax you pay, depending on how much you buy.

This is linear, not exponential. The sales tax T depends on the amount S of stuff you buy using the formula

$$T = rS$$

where r is the sales tax rate. In Massachusetts in 2014, $r = 6.25\% = 0.0625$.

(e) The amount of dirt left in your kid's filthy jeans when you wash them over and over again.

If each washing gets rid of the same percentage of the remaining dirt then the amount of dirt left decreases exponentially.

(f) The population of the world as the years go by.

Population growth is (approximately) exponential. The difference between the birth rate and death rate determines the percentage change each year.

(g) Your credit card balance if you stop making payments.

This is exponential growth. The monthly growth rate is the annual rate divided by 12. In reality, the credit card company won't let you completely stop making payments. You must always pay some minimum amount. Your balance will slowly decrease over time.

(h) The height of the snow as it accumulates in a big storm.

This is linear. The height depends on the time; the slope is measured in (inches of snow) per hour.

(i) The number of people sick in the first weeks of the flu season.

This is exponential because each person with the 'flu spreads it to some number of healthy people he or she comes in contact with. That means the rate of spread depends on the number of people already sick, so the rate is increasing.

(j) The number of subscribers to a hot new social network in its first days.

This is like an infection. The number of new subscribers grows faster and faster since each new subscriber invites his or her friends. This can only be true when the social network is new. After a while the people being invited will already have joined.

Exercise 9.7.3. [S][Section **??**][Goal **??**] Is it really exponential?

In everyday usage the phrase "growing exponentially" is just a vibrant synonym for "growing rapidly". It's rare that it really means a constant relative change.

Find instances of "exponential" growth in the media where what's meant is just very rapid growth.

From *The New York Times* on May 31, 2019:

> Infields have been altering their positioning based on their opponents for decades, but the frequency and precision have risen exponentially in recent years. [R**??**]

Searching for "exponential" on *The Denver Post* website on November 19, 2014 found fifteen articles for the year to date — more than one a month. Here are a few snippets. Only the first is actually about growth.

> Reverb: It seems like with you [Greensky Bluegrass guitarist Dave Bruzza] playing two nights at the Ogden, you have just been exponentially growing over the last few years here in Denver.

> The two ballot measures [on gaming and on genetically modified organisms] show a vivid contrast: one sports competitive fundraising efforts, and the other displays an exponential gap in contributions from the two sides.
>
> ‘Any amount of additional education is valuable. It’s essential we get all students to the finish line and get their degrees, but along the way, students’ eyes are opened up,” Chavez said. “Every step, every day, every year further opens up their eyes, and they are also then sharing these experiences with younger siblings, friends and peers, so it’s an exponential positive benefit to the public.
>
> The bond is designed to alleviate exponential crowding in 27J schools, and it will be on the Nov. 4 ballot.

Here are the first four entries when I searched *The New York Times* web site for “exponential” on June 9, 2011:

> What’s Next? The Cognocene — Room for Debate
> But the human influence on Earth systems, while present at low levels for perhaps millennia, has really only gone exponential with the ...
> May 19, 2011 — Room For Debate
>
> Toyota Tries to Loosen Grip on U.S. Product Development
> `wheels.blogs.nytimes.com/2011/06/01/chastised-toyota-tries-to-loosen-grip-on-product-development-for-u-s/`
> I’m in my 19th year at Toyota, and I’ve definitely seen an exponential growth in responsibility for North American operations. ...
> June 1, 2011 — Wheels
>
> Ricardo Sanchez, Retired Army General, Files to Run for Senate in...
> `thecaucus.blogs.nytimes.com/2011/05/11/general-in-abu-ghraib-scandal-files-to-run-for-senate-in-texas/`
> The Democratic Senatorial Campaign Committee has identified Texas, which has experienced exponential growth in Hispanic residents over the ...
> May 11, 2011 — The Caucus
>
> ... plan for making money from advertising, finally answering the question of how the company expects to turn its exponential growth into revenue.

The first of these really is describing the exponential growth or the world’s population since the Industrial Revolution.

The last three may be accurate assertions about the growth they claim, but there’s no evidence that the authors looked for data.

Here are more, from a student.

> September 21, 2011 — *New York Times Online*
> Title: At Facebook, Exponential Sharing

Author: Somini Sengupta
Quote: “Facebook’s latest features will allow for an exponential expansion of shared information.”

August 4, 2011 — *New York Times Online*
Title: Classical Music Moves From Concert Halls to Cafes
Author: Chloe Veltman
Quote: “The exponential growth of Classical Revolution and other innovative chamber music entities, like the San Francisco Friends of Chamber Music and Opera on Tap, point to a broader trend in classical music — with its traditionally lofty image and high ticket prices — of making it more approachable.”

October 23, 2011 — Letters to the Editor NYT Online
Title: The Road to Ruin
Author: Frances Hawxwell
Quote: “The editorial ’self-inflicted misery’ (Oct. 17) ridicules the British coalition government’s attempt at tackling the national debt through controlling our (exponential) rate of increase in government spending and borrowing.”

September 12, 2011 — *Boston Globe Online*
Title: Creativity in the Age of Psychiatric Medication
Author: Claudia Gold
Quote: “I discuss the problems associated with the exponential rise in prescribing of psychiatric medication for children.”

October 10th, 2011 — *Washington Post Online*
Title: Why University Presidents Refuse Reform
Author: Mark Taylor
Quote: “As the exponential increase in the cost of education translates into ever more limited resources, it is impossible to avoid conflicts of interest among these constituencies.”

November 13th, 2011 — *The Miami Herald Online*
Title: Climate change, beetle may doom rugged pine
Author: Craig Welch
Quote: “We know the incidence of blister rust infection and mountain pine beetle outbreaks is increasing exponentially [.]”

Exercise 9.7.4. [S][Section **??**][Goal **??**] Health care spending.

In Chapter **??**, Exercise **??** , we used data from the 2010 National Health Expenditures report to compute the absolute and relative changes in health care spending per person from 2007 to 2008.

(a) Use the results of those calculations to build linear and exponential models for the growth of health care spending per person.

(b) Use each model to predict when health care spending will reach \$10,000 per person per year.

(a) Use the results of those calculations to build linear and exponential models for the growth of health care spending per person.

Let H represent health care spending, in dollars per person, and y the number of years since 2006. Then the equations are

$$H = 7421 + 260y$$

and

$$H = 7421 \times 1.035^y.$$

(The slope for the linear equation is $7681 - 7421 = 260$. The relative change for the exponential equation is $7681/7421 = 1.035$.)

(b) Use each model to predict when health care spending will reach \$10,000 per person per year.

I put these numbers into the spreadsheet that compares linear and exponential growth and found that the linear model predicts that H will reach \$10,000 in ten years from 2006 — that is, in 2016. The exponential model predicts the same thing will happen about one year sooner, in 2015.

I could have found the same answers without even writing down the equations by asking Excel to make a scatterplot of the points $(0, 7421)$ and $(1, 7681)$, then constructing both linear and exponential trendlines and asking for a 10 year forecast.

The exponential trendline has equation

$$H = 7421\ e^{0.0344y}.$$

I checked with `=EXP(0.0344)` and found that

$$e^{0.0344} = 1.034998523 \approx 1.035,$$

which matches the 3.5% relative change I computed in part (a).

Exercise 9.7.5. [S][Section **??**][Goal **??**] Malthus.

In 1798 Thomas Malthus, an English economist and clergyman, wrote "An Essay on the Principle of Population". He said there:

> I think I may fairly make two postulata.
>
> First, That food is necessary to the existence of man.
>
> Secondly, That the passion between the sexes is necessary and will remain nearly in its present state.
>
> These two laws, ever since we have had any knowledge of mankind, appear to have been fixed laws of our nature, and, as we have not hitherto seen any alteration in them, we have no right to conclude that they will ever cease to be what they now are, without an immediate act of power in that Being who first arranged the system of the universe, and for the advantage of his creatures, still executes, according to fixed laws, all its various operations.
>
> ...
>
> Assuming then my postulata as granted, I say, that the power of population is indefinitely greater than the power in the earth to produce subsistence for man.
>
> Population, when unchecked, increases in a geometrical ratio. Subsistence increases only in an arithmetical ratio. A slight acquaintance with numbers will shew the immensity of the first power in comparison of the second.

> ...
>
> The power of population is so superior to the power in the earth to produce subsistence for man, that premature death must in some shape or other visit the human race. The vices of mankind are active and able ministers of depopulation. They are the precursors in the great army of destruction; and often finish the dreadful work themselves. But should they fail in this war of extermination, sickly seasons, epidemics, pestilence, and plague, advance in terrific array, and sweep off their thousands and ten thousands. Should success be still incomplete, gigantic inevitable famine stalks in the rear, and with one mighty blow levels the population with the food of the world. [R**??**]

Malthus claimed that the food supply grows in a linear fashion. As a unit of food supply he used the amount of food needed for one person for one year. He estimated food production in Britain in 1798 as 7,000,000 food units and that food production might increase by a constant 280,000 units each year.

