Rubik's Stellated Dodecahedron

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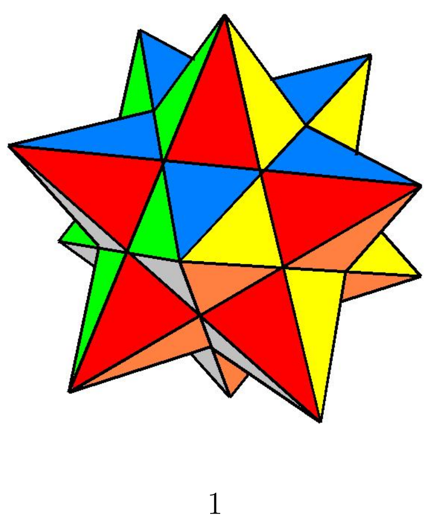
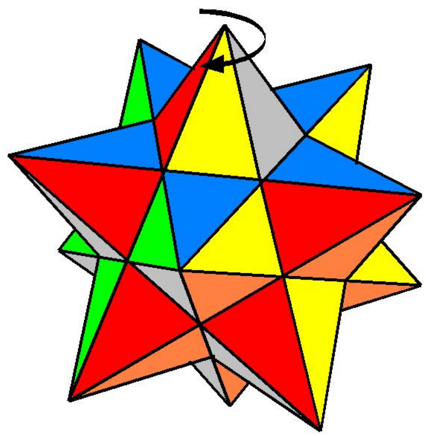
1 The Gadget

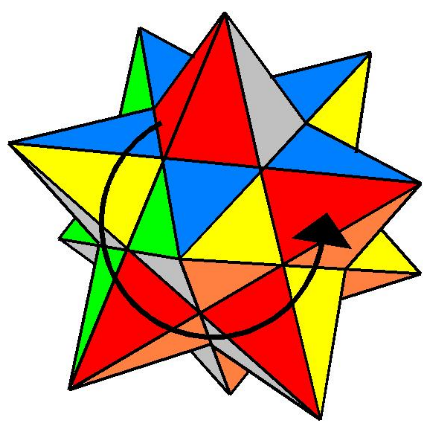
Here's a stellated dodecahedron. Its 60 triangles naturally group themselves into twelve pentagrammatical stars, which you should think of as the real faces. I've colored them using six colors, one for each pair of parallel faces (you only see one of each pair).



Now define two Rubik like transformations on this gadget. The first is the *pyramid move*, that rotates one of the stellations through a fifth of a circle. Here's an example:



The second is the *star move* that rotates one of the pentagram faces through a fifth of a circle. Here's what happens if you perform a star move on the previous figure:



I thought of this gadget during the night of July 5, 2006, after working late on the rhombic dodecahedron section of the book I'm writing about my experiences at the Manning elementary school.

Now, as Milton wrote, "The World was all before them." [1]

[1] John Milton, Paradise Lost, fourth line from the end of Book X.

2 Hardware

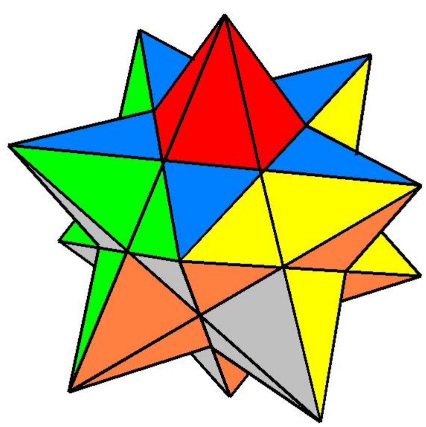
Can this gadget be built? Probably. I have some ideas about how to go about the design, but I think I can't actually make a working puzzle. I will ask a few friends who might be intrigued. I suspect the best way to get it built is to contact a company that makes Rubik's cubes and try to sell them my idea. But I don't want to give it away!

3 Software

Whether or not you can build the real thing it's clear that a virtual one that can be manipulated with software is feasible. It can't be harder than the four dimensional Rubik's cubes that are out there on the web. [2] I've built a virtual gadget - manipulable on line.

