DECIDABILITY OF THE TWO-QUANTIFIER THEORY OF THE RECURSIVELY ENUMERABLE WEAK TRUTH-TABLE DEGREES AND OTHER DISTRIBUTIVE UPPER SEMI-LATTICES

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Abstract. We give a decision procedure for the $\forall\exists$-theory of the weak truth-table (wtt) degrees of the recursively enumerable sets. The key to this decision procedure is a characterization of the finite lattices which can be embedded into the r.e. wtt-degrees by a map which preserves the least and greatest elements: A finite lattice has such an embedding if and only if it is distributive and the ideal generated by its cappable elements and the filter generated by its cuppable elements are disjoint.

We formulate general criteria that allow one to conclude that a distributive upper semi-lattice has a decidable two-quantifier theory. These criteria are applied not only to the weak truth-table degrees of the recursively enumerable sets but also to various substructures of the polynomial many-one (pm) degrees of the recursive sets. These applications to the pm degrees require no new complexity-theoretic results. The fact that the pm-degrees of the recursive sets have a decidable two-quantifier theory answers a question raised by Shore and Slaman in [21].

1. Introduction

If r is a reducibility between sets of natural numbers, we let $\mathcal{D}_r$ denote the set of r-degrees, ordered by $\leq_r$, and $\mathcal{R}_r$ denote the set of recursively enumerable r-degrees, also ordered by $\leq_r$. For the commonly studied reducibilities r, except for 1- reducibility, $\mathcal{D}_r$ is an upper semi-lattice with least element, and $\mathcal{R}_r$ is a bounded upper semi-lattice. (For many-one (m-) reducibility, we must ignore the m-degrees of $\emptyset$ and ω in order to get a least element.) It is natural to ask, for each of these structures, whether the structure (in the language $\{\leq\}$) is decidable. For the commonly studied structures, the answer is no. For $\mathcal{R}_{wtt}$, (wtt stands for weak truth-table reducibility) this undecidability is a recent result of Ambos-Spies, Nies and Shore [6].

The methods used to show the undecidability of these structures in fact show that some quantifier level of the theory of the structure is undecidable, and, thus, an obvious next step is to try to find the exact quantifier level at which the theory of the structure becomes undecidable.

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Any finite partial order can be embedded, as a partial order, into any of these structures and this easily shows that the one-quantifier theory of these structures in the language $\{\leq\}$ is decidable. Lattice-embedding results allow one to conclude that for many of these structures the one- quantifier theory remains decidable if $\vee, \wedge$ and 0 (and, in the case of $\mathcal{R}_r$, sometimes 1 as well) are added to the language. ($\wedge$ must be added as a three-place relation symbol.) However, even at this seemingly simple level, our knowledge is incomplete – it is not known whether or not the one-quantifier theory of $\mathcal{R}_T$ (where T stands for Turing reducibility) in the language $\{\leq, \vee, \wedge\}$ is decidable.

At the two-quantifier level, there are only a few results known so far. In [9], Degtev showed that the two-quantifier theory of $\mathcal{D}_m$ in the language $\{\leq, \vee, 0\}$ and the two-quantifier theory of $\mathcal{R}_m$ in the language $\{\leq, \vee, 0, 1\}$ are decidable. Lerman [18] and Shore [20] showed that the two-quantifier theory of $\mathcal{D}_T$ in the language $\{\leq, 0\}$ is decidable and, recently, Jockusch and Slaman [15] extended this result by showing that the two-quantifier theory of $\mathcal{D}_T$ in the language $\{\leq, \vee, 0\}$ is decidable.