Malthus also believed that the population of Britain was growing at a rate of 2.8% each year. In 1798, the population was about 7,000,000.

(a) Write a linear function that models food production.

(b) Write an exponential function that models population growth.

(c) Was there enough food for each individual in Britain in 1798?

(d) Using Malthus's models, determine whether there would be enough food for each individual in Britain in 1800.

(e) Malthus claimed that the population in Britain would eventually outstrip the food supply — a prediction we now call "the Malthusian dilemma." He didn't have Excel to do the arithmetic for him, but we do. Use it to estimate when Malthus's predicted disaster would occur. Was Malthus right?

(a) Write a linear function that models food production.

Let F stand for food production, measured in thousands of food units (so that we don't use too many zeroes) and T for the number of years since 1798. Then Malthus's model for food production is the equation

$$F = 280T + 7000.$$

It's a straight line with slope 280K units/year and intercept 7000K units.

(b) Write an exponential function that models population growth.

Let P stand for the population of Britain, in thousands of people, and T for the number of years since 1798. Then Malthus's model for population growth is the exponential growth equation

$$P = 7000 \times (1.028)^T.$$

(c) Was there enough food for each individual in Britain in 1798?

There was sufficient food in Britain in 1798 since Malthus arranged his models so that they started out in perfect agreement — the population of 700K people needed exactly the 700K food units produced that year.

(d) Using Malthus's models, determine whether there would be enough food for each individual in Britain in 1800.

I could do this problem with a calculator, plugging in the value $T = 2$ in each of the equations above. But reading ahead to the next problem I decided to use the LinearExponential.xlsx spreadsheet instead. I set

```
 7000   START: Starting value (Kunits)
  280   ABSCHANGE: Absolute increase (Kunits/year)
1.028   RELCHANGE: Relative increase ( /year)
```

Then the spreadsheet shows

```
year 2     7560 Kunits food    7397 Kpeople
```

so there's more than enough food for the population.

(e) Malthus claimed that the population in Britain would eventually outstrip the food supply — a prediction we now call "the Malthusian dilemma." He didn't have Excel to do the arithmetic for him, but we do. Use it to estimate when Malthus's predicted disaster would occur. Was Malthus right?

By adding a few lines to the table in the spreadsheet from 1 to 2 I could see that the exponential function caught up with the linear function at about 26 years. So Malthus predicted that in 1824 people in Britain would start to go hungry.

History says that didn't happen — famine hasn't hit Britain ever (there were short rations during and right after World War II, but that's not what Malthus had in mind).

Exercise 9.7.6. [S][Section **??**][Goal **??**] [Goal **??**] The pawn shop business model.

On April 9, 2011 *The New York Times* reported on a pawn shop that opened in an ex-Blockbuster store:

> The borrowers are given 60 days to pay back the loan, and La Familia charges a 20 percent interest rate per month. (So for a $100 loan, the borrower would need to pay back $140 after 60 days.) [R**??**]

(a) Explain why 20% interest per month on a $100 loan for two months would actually require repayment of a little more than $140.

(b) What is the annual interest rate when this business lends money?

(a) Explain why 20% interest per month on a $100 loan for two months would actually require repayment of a little more than $140.

After the first month the debt is $1.2 \times \$100 = \$120$. At the end of two months it's $1.2 \times \$120 = \$144$. You have to pay interest on the interest.

(b) What is the annual interest rate when this business lends money?

The compound interest formula (coming from the 1+ trick) says that to find the total owed after a year I should multiply the original loan amount by $(1.2)^{12} = 8.92$. That corresponds to an interest rate of 792% — almost 800%!

Exercise 9.7.7. [S][W][Section **??**][Goal **??**] [Goal **??**] Playing with exponential growth.

Open the spreadsheet `exponentialGrowth.xlsx` and describe what happens to the graph when you make each of the following experiments. If you can see easily what happens to the numbers, describe that too.

(a) Change the value of `START` from 1,000 to 10, then 100, then 10,000. Change it to some other random positive values that aren't as nice.

(b) Change the value of `START` from 1,000 to $-1,000$.

(c) Change the value of `RELCHANGE` to 1.

(d) Change the value of `RELCHANGE` to 1.01 (1% growth). Fit a linear trendline to the data. What is the R-squared value? What does it tell you?

(e) Change the value of `RELCHANGE` to 2. Why does the graph look flat at 0 as far as $T = 20$? Is it really flat?

(f) Change the value of `RELCHANGE` to 10.

(g) Change the value of `RELCHANGE` to 0.9 (a 10% decrease).

(h) (Optional) Can you figure out how we got the label on the graph to incorporate the values of `START` and `RELCHANGE`?

Here's one student's solution.

(a) When I change the `START` value to 10, then to 100 and then to 10,000 the graph keeps the exact same shape. The values also contain the exact same numbers. The decimal place just moves a place or two depending on the value of `START`.

(b) When I change the `START` value to $-1,000$ the graph flips upside down to represent negative numbers moving away from 0 exponentially.

(c) When I change `RELCHANGE` to 1 the graph becomes a flat line and all of the values remain 1,000 because there is no change.

(d) When I change `RELCHANGE` to 1.01 the linear trend line shows an R^2 value of 0.99842. This shows that a relative change of 1.01, while technically exponential growth, is very close to linear growth because a 1% growth will take many years to show a large growth year to year.

(e) When I change `RELCHANGE` to 2 the graph flat until $T = 20$ but it really isn't. The reason it looks flat is because exponential growth doubles and the doubling gets so large at that time that the numbers before it are too small on the graph to show a visible change.

(f) When I change `RELCHANGE` to 10 the graph is like the previous one but more dramatic, In other words, showing even larger increases over time.

(g) Setting `RELCHANGE` to 0.9 shows exponential decay. The interesting part is that unlike growth where the numbers start with relatively small increases and they get larger, with decay the decrease starts at a fixed large number decreases at a small rate.

Exercise 9.7.8. [S][Section **??**][Goal **??**] Five percent.

If you try to use `linearExponential.xlsx` to see when exponential growth at 5% catches linear you see that it's still behind at 20 years, which is as far as the table goes.

Modify the spreadsheet to determine when it catches up.

When I changed `RELCHANGE` from `1.08` to `1.05` the curves on the chart no longer crossed.

I selected cells `A34:C34`, copied, then pasted into the next 10 rows. Looking at the numbers, I could see that columns `B` and `C` were about equal at 27 years.

Unfortunately, the chart didn't update itself! There are several tricks to make that happen. One is to right click on one of the curves in the chart, choose `Select Data ...` from the menu, and edit the `Chart data range:`. Another is to `Insert` a whole bunch of rows between rows 33 and 34 and then copy row 33 to those empty rows.

Exercise 9.7.9. [U][Section **??**][Goal **??**] [Goal **??**] Playing with linear vs exponential growth.

Use the spreadsheet `linearExponential.xlsx` to answer these questions.

(a) How does changing the initial investment change the time it takes for the exponential function to catch up with the linear function?

(b) Does doubling or tripling both the initial investment and the absolute change affect the time it takes for the exponential function to catch up to the linear function?

Exercise 9.7.10. [S][Section **??**][Goal **??**] [Goal **??**] Deals you can't believe.

The data in this problem aren't real. But the problem is interesting and instructive, so it's worth spending time on.

Suppose you are shopping for a car and find three deals advertised:

- Make a $10,000 down payment and pay only $100 per month for two years.
- Just $5000 down, monthly payments start at a low $50 and increase by $50 each month for two years.
- Give me $1.00 today and take the car home! Pay 1 penny for the first month. Then double your payment each month. After two years, the car is yours.

(a) Before you do any calculating, which deal do you think is best? Why?

(b) What would your monthly payments be in the second and tenth months if you take the second dealer's offer?

(c) What would your monthly payments be in the second and tenth months if you take the third dealer's offer?

Month	Payment		
	Deal 1	Deal 2	Deal 3
(down) 0	10,000	5,000	1.00
1	100	50	0.01
2			
⋮			
24			
Total			

Table 9.12. Three car deals

(d) For each deal, write an algebraic expression that gives the monthly payment.

(e) Use Excel to calculate your total payments for the 24 months. Set up four columns as in Table **??**. Then tell Excel how to fill in the columns to 24 months. Finally, use the `SUM` function to add up the payments.

(f) Now use what your calculations tell you to compare the three deals. Which is best? Which is worst?

(a) Before you do any calculating, which deal do you think is best? Why?

Most people think that the last deal might be the best since the monthly payments start out so small.