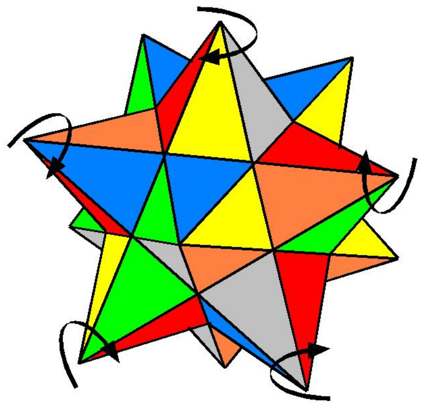
4 What configurations are possible?

For example, continuing the alternating sequence of pyramid and star moves above sends all the red triangles on the front face in the first figure to the top pyramid. Here's what the result will be (if I got it right):

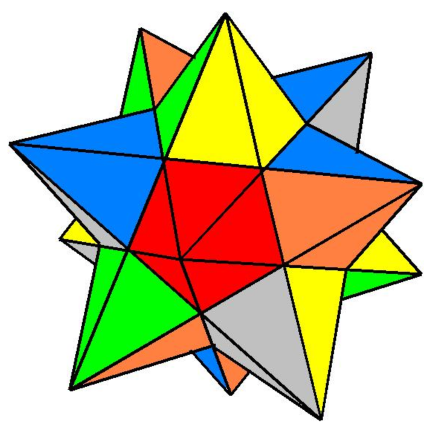


Another way to create a monochromatic red pyramid is to start with five independent pyramid moves on the arms of the red star:

[2]See http://www.superliminal.com/cube/cube.htm.



followed by five independent star moves on the stars surrounding the pyramid you are looking down on:



I think that once you have a monochromatic pyramid you can make its supporting star the same color too. I don't think it would be hard to find out.

Can you find a sequence of moves that makes all the pyramids monochromatic? I doubt it, but I don't have any real justification for that guess yet since I've been imagining this gadget for just a few days. If you can't, what's the maximum number of monochromatic pyramids you can create simultaneously? How will they be distributed?

Is there more than one solution? Suppose that you have reached a configuration in which all the pentagrammatical faces are monochromatic, and occur in opposite parallel pairs. (The second assertion might be a consequence of the first.) Is the configuration unique with those properties, or are there permutations of the five colors on the pyramid above, say, the red star?

There may be permutations possible in the geometry that can't be reached by Rubik moves.

5 What is the gadget group? How hard is the gadget?

How does the difficulty of unscrambling a scrambled gadget compare to the difficulty of unscrambling a Rubik's cube? I suspect the gadget is harder.

First, a disclaimer. I've never solved a Rubik's cube (nor tried). I have only browsed occasional discussions of the group theory and combinatorics on the subject. If I really want to pursue this I should check out David Joyner's *Adventures in Group Theory: Rubik's Cube, Merlin's Machine, and Other Mathematical Toys*

Here are some preliminary observations.

- Each Rubik's cube move is a rotation around one of three axes - the coordinate axes. Each gadget move is a rotation about one of six axes - the long diagonals of an icosahedron.

- A Rubik's cube has $6 \times 9 = 54$ separate small colored squares (sometimes called "stickies" in the literature). The gadget has $12 \times 5 = 60$ separate colored isosceles triangles.

- The group of Rubik's cube moves is far from transitive on the cube's squares. The group of gadget moves is transitive on the set of triangles. In fact with gadget moves you can make any two triangles share either a short edge or a long edge.

- Rubik's cube moves are all of degree 4, gadget moves of degree 5.

  Rubik's cube moves commute rarely because each move changes the positions of a large set of small squares. That means those sets are likely to intersect. In fact those moves commute only when they are about the same axis.

  Gadget moves commute frequently since each involves only 5 of the 60 triangles. So all pyramid (respectively star) moves commute with one another, and most pyramid moves commute with most star moves.

The commutativity tends to make the gadget easier to solve. Whether it makes easy enough so that it's no harder than the Rubik's cube remains to be seen.

6 Generalizations

Think about all these questions for the other stellated dodecahedron with twenty triangular pyramids. (I always thought this one should be called the stellated icosahedron.) I might actually draw some pictures for this.

Investigate higher dimensional analogues (if there are any).

7 What's in a name

What should this puzzle/gadget be called? Something catchy, descriptive, easy to say and easy to market.