(b) What would your monthly payments be in the second and tenth months if you take the second dealer's offer?

The payments would be \$100 in the second month and \$500 in the tenth.

(c) What would your monthly payments be in the second and tenth months if you take the third dealer's offer?

The payments would be \$0.02 in the second month and \$5.12 in the tenth.

(d) For each deal, write an algebraic expression that gives the monthly payment.

Writing M for the month and P for the payment, the three expressions are

$$P = 100$$

for the first deal,

$$P = 50M$$

for the second and

$$P = 0.01 \times 2^M$$

for the third.

(e) Use Excel to calculate your total payments for the 24 months.

The spreadsheet at `../Answers/UnbelievableDealSolution.xlsx` tells me that the three totals are \$12,400, \$20,000 and \$167,773.15 for the three deals.

(f) Now use what your calculations tell you to compare the three deals. Which is best? Which worst?

Clearly the first deal is the best and the third is the worst — by a lot!

Exercise 9.7.11. [S][C][Section **??**][Goal **??**] Green energy in China.

In the December 21 & 28 2009 issue of *The New Yorker* Evan Osnos wrote in his essay "Green Giant: Beijing's crash program for clean energy" that China's spending on R & D, now seventy billion dollars a year, has been growing at an annual rate of about twenty percent for two decades. [R**??**]

(a) What does "R&D" stand for?

(b) Use Excel to build a chart of annual Chinese R&D expenditures for the years 1989-2008.

(c) Add a data column showing the annual expenditures adjusted for inflation (use the United States cost of living index) and display that data on your chart.

"R&D" stands for "Research and Development".

See `../Answers/ChinaRandDSolution.xlsx`.

I did a little bit of extra work in that spreadsheet: I found an exponential trendline for the inflation adjusted expenditures. That trendline shows that the increase adjusted for inflation was about 17% per year, not 20%. That makes sense, since the average inflation rate was about 3% per year.

Exercise 9.7.12. [S][Section **??**] [Goal **??**] Car excise tax.

In Massachusetts you pay excise tax each year on the current value of your automobile. Assume for the sake of this problem that the rate is 3%, so you would pay $600 in excise tax in the first year you owned a new $20,000 car.

Use Excel to answer the following questions.

(a) Suppose the car depreciates linearly at a rate of $1,800 per year. What formula calculates the amount of excise tax you pay as a function of the age of the car?

(b) If you own the car for ten years, what will the car be worth then and how much total excise tax will you have paid?

(c) Answer the same questions if it depreciates at the rate of 13% per year.

(d) Find real data on the way a new car depreciates in value. Is an exponential model a good approximation?

(a) Suppose the car depreciates linearly at a rate of $1,800 per year. What formula calculates the amount of excise tax you pay as a function of the age of the car?

Writing E for excise tax and A for the age of the car in years, the formula is

$$E = 0.03 \times (20,000 - 1,800A).$$

(b) If you own the car for ten years, what will the car be worth then and how much total excise tax will you have paid?

After 10 years the car will be worth \$3,800 and I will have paid a total of \$3,570 in excise taxes.

You can see my work in the spreadsheet at `../Answers/ExciseTaxSolution.xlsx`.

(c) Answer the same questions if it depreciates at the rate of 13% per year.

The formula is

$$E = 0.03 \times (20{,}000(0.87)^A).$$

After 10 years the car will be worth \$5,711 and I will have paid a total of \$3,469 in excise taxes.

That's interesting. The car is worth more and my total tax bill is a little bit less. That's because the exponential depreciation cuts the value of the car rapidly at the start, then more slowly.

(d) Find real data on the way a new car depreciates in value. Is an exponential model a good approximation?

The car depreciation calculator at `www.money-zine.com/Calculators/Auto-Loan-Calculators/Car-Depreciation-Calculator/` says that with a low depreciation rate the first year depreciation a \$20,000 car would be \$3,600 and the total ten year depreciation would be \$14,254. That's pretty close to the exponential depreciation in this problem. If I took the trouble to look at the depreciation a year at a time I would probably see something like a big hit in the first year and then perhaps something between 10% and 12% a year after that.

Exercise 9.7.13. [S][Section **??**][Goal **??**] Iodine 131.

An article in *The New York Times* on April 6, 2011 soon after the Fukushima disaster discussed levels of radioactive iodine (iodine 131) in fish caught near Japan. The article noted that Japan recently revised the safety limit for iodine 131 in fish to 2,000 becquerels per kilogram. (A becquerel is a measure of radiation.)

Radioactive iodine has a half-life of about 8 days.

If a fish contained 10,000 becquerels of iodine 131 per kilogram, how long would it take for the iodine to decay to a "safe" level?

If a fish contained 10,000 becquerels of iodine 131 per kilogram, how long would it take for the iodine to decay to a "safe" level?

In 8 days the level would be 5,000 becquerels per kg. In 16 days it would be 2,500, so almost safe. In 24 days it would be only 1,250. So the answer is somewhere between 16 and 24 days — I estimate about 19 days. I could find the answer exactly with the formula and trial and error guesses but this estimate is good enough.

Exercise 9.7.14. [U][Section **??**][Goal **??**][Goal **??**] Quadrupling time.

(a) Explain why the quadrupling time in exponential growth is just twice the doubling time.

(b) Show that quadrupling time is given by a “Rule of 140” analogous to the rule of 70.

Exercise 9.7.15. [S][Section **??**][Goal **??**][Goal **??**][Goal **??**] Tripling time.

Suppose you invest $1000 at 10% interest compounded every year. (That’s a pretty good rate of return if you can get it — don’t trust a Madoff promise!)

(a) How long will it be until your balance is $3000?

[See the back of the book for a hint.] Use the `ExponentialGrowth.xlsx` spreadsheet, or find the answer by trying different values for T (# of years) in the formula until you find one that gets you close to $3000.)

(b) Find the tripling time for some other interest rates.

(c) Check that the tripling time in exponential growth is given (approximately) by a “Rule of 110.”

(d) Check that $e^{1.1} \approx 3$.

$$\$1000 \times (1.07)^{15} = \$4177.24817$$

so 15 years is too long. Try 10:

$$\$1000 \times (1.07)^{1}0 = \$2593.74246.$$

That’s too short a time. After 11 years:

$$\$1000 \times (1.07)^{1}1 = \$2853.11671.$$

Close. How about 12 years?

$$\$1000 \times (1.07)^{1}2 = \$3138.42838$$

so the investment will triple in between 11 and 12 years.

That’s consistent with a Rule of 110, which predicts a tripling time of $110/10 = 11$ years.

Exercise 9.7.16. [U][Section **??**][Goal **??**][Goal **??**] Compounding very frequently.

(a) Calculate the effective rate for 8% annual interest when it’s compounded weekly, daily, hourly and once every second.

(b) Estimate the effective rate if the interest is compounded every instant.

[See the back of the book for a hint.] Your computations should suggest that it won’t be infinite, which might have been your first guess.

(c) Redo the calculations starting with a 25% annual increase. (Not realistic for interest on a bank account!) Show that the Rule of 70 for doubling times is more accurate the more frequently you compound the interest.

Exercise 9.7.17. [S][C][W][Section **??**] [Goal **??**][Goal **??**] How fast does information double?

In the Preface to the Carnegie Corporation report *Writing to Read* Vartan Gregorian wrote that he's been told that the amount of available information doubles every two to three years. [R**??**]

(a) What growth rate in percent per year would lead to a doubling time of two to three years?

(b) Who is Vartan Gregorian?

(c) Can you verify his assertion?

[See the back of the book for a hint.] You can answer the first question using guess-and-check until you're close enough. You'll need a web search for the second and a very good web search for the third, which is open-ended.

(a) What growth rate in percent per year would lead to a doubling time of two to three years?

I experimented using the exponential growth spreadsheet from the text and found that a growth rate of about 41% per year doubles things in two years. (That's because $1.41 \times 1.41 \approx 2$: the square root of 2 is the exact answer, but not needed in this problem or this book.)

For three year doubling the spreadsheet experiment gives a growth rate of about 26% per year. (1.26 is close to the cube root of 2.)

So doubling every two to three years corresponds to an annual increase of between 26 and 41 percent.

(b) Who is Vartan Gregorian?

Wikipedia tells me that Vartan Gregorian is the president of Carnegie Corporation of New York (in 2011).

(c) Can you verify his assertion?

From `solace.com/blog/handling-data-growth-as-digital-information-doubles-every-18-24-months/`

> Various studies have claimed that the amount of digital information in the world doubles every 18 to 24 months.

. . . but where are the actual studies?

Then there's this:

> It is projected that just four years from [2010], the world's information base will be doubling in size every 11 hours. [R**??**]

So the actual doubling time is hard to pin down even approximately.

Exercise 9.7.18. [S][Section **??**] [Goal **??**][Goal **??**] Bacteria doubling time.

Find the approximate doubling times for strains R and S in the bacteria growth example in Section **??**.

I tried to make an estimate just by looking at the tables. The best I could do was to say that for both the doubling time seemed to be about 3 hours.

To get better estimates I looked at the exponential trendlines. For strain R the exponential factor is

$$e^{0.2732} = 1.314163051,$$

which corresponds to a 31% increase per hour. Then I used the Rule of 70 to get the doubling time:

$$\frac{70}{31.4} \approx 2.23$$

so the doubling time is about two and a quarter hours, so two hours and 15 minutes.

For strain S

$$e^{0.2558} = 1.291494403$$

which corresponds to a 29% increase per hour. Then the Rule of 70 says

$$\frac{70}{29.1} \approx 2.40$$

so the doubling time is about two hours and 25 minutes — just ten minutes more than for strain R.

Exercise 9.7.19. [S][Section **??**][Goal **??**] [Goal **??**] When will R catch S?

The population of strain S outnumbers that of strain R for the entire first 17 hours of the experiment discussed in Section **??**. But the exponential trendline equation shows that strain R is growing faster. If the exponential growth were to continue (which it didn't) when would strain W catch up?

[See the back of the book for a hint.] Guess a number of hours, try your guess in the two exponential equations, then adjust your guess up or down until the answers match.

I built a small spreadsheet with these formulas in columns `A:D`

Time	S	R	R/S
64	=6.5913*EXP(0.2558*A4)	=2.13*EXP(0.2732*A4)	=C4/B4
=A4+1	=6.5913*EXP(0.2558*A5)	=2.13*EXP(0.2732*A5)	=C5/B5
=A5+1	=6.5913*EXP(0.2558*A6)	=2.13*EXP(0.2732*A6)	=C6/B6

I experimented with the value in cell `A4` to find out where the ratio R/S switched from less than 1 to greater.

Here are the numbers.

Time	S	R	R/S
64	84897108.9	83547173.28	0.984099157
65	109644141	109794608.1	1.001372323
66	141604794.4	144288017.2	1.018948672

They show that strain R would catch up at about 65 hours.

Exercise 9.7.20. [U][Section **??**][Goal **??**] The magic number *e*.

(a) Find the value of *e* in Excel using the formula `=EXP(1)`.

(b) Find the value from Google with the same formula (without the equal sign). Check that the answers agree as far as they go together.

(c) Which provides more digits?

(d) Can you get more precision from Excel by formatting the cell in which the number appears?

(e) Find even more digits with an internet search.

Exercise 9.7.21. [S][Section **??**][Goal **??**] Educating mothers saves lives, study says.

On September 17, 2010 *The Boston Globe* carried an Associated Press report on a study that found that the death rate for children under 5 dropped by nearly 10 percent for every extra year of their mother's education. That education saved 4.2 million children in the developing countries in 2009.

The story continued with these baseline numbes:

> In 1970, women aged 18 to 44 in developing countries went to school for about two years. That rose to seven years in 2009. [R**??**]

(a) How much did the death rate for children under 5 decline from 1970 to 2009?

(b) Build as much as you can of the exponential model implicit in this quotation. What are the independent and dependent variables? What is the annual relative change?

(a) How much did the death rate for children under 5 decline from 1970 to 2009?

Since the number of years of schooling increased by $7-2=5$ years the death rate decreased by 10% five times. Since $0.9^5 = 0.59049$ the death rate in 2009 was only 60% of what it was in 1970.

(b) Build as much as you can of the exponential model implicit in this quotation. What are the independent and dependent variables? What is the annual relative change?

The independent variable is y, the number of years of education for women. The dependent variable is D, the death rate for children under five. The annual change is a decrease of 10%, which means the equation looks like

$$D = S \times 0.9^y.$$

S is the death rate when women don't have any schooling at all. There isn't enough information to find its value.

Exercise 9.7.22. [S][Section **??**][Goal **??**] [Goal **??**] Email.

On May 30, 20ll Virginia Heffernan blogged in *The New York Times* that the number of email accounts grew from about 15 million in the early 1990s to 569 million by December 1999, and that today [when she was writing] there are [were] more than 3 billion. [R**??**]

(a) Is this exponential growth?

(b) Can you use these numbers to make predictions?

[See the back of the book for a hint.] Compare the relative change in the number of email accounts for the two time periods (early 1900s to end of 1999, end of 1999 to the time of the blog post).

(a) Is this exponential growth?

I'll try 1990 for "the early 1990s". Then the data tell me that in those 10 years the number of email accounts grew by a factor of

$$\frac{569 \text{ million}}{15 \text{ million}} \approx 38.$$

(The relative change is what matters here.)

In the eleven and a half years from the end of 1999 to May of 2011 the number grew by a factor of only

$$\frac{3 \text{ billion}}{569 \text{ million}} \approx 5.3.$$

If the growth were truly exponential throughout the time then the figure for 2011 would have to be more than 38 times as large as the figure for the end of 1999.

(b) Can you use these numbers to make predictions?

No. The number of email accounts grew very rapidly at the beginning and more slowly after that. I don't have enough data to make predictions.

Exercise 9.7.23. [S][Section **??**][Goal **??**] When will India pass China?

In an article dated April 1, 2011 on the website About.com you could read that India, the world's second largest country, had a population of 1.21 billion. India was expected to pass China by 2030, when it would have more than 1.53 billion people. China's population then would be just 1.46 billion.

The article noted that India's growth rate of 1.6% per year doubles its population in less than 44 years. [R**??**]

(a) Is the article correct in stating that an annual growth rate of 1.6% means India's population will double in 44 years?

(b) Assuming that India's growth rate remains 1.6% annually, what will its population be in 2030 when it surpasses China's?

(c) Assuming that India's growth rate remains 1.6% annually from 2011 on, what will its population be in the year 2100? Compare that figure to the current population of the world. Do you think India's growth rate can in fact continue at 1.6% for the 89 years from 2011 to 2100?

(a) Is the article correct in stating that an annual growth rate of 1.6% means India's population will double in 44 years?

The rule of 70 says that a growth rate of 1.6% per year has a doubling time of about $70/1.6 = 43.75$ or about 44 years.

(b) Assuming that India's growth rate remains 1.6% annually, what will its population be in 2030 when it surpasses China's?

2030 is 19 years from 2011. Then India's population will be $1.21 \times 1.016^{19} = 1.64$ billion.

That's a lot larger than the predicted peak for China in that year, so I suspect the prediction is based on an assumption that India's growth rate will slow down, although not enough to keep it from passing China.

(c) Assuming that India's growth rate remains 1.6% annually from 2011 on, what will its population be in the year 2100? Compare that figure to the current population of the world. Do you think India's growth rate can in fact continue at 1.6% for the 89 years from 2011 to 2100?

At the 2011 growth rate, India's population in 89 years would be $1.21 \times 1.016^{89} = 4.97$ billion, or just about 5 billion. The 2011 world population is 7 billion. I don't believe that India could hold nearly as many people then as the whole world held in 2011, so I don't think the growth rate can be 1.6% per year for the rest of the century.

Here's another interesting way to do this problem. The doubling time is about 44 years. Twice that is 88 years, so almost to the end of the century. At the 2011 rate of increase, by then the population will have quadrupled, to about 4.8 billion.

Exercise 9.7.24. [S][Section **??**] [Goal **??**] Health care expenditures grow.

The National Health Expenditures report, released in January 2009, stated that overall health care spending in the United States rose from $7062 per person in 2006 to $7421 per person in 2007.

(a) Calculate both the absolute change and percentage change in health care spending per person from 2006 to 2007.

(b) Using 2006 as your starting year (2006 = year 0), determine an exponential equation that calculates the amount of health care spending over time assuming the annual percentage change stays the same. Clearly identify the variable names and symbols in your equation.

(c) Using 2006 as your starting year (2006 = year 0), determine a linear equation that calculates the amount of health care spending over time assuming the annual absolute change stays the same. Clearly identify the variable names and symbols in your equation.

(d) Create an Excel spreadsheet to compare the two growth models' predictions for health care spending through the year 2021. Include a chart showing both models.

(e) Which model first predicts that U.S. health care spending will reach a level of \$10,000 per person? In what year will that occur?

(a) Calculate both the absolute change and percentage change in health care spending per person from 2006 to 2007.

The absolute change in health care spending was

$$\$7421 - \$7062 = \$359.$$

To find the percentage change I computed

$$\frac{\$7421}{\$7062} = 1.05083546,$$

which corresponds to a percentage change of about 5.1%.

(b) Using 2006 as your starting year (2006 = year 0), determine an exponential equation that calculates the amount of health care spending over time assuming the annual percentage change stays the same. Clearly identify the variable names and symbols in your equation.

Let H represent health care spending, in dollars per person, and y the number of years since 2006. Then the equation is

$$H = 7062 \times 1.0508^y.$$

(c) Using 2006 as your starting year (2006 = year 0), determine a linear equation that calculates the amount of health care spending over time assuming the annual absolute change stays the same. Clearly identify the variable names and symbols in your equation.

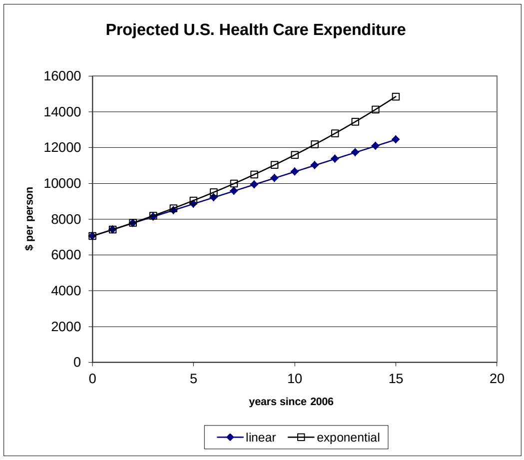
Using H and y as above, the equation is

$$H = 7062 + 359y.$$

The slope is the constant absolute change, in dollars per year.

(d) Create an Excel spreadsheet to compare the two growth models' predictions for health care spending through the year 2021. Include a chart showing both models.

I modified the spreadsheet from the book that compares linear and exponential growth to create `../Answers/HealthCareSolution.xlsx` where I drew this chart:



(e) Which model first predicts that U.S. health care spending will reach a level of $10,000 per person? In what year will that occur?

The exponential growth model predicts higher values for all years after 2007, so it predicts $10,000 sooner — in 2014. The linear model predicts $10,000 a year later.

Exercise 9.7.25. [S][Section **??**][Goal **??**][Goal **??**] Joe Seeley, *in memoriam.*

Joe Seeley died at age 50 in the fall of 2012.

He was a brave and witty blogger at `joes-blasts.blogspot.com/` throughout his hospitalization, creating virtual lemonade from the sourest of lemons. I think his words helped him; I know they helped those who cared for him to cheer him on. They will help the hospital staff care better for patients who come after him. And they will help you learn a little mathematics.

Figure **??** appeared in the blog at a hopeful moment in his odyssey. It shows Joe's white blood cell counts on days following a stem cell transplant. He chose white for the bars, to symbolize white blood cells, and red for the background, for blood in general. I wrote him about it:

> March 18, 2011 6:58 AM
> Ethan Bolker said . . .
> Exponential growth is good! . . . Will you still have a daily double after the predicted short dip? May I use your data for my quantitative reasoning class at UMass Boston?
> March 18, 2011 9:48 AM
> Joseph Seeley said . . .
> I will not see doubling again, unless something is wrong. Over the next few months, the counts will rise and fall, sometimes for no reason that the doctors can determine.
> I hereby authorize the use of my blood count data for any and all educational purposes.

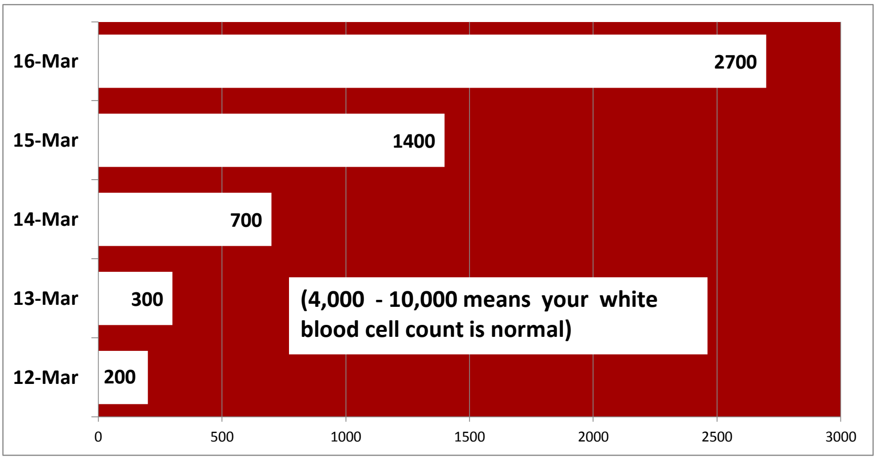


Figure 9.13. Proliferating white blood cells

(a) Enter the data in Excel. Reproduce Joe's chart. Match the formatting (labels, colors, sizes, fonts) as well as you can.

(b) Create an exponential trendline for the data.

(c) Use common sense or your trendline to predict when Joe's white blood cell count will be in the normal range.

(d) His white blood count on March 17 was 5100. Does that match your prediction?

(e) Modify your chart to include this new information. Mark it with a suitable exclamation!

See `../Answers/JoeSeeleySolution.xlsx`. I answered all the questions in the spreadsheet.

Exercise 9.7.26. [S][Section **??**][Goal **??**] How impatient are you?

The Boston Globe reported on September 24, 2012 that:

> [MIT grad Andy Berkheimer] found that [YouTube] viewers start closing out if there's even a two-second delay. Every one-second delay after that results in a 5.8 percent increase in the number of people who give up. A 40-second wait costs a video nearly a third of its audience. [R**??**]

Show that at this rate more than ninety percent of the viewers would give up after 40 seconds — not the "nearly a third" in the quote.

Since

$$0.942^{40} = 0.0916296677 < 10\%,$$

fewer than 10% of the viewers are still around after 40 seconds. More than 90% have given up.

Exercise 9.7.27. [U][Section **??**][Goal **??**] Does compounding always matter?

On page 32 of his excellent and highly recommended *The Signal and the Noise* Nate Silver wrote

> [Over the] 100-year-period from 1896 through 1996 ... sale prices of houses had increased by just 6 percent *total* after inflation, or about 0.06 percent annually. [R**??**]

Silver clearly divided six percent by 100 to reach his 0.06 percent annually conclusion. But that's not how percent increases work. The 0.06 percent annual increase must be compounded.

Would taking compounding into account change Silver's fundamental point?

Would taking compounding into account change Silver's fundamental point?

To calculate the result of compounding 0.06 percent per year for 100 years I found

$$1.0006^{1}00 = 1.0618174413.$$

That rounds to 6 percent. The extra two tenths of a percent from compounding doesn't really matter in the argument Silver is making.

Exercise 9.7.28. [S][Section **??**][Goal **??**] Cuba, you owe us \$7 billion.

On April 18, 2014 Leon Neyfakh wrote in *The Boston Globe* that property confiscated by the Cuban government in the 1959 revolution was

> ...originally valued at \$1.8 billion, which at 6 percent simple interest translates to nearly \$7 billion today. [R**??**]

(a) Is the simple interest calculation in the quotation correct?

(b) What would the value be today at 6 percent compound interest?

(c) What would the value be today simply taking inflation into account?

(d) Discuss which of the three valuations makes the most sense.

(a) Is the simple interest calculation in the quotation correct?

At 6 percent simple interest the value grows linearly at a rate (slope) of $0.06 \times 1.8 = 0.108$ billion dollars per year. In the 55 years from 1959 to 2014 it would grow to

$$\$1.8\text{b} + 0.108 \frac{\$\text{b}}{\text{year}} \times 55 \text{ years} = \$7.74\text{b}$$

so the quotation seems to underestimate the 2014 value. It's almost 8 billion dollars.

(b) What would the value be today at 6 percent compound interest?

Compounding at a rate of 6 percent per year the value after 55 years would be

$$\$1.8\text{b} \times 1.06^{55} = \$44.4\text{b}.$$

(c) What would the value be today simply taking inflation into account?

The inflation calculator tells me that $1.8 billion in 1959 would be worth $14.2 billion in 2014.

(d) Discuss which of the three valuations makes the most sense.

Simple interest isn't a realistic way to think about how the value of an investment changes over time. The "six percent" in the compound interest calculation is a made up growth factor. I think the most sensible way to think about what that 1.8 billion dollars would be worth in 2014 is simply to adjust it for inflation.

I could make a case for something larger than the inflation adjusted figure.

Exercise 9.7.29. [S][Section **??**][Goal **??**] "As Time Goes By".

On December 13, 2012 you could read in *The New York Times* that the piano from Rick's place in the 1942 movie *Casablanca* is up for auction.

> Sotheby's expects [it] to sell from $800,000 to $1.2 million in the auction on Friday. That is between 34 to 48 times what [Ingrid] Bergman was paid for sharing top billing with Humphrey Bogart. [R**??**]

(a) How much was Ingrid Bergman paid for her role in the film? Calculate this two ways using the data in the quotation and comment on what you discover.

(b) Would adjusting her pay to take inflation into account allow her to bid on the piano in 2012?

(c) What compound interest rate would she have to have earned on her pay to bid on the piano in 2012?

(d) Find out what happened at the auction.

(a) How much was Ingrid Bergman paid for her role in the film? Calculate this two ways using the data in the quotation and comment on what you discover.

Using the high end estimate, Ingrid Bergman was paid $\$1,200,000/48 = \$25,000$ for her role as Ilsa in *Casablanca*.

Just to check, I get the same answer for the low end of the range — and found $\$800,000/34 = \$23,529$. That's a weird number, and doesn't match the nice looking $25K from the high estimate. I think that one is right. So I calculated $\$800,000/\$25,000 = 32$!

I think the *Times* made a typographical error. The "34" should have been "32".

(b) Would adjusting her pay to take inflation into account allow her to bid on the piano in 2012?

The inflation calculator tells me that $25K in 1942 would be $352K in 2012 — not nearly enough to buy the piano.

(c) What compound interest rate would she have to have earned on her pay to bid on the piano in 2012?

There were $2012 - 1942 = 70$ years between the film and the auction. For the low end, I need to find the value of r that makes

$$25,00 \times (1+r)^{70} = 800,000,$$

or, equivalently,

$$(1+r)^{70} = \frac{800,000}{25,000} = 32.$$

I experimented in Excel. With $r = 0.05$ I get 30.43. With $r = 0.06$ I get 59 — way too big. $r = 0.051$ gives me 32.5. Close enough.

(d) Find out what happened at the auction.

On December 14, 2012 *The New York Times* reported that

> "Casablanca" Piano Is Sold for $602,500 at Auction

On November 24, 2014 *The New York Times* reported on another Casablanca piano auction, that referred to this one too:

> The other piano, the one in the flashback scene, sold for $602,500 at Sotheby's in December 2012. (Dr. Milan once owned that piano, too, but sold it in 1988 for $154,000. [R**??**]

So now we can compute Dr. Milan's gain ... and take inflation into account.

Review exercises.

Exercise 9.7.30. [A] You invest $500 in an account that earns $10 in interest each year.

(a) At the end of 24 months, what is the balance?

(b) At the end of 30 months, what is the balance?

(c) At the end of 5 years, what is the balance?

(d) Find the linear equation that gives the balance after t years.

Exercise 9.7.31. [A] You buy a car for $15,000 and for tax purposes you depreciate it at a rate of 11% per year.

(a) At the end of 24 months, what is the value of the car?

(b) At the end of 5 years, what is the value of the car?

(c) Find the exponential equation that gives the value of the car after t years.

(d) Does the value of the car ever reach $0?

Exercise 9.7.32. [A] Calculate the percentage.

(a) What is 8% of \$2000?

(b) What is 108% of \$2000?

(c) What is 3.25% of \$800?

(d) What is 103.25% of \$800?

Exercise 9.7.33. [A] Use a calculator to evaluate these expressions using exponents. (You may find typing into the Google or Bing calculator much faster than using one that requires you to press keys, either with your fingers or with a mouse.)

(a) 1.03^4

(b) 0.89^5

(c) 140×1.03^4

(d) 80×0.89^5

(e) $\frac{1}{3^8}$

(f) $(\frac{1}{3})^8$

(g) 1.25^0

(h) 1.25^1

(i) e^2

(j) e^{15}

Exercise 9.7.34. [A] In the exponential functions below, identify the relative change and the initial amount.

(a) $P = 100 \times (1.05)^T$.

(b) $y = 400 \times (0.88)^x$.

(c) $S = 550 \times (1.22)^Q$.

(d) $P = 96 \times (0.50)^T$.

Exercise 9.7.35. [A] Excel gives the following best-fit exponential function for a set of data: $y = 2.099 \times e^{1.344x}$. Find the constant growth rate and rewrite the function without using e.

Exercises added for the second edition.

Exercise 9.7.36. [U][S][Section **??**][Goal **??**][Goal **??**] There's more next year.

Lewis Carroll's Professor talks with Sylvie and Bruno:

> "Come in!" [said the Professor]
>
> "Only the tailor, Sir, with your little bill," said a meek voice outside the door.
>
> "Ah, well, I can soon settle his business," the Professor said to the children, "if you'll just wait a minute. How much is it, this year, my man?" The tailor had come in while he was speaking.
>
> "Well, it's been a doubling so many years, you see," the tailor replied, a little gruffly, "and I think I'd like the money now. It's two thousand pound, it is!"
>
> "Oh, that's nothing!" the Professor carelessly remarked, feeling in his pocket, as if he always carried at least that amount about with him. "But wouldn't you like to wait just another year, and make it four thousand? Just think how rich you'd be! Why, you might be a King, if you liked!"
>
> "I don't know as I'd care about being a King," the man said thoughtfully. "But it dew sound a powerful sight o' money! Well, I think I'll wait —"
>
> "Of course you will!" said the Professor. "There's good sense in you, I see. Good-day to you, my man!"
>
> "Will you ever have to pay him that four thousand pounds?" Sylvie asked as the door closed on the departing creditor.
>
> "Never, my child!" the Professor replied emphatically. "He'll go on doubling it, till he dies. You see it's always worth while waiting another year, to get twice as much money!" [R**??**]

(a) Use the fact that $2^{10} \approx 1000$ to estimate the original tailor's bill if it's been doubling for five years.

(b) Check your estimate with a precise calculation.

(a) Use the fact that $2^{10} \approx 1000$ to estimate the original tailor's bill if it's been doubling for five years.

$$2000 = 2\ times 1000 \approx 2^{11} = 2^5 2^6$$

so in five years an original bill of $2^6 = 64$ pounds will grow to about 2000 pounds.

(b) Check your estimate with a precise calculation.

If B is the original bill then

$$2^5 B = 2000$$

so

$$B = \frac{2000}{2^5} = 62.5 \text{ pounds} .$$

Exercise 9.7.37. [U][S] Light pollution.

On August 18, 2019 Kelsey Johnson wrote in *The New York Times* that

> [T]he global amount of artificial light at night has been growing by at least 2 percent per year. At this rate the amount of light pollution originating from Earth-based sources alone will double in less than 50 years. [R??]

In a more technical discussion linked from that article you can read Jeff Hecht's observation that

> A new analysis of satellite data from the past four years shows that the total acreage lit by artificial light at night increased by an average of 2.2 percent a year. The brightness of the areas lit at the start of the study also increased by the same rate — 2.2 percent annually — around the globe. [R??]

(a) Use the Rule of 70 to find a better estimate of the light pollution doubling time.

(b) How do you think Johnson arrived at the 50 year doubling time?

(c) Use the data in the quote from Hecht to explain why the annual light pollution increase might be better reported as almost four and a half percent.

(a) Use the Rule of 70 to find a better estimate of the light pollution doubling time.

Thirty or 32 years would be a better estimate since $70/2.2 \approx 31.8$.

(b) How do you think Johnson arrived at the 50 year doubling time?

By thinking that 2 percent per year would lead to 100 percent in 50 years.

(c) Use the data in the quote from Hecht to explain why the annual light pollution increase might be better reported as almost four and a half percent.

Since both lit area and the light pollution from lit area are increasing by 2.2 percent annually the total annual light pollution increases by a factor of

$$1.022^2 = 1.044484,$$

which corresponds to almost 4.5 percent.

Exercise 9.7.38. [S][U] Thirty four years at seven percent.

From *Swan Boats at Four*, a novel by George V. Higgins:

> Rutledge said "In other words, if we'd painted over that damned picture in the summer of nineteen seventy-eight, we would've made the club, and ourselves individually, liable for a hundred thousand bucks, plus interest at, say, an average of seven percent per annum, compounded for thirty-four years. ..."
>
> "Offhand," [David] said, "I can't even imagine how much that would've been."
>
> "At the time, I couldn't either," Rutledge said, " ...so we looked it up — I don't mean we figured it out. ...I don't recall the exact figure, but it came out to around a million and a half dollars." [R**??**]

David is a banker. He would know the Rule of 70 and figure it out offhand, without pencil and paper. Higgins should have known that.

(a) Use the Rule of 70 to decide whether Rutledge was right when he said the figure was "around a million and a half dollars".

(b) Calculate the liability accurately.

(a) Use the Rule of 70 to decide whether Rutledge was right when he said the figure was "around a million and a half dollars".

The rule tells me that seven percent interest doubles a debt in ten years. In twenty years it will quadruple, in thirty it will be eight times as large. So the \$100,000 would be \$800,000 after 30 years. It would be \$1,600,000 after 40 years. That's barely over a million and a half, so I don't think it would be a million and a half after just 34 years.

(b) Calculate the liability accurately.

$$100,000 \times 1.07^{34} = 997811.353702$$

so the \$100,000 debt would grow to just about a million dollars in 34 years.

Exercise 9.7.39. [U][S] Disrupting the cow.

A November 1919 article in *The Boston Globe* reported on the falling cost of precision fermentation (PF), a process for growing meat in a laboratory:

> Due to rapid improvements in underlying biological and information technologies, the cost of PF is falling exponentially — from \$1

> million per kilogram in 2000 to about \$100 today. With the technologies we have today, we project these costs will fall even lower — to \$10 per kilogram by 2023-25. [R**??**]

(a) Use the data from the Globe to verify the statement

The cost of PF falls by a factor of about $1/100$ per decade.

(b) Rewrite the statement in (a) using an assertion about percentage change.

(c) Explain why the cost of PF is falling by a factor of about $1/10$ every five years, to 10\$ of what it starts at.

(d) Is the Globe article right when it projects \$10 per kilogram by 2023-25?

(a) Use the data from the Globe to verify the statement

The cost of PF falls by a factor of about $1/100$ per decade.

From 2000 to 2019 is about two decades. If the cost falls by a factor of $1/100$ in one decade it will fall by

$$\frac{1}{100} \times \frac{1}{100} = \frac{1}{10000}$$

in two decades. Since a million is six zeroes,

$$\frac{1}{10000} \times \frac{\$1\text{ million}}{\text{kg}} = \frac{\$100}{\text{kg}}.$$

(b) Rewrite the statement in (a) using an assertion about percentage change.

In a decade the cost of PF falls by about 99% to 1% of what it started at.

(c) Explain why the cost of PF is falling by a factor of about $1/10$ every five years, to 10\$ of what it starts at.

That's because two five year periods is a decade and

$$.\frac{1}{10} \times \frac{1}{10} = \frac{1}{100}.$$

(d) Is the Globe article right when it projects \$10 per kilogram by 2023-25?

Yes. 2023-2025 is about five years from 2019, so the cost should drop to 10% of \$100/kg.

Exercise 9.7.40. [S] So many words!

On December 1, 2012 R. Alexander Bentley and Michael J. O'Brien wrote in *The New York Times* that

> [F]or the last 300 years, the number of words published annually grew exponentially by about 3 percent per year. From about 20 million words for 1700, the annual word count grew to several trillion for 2000. [R**??**]

(a) Check the authors' arithmetic.

(b) If the growth continues at the same rate how many words will be published in the year 3000?

(c) How much confidence do you have in your prediction?

(a) Check the authors' arithmetic.

The coolest way to do this is with the rule of 70. A 3% annual increase has a doubling time of $70/3 \approx 25$ years. The question asks about 300 years of growth, which would be 12 doublings.

But to get from 20 million to 20 trillion you must multiply by 10^6. Since $2^{10} \approx 1,000$, that's 20 doublings. It would take 19 doublings to get to 10 trillion, and 16 to get to 1 trillion. So 12 is not enough.

You can, of course, solve this problem the boring way with some arithmetic:

$$(20 \text{ million}) \times (1.03)^{300} \approx 140,000 \text{ million} = 140 \text{ billion}.$$

That's at least an order of magnitude short of "several trillion".

To get to 2 trillion you'd need 19 doublings in 300 years. That's a doubling time of about 16 years. Then the rule of 70 says you'd have to have had an annual growth rate of about 4.3%.

(b) If the growth continues at the same rate how many words will be published in the year 3000?

Another 100 years at 4.3% would be about 6 more doublings, so each trillion words would grow to $2^6 = 64$ trillion.

(c) How much confidence do you have in your prediction?

Not much!

Exercise 9.7.41. [S] World population.

According to a Harvard School of Public Health press release the world's population has grown slowly for most of human history. It took until 1800 for the population to hit 1 billion. However, in the past half-century, population jumped from 3 to 7 billion. In 2011, approximately 135 million people will be born and 57 million will die, a net increase of 78 million people.

(a) By what percent did world population increase in 2011?

(b) Write the equation for a mathematical model for world population growth for years since 2011 if the annual net increase seen in 2011 remains constant for the remainder of the century. (Use 7 billion as the 2011 population.)

(c) Write the equation for a mathematical model for world population growth for years since 2011 if the annual percentage increase seen in 2011 remains constant for the remainder of the century. (Use 7 billion as the 2011 population.)

(d) Construct an Excel spreadsheet predicting world population for the years through 2100 using both models. Graph both predictions on the same chart.

(e) The table below gives the United Nations high estimate for world population growth during the remainder of this century.

Year	Population (billions)
2011	7
2025	8.5
2050	10.6
2100	15.8

Which of your models most closely matches the UN high estimate?

(a) By what percent did world population increase in 2011?

The percentage increase was

$$\frac{78\times 10^6}{7\times 10^9} = 0.01114285714 \approx 1.1\%.$$

(b) Write the equation for a mathematical model for world population growth for years since 2011 if the annual net increase seen in 2011 remains constant for the remainder of the century. (Use 7 billion as the 2011 population.)

Let Y be the number of years since 2011 and P the world population, in billions. Then the equation is

$$P = 7 + 0.078Y.$$

(c) Write the equation for a mathematical model for world population growth for years since 2011 if the annual percentage increase seen in 2011 remains constant for the remainder of the century. (Use 7 billion as the 2011 population.)

With the same variables as above, the exponential equation is

$$P = 7\times(1.011)^Y.$$

(d) Construct an Excel spreadsheet predicting world population for the years through 2100 using both models. Graph both predictions on the same chart.

(e) The table below gives the United Nations' high estimate for world population growth during the remainder of this century.

Which of your models most closely matches the UN high estimate?

I've added the predictions to the table. Numbers are in billions. The last two columns show the relative errors in the predictions.

Year	UN	linear	exp	lin/UN	exp/UN
2011	7.0	7.00	7.00	1.00	1.00
2025	8.5	8.19	8.17	0.96	0.99
2050	10.6	10.32	10.78	0.97	1.02
2100	15.8	14.57	18.77	0.92	1.19

The linear and exponential predictions are both pretty close for the first half of the 21st century. By 2100 the linear prediction is 8% lower than the UN's, the exponential prediction 19% higher.

Exercise 9.7.42. [N] Making it into the Hall of Fame.

On January 12, 2013, Nate Silver wrote in his blog at *The New York Times* that individual voting totals for the baseball Hall of Fame seemed to increase by about ten percent each year.

> Thus, a player who received 10 percent of the vote in his first year would be expected to receive about 11 percent on his second try, while a player who got 50 percent of the vote would go up to 55 percent. [R**??**]

Explain why this is exponential growth. Look up the original. Make some projections.

Exercise 9.7.43. [U][N] Save another one percent.

The interactive web based calculator provided by *The New York Times* at `www.nytimes.com/interactive/2010/03/24/your-money/one-pct-more-calculator.html` suggests many questions about savings, compound interest and inflation.

Exercise 9.7.44. [U][N] The London olympics.

On July 22, 2012 *The Boston Globe* wrote about the third Olympic games to be hosted by London:

> The Games have grown geometrically during the past 104 years — from 2,023 athletes representing 22 countries competing in 109 events in 24 sports in 1908 to 4,064 athletes, 59 countries, 136 events, and 19 sports in 1948 to 10,500 athletes, 204 countries, 302 events and 37 sports in 2012. [R**??**]

Note: "geometric" is a synonym for "exponential".

Possible questions: find exponential regression curves for the numbers of athletes, countries and events.

Exercise 9.7.45. [U][N][Section **??**] [Goal **??**] Radioactive waste.

The web site `www.nirs.org/factsheets/hlwfcst.htm` we quoted earlier offers much more information about radioactive waste.

Ask and answer some interesting Fermi problems suggested by the data there. You could consider what it says about a nuclear power plant near you.

We could construct the Fermi problems based on this radioactive waste data ourselves, and ask the students to solve them. But by this time in the course we hope they can start from the numbers and create their own.

Exercise 9.7.46. [U][N] Payday loans 2016.

> People who borrow money against their paychecks are generally supposed to pay it back within two weeks, with substantial fees piled on: A customer who borrows $500 would typically owe around $575, at an annual percentage rate of 391 percent. [R**??**]

Many more juicy numbers in the article.

Exercise 9.7.47. [U][N] Was this a good deal on a mortgage?

On July 6, 2016 Max Jacob posted this question at `money.stackexchange.com/questions/66978/was-this-a-good-deal-on-a-mortgage` .

> I just graduated college last year, and was looking to buy a house. Clearly I wouldn't be able to buy it outright, so I was thinking of getting a mortgage. I have a steady job ($55,000 annual before taxes) and an okay credit score (740). Here's the deal that a mortgage consultant gave me:
>
> $2,000 per month for 30 years on a $300,000 home (he didn't mention any other fees)
>
> It seems terrible to me because it comes out to a total of $720,000 over that 30-year period. I'm completely new to the concept, so maybe I'm just being naive. [R**??**]

Possible uses: classroom discussion, group homework assignment, think about the answers informative answers at the site, appended here.

From user Aganju (`money.stackexchange.com/users/35405/aganju` :

> That seems a very bad offer, it borders on fraud.
>
> In the current US economy, you should be able to get between 3 and 4 % APR (and that number is what you should look at). That means that for 300000 over 30 years, you'd pay $1265 to $1432 per month.
>
> If you are able to pay more than that monthly rate, you should go for less than 30 years - 20, 15, 10, whatever you can afford - but don't overextend yourself.
>
> Google "mortgage calculator" to do your own calculations.

From user quid (`money.stackexchange.com/users/22881/quid` :

> I'm calculating that to about a 7% apr, which given loan rates available today seems a bit high.
>
> I wouldn't get too caught up on what that equates to over the life of the loan. There are a lot of forces in play over a 30 year period, namely the time value of money. 30 years from now a dollar will be less valuable in real terms due to the forces of inflation. At 2% per year in inflation today's $1 will be worth about $0.55 in 30 years.

From user keshlam (`money.stackexchange.com/users/12439/keshlam`):

> Some part of the payment is probably also going for tax escrow, insurance payments, probably PMI if you aren't putting at least 20% down. Get a complete breakdown of the costs.
>
> Remember to budget for upkeep.
>
> And please see past discussion of why buying a home at this point in your career/life may be very, very premature.

Exercise 9.7.48. [U][R][S] Get rich quick.

Tad Friend wrote in the October 10, 2016 issue of *The New Yorker* that

> YC's gold standard for revenue growth is ten percent a week, which compounds to 142x a year. [R**??**]

Check his calculation.

He's right. The Google calculator tells me

$$1.1^{52} = 142.042931984$$

which will multiply any starting amount by 142.

Exercise 9.7.49. [U][N] Double, double, ...

In April, 2017 Tad Friend wrote in *The New Yorker* that

> Progress in computers, or anyway in semiconductors, has been subject to Moore's Law, the exponential flywheel that has doubled capacity every two years. In linear progress, after thirty iterations you've advanced thirty steps; in exponential progress, you've advanced 1.07 billion steps. Our progress in mapping the human genome looked like it was linear — and then was revealed, once the doublings grew significant, as exponential. [R**??**]

(a) Where does that 1.07 billion come from?

(b) Question about Moore's Law?

Exercise 9.7.50. [U] Free meat!

On July 2, 2017 *The Boston Globe* published an article about bioengineered meat grown from animal stem cells.

> Steve Myrick, vice president of business development at Memphis [Meets], said the company is now producing beef, chicken, and duck in stainless

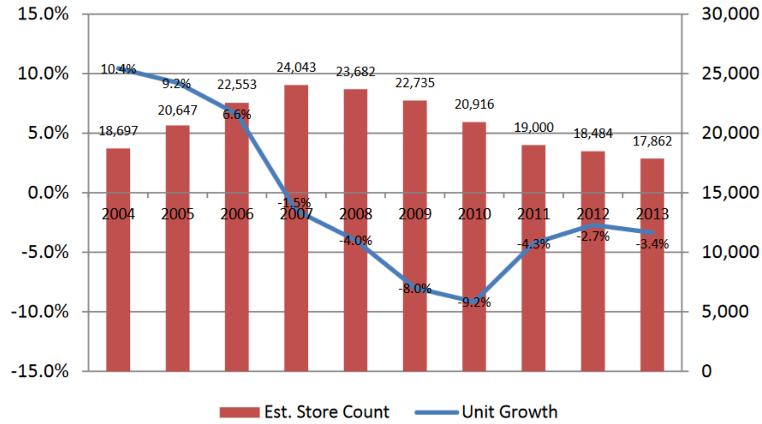


Figure 9.14. Payday Loan Stores 2004-2013 [R**??**]

> steel tanks for roughly $2,400 per pound. That'a lot cheaper than it was even a year ago, and the price is falling about 15 to 20 percent per month, as the company improves yields and finds less expensive nutrients. [R**??**]

(a) Estimate the cost "a year ago".

(b) When will the cost be competitive with what meat sells for in your local market?

Exercise 9.7.51. [U] Payday loans 2017.

On October 5. 2017 *The New York Times* reported that the Consumer Financial Protection Bureau announced new rules applicable to payday loans. The article read (in part)

> Currently, a cash-strapped customer might borrow $400 from a payday lender. The loan would be due two weeks later — plus $60 in interest and fees. That is the equivalent of an annual interest rate of more than 300 percent, far higher than what banks and credit cards charge for loans.
>
> ...
>
> The payday-lending industry is vast. There are now more payday loan stores in the United States than there are McDonald's restaurants. The operators of those stores make around $46 billion a year in loans, collecting $7 billion in fees. Some 12 million people, many of whom lack other access to credit, take out the short-term loans each year, researchers estimate. [R**??**]

(a) Verify the interest calculation in the first paragraph of the quotation.

(b) Are the assertions in the second paragraph reasonable? The data in Figure **??** should help answer this question.

Exercise 9.7.52. [U][N] R&D in China.

> China is the clearest example. Since 2000, China's spending on research and development has grown by an average of 18 percent each year, while ours grew by only 4 percent. This has placed China a decisive second in R & D expenditures behind the United States, where the government and private sector together invest far more than any other country. Even so, the share of R & D funded by the federal government declined to about 25 percent from just over 30 percent from 2010 to 2015. [R**??**]

Exercise 9.7.53. [U][S] Endangered apes.

On April 26, 2018 *The Washington Post* summarized a study from the journal *Science Advances* that reported

> ...estimates that there were 361,900 gorillas and 128,700 chimpanzees as of 2013. That's about one-third more gorillas and one-tenth more chimpanzees than previous surveys estimated, though those calculations were performed differently and were not designed to count the animals across their entire range.
>
> That is the good news. Now the bad: Researchers found that gorilla populations are dropping faster than they believed, at a rate of nearly 3 percent per year, said Fiona Maisels, a conservation scientist with the Wildlife Conservation Society and the University of Stirling in Scotland. At this rate, half of the world's gorillas could be gone by about 2040, she said. [R**??**]

(a) How many gorillas and chimpanzees were estimated in the previous surveys?

(b) This article updates numbers from 2013. Estimate the gorilla population in 2018, when this study was published.

(c) Is Dr. Maisel's prediction about the 2040 gorilla population reasonable?

(a) How many gorillas and chimpanzees were estimated in the previous surveys?

"One third more gorillas" means that

$$\frac{4}{3} \times \text{previous estimate} = 361,900$$

so

$$\text{previous estimate} = \frac{3}{4} \times 361,900 = 271,425 \approx 270,000 \text{ gorillas.}$$

A similar calculation says the previous estimate was about 117,00 chimpanzees.

(b) This article updates numbers from 2013. Estimate the gorilla population in 2018, when this study was published.

Given three percent per year decrease the 2018 gorilla population is about

$$270,000 \times 0.97^5 \approx 232,000.$$

Figure 9.15. Connected Homes, 2015-2017 [R**??**]

(c) Is Dr. Maisel's prediction about the 2040 gorilla population reasonable?

The rule of 70 says that the half life of a population declining at a rate of 3 percent per year is about $70/3 \approx 23$ years. Since 2040 is 22 years from now Dr. Maisel's prediction is reasonable.

Exercise 9.7.54. [U] An extra percentage point.

On May 22, 2018 *The Boston Globe* reported that

> Over the past 18 months, the average rate on 30-year fixed mortgages has climbed a full percentage point, which translates into an additional $100,000 in interest over the life of a $500,000 loan. [R**??**]

(a) Verify this assertion.

(b) Does the difference in interest paid depend on the actual mortgage rate, or only on the increase?

Exercise 9.7.55. [U] The Internet of Things.

Figure **??** shows the growth of U. S. households with devices like appliances connected to the internet.

(a) Is the 31% compound annual growth rate correct?

(b) Estimate when every home in the United States will be connected.

Exercise 9.7.56. [U][W][N] 5 miilion Bostonians?

On August 19, 2019 Martin Finucane wrote in *The Boston Globe* that in 1900 Globe reporter Thomas F. Anderson reported a prediction that Boston's population would be 5,251,330 in 2000. [R**??**]

The original article said

> This population is estimated on the rate of increase of the last 20 years, and on the same basis (the population according to the census of 1900 being 560,892), the figures for consecutive 10-year periods will probably be as follows. . .

The current story continued

> Anderson then cited a series of population figures that appeared to reflect an approximately 25 percent increase per decade for 100 years [. . . because he] had just seen the population of the city explode by 54 percent from 1880 to 1900.

$1.25^2 = 1.5625$

$1.25^{10} = 9.31$

$560,000 * 1.25^1 0 \approx 5,200,000$

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