In this paper, we show that the two-quantifier theory of $\mathcal{R}_{wtt}$ in the language $\{\leq, 0, 1\}$ is decidable. The structure $\mathcal{R}_{wtt}$ is quite different from $\mathcal{D}_m$, $\mathcal{R}_m$ and $\mathcal{D}_T$. Every finite lattice is isomorphic to an initial segment of $\mathcal{D}_T$, and for $\mathcal{D}_m$ and $\mathcal{R}_m$, every finite distributive lattice is isomorphic to an initial segment, while no nondistributive lattice can be lattice-embedded into the structure. These initial segments results play a strong role in the two-quantifier decision procedures for these structures. By contrast, in $\mathcal{R}_{wtt}$, one has both density and Sacks Splitting; in fact, these two results can be combined ([16]). Thus, each nontrivial interval of $\mathcal{R}_{wtt}$ has a rather complicated structure, and, in particular, cannot be finite. These differences mean that our decision procedure requires new techniques.

One advantage we have in deciding the two-quantifier theory of $\mathcal{R}_{wtt}$ is the fact that it is a distributive upper semi-lattice, i.e., it satisfies

$$(\forall a, b, c)(c \leq a \vee b \rightarrow (\exists a_0 \leq a)(\exists b_0 \leq b)(c = a_0 \vee b_0))$$

(the structures $\mathcal{D}_m$ and $\mathcal{R}_m$ are also distributive) and, hence, no nondistributive lattice can be lattice-embedded into it. In addition to distributivity, the main ingredients in our decision procedure are a characterization of the lattices that can be lattice-embedded into $\mathcal{R}_{wtt}$ preserving 0 and 1, given in Section 2, and the extension-of-embeddings result for $\mathcal{R}_{wtt}$ given in [12]. In Section 3, we give general criteria under which a distributive upper semi-lattice for which the extension-of-embeddings result of [12] holds has a decidable two-quantifier theory in the language with $\leq, 0$ and, if appropriate, 1. We apply these criteria not just to $\mathcal{R}_{wtt}$, but also to various complexity-theoretic structures. In particular, we answer a question of Slaman and Shore by showing that the two-quantifier theory of the polynomial many-one degrees of the recursive sets in the language $\{\leq, 0\}$ is decidable. Our complexity-theoretic applications require no new results in complexity theory. All that was missing was the algebraic analysis of Section 3.

The best undecidability result for quantifier levels of the theory of $\mathcal{R}_{wtt}$ is Lempp and Nies's recent result [17] that the four-quantifier theory is undecidable. Thus, the exact point at which the theory of $\mathcal{R}_{wtt}$ becomes undecidable is unknown, but the gap is small. A reasonable next step would be to try to decide the two-quantifier theory of the structure in the language $\{\leq, \vee, 0, 1\}$.

We refer the reader to Soare [22] for undefined terms and notations. If $A, B \subseteq \omega$, we say that A *is weak truth-table reducible to* B ($A \leq_{wtt} B$) if for some e, $A = \{e\}^B$ and there is a recursive function f such that for all x, $u(B; e, x) \leq f(x)$. If $e = \langle e_0, e_1 \rangle$, and A is any set, we define

$$[e]^A(x) = \begin{cases} \{e_0\}^A(x) & \text{if } \{e_0\}^A(x)\downarrow \text{ and } \{e_1\}(x)\downarrow \text{ and} \\ & u(A; e_0, x) \leq \{e_1\}(x), \\ \uparrow & \text{otherwise,} \end{cases}$$

and for all s, we define

$$[e]^A_s(x) = \begin{cases} \{e_0\}^A_s(x) & \text{if } \{e_0\}^A_s(x)\downarrow \text{ and } \{e_1\}_s(x)\downarrow \text{ and} \\ & u(A; e_0, x) \leq \{e_1\}(x), \\ \uparrow & \text{otherwise.} \end{cases}$$

Then, $A \leq_{wtt} B$ if and only if for some e, $[e]^B = A$, and if $\{A_s\}_{s\in\omega}$ is a recursive enumeration of an r.e. set A, then, for all x, $\lim_{s\to\infty}[e]^{A_s}_s(x) = [e]^A(x)$.

We assume that $\langle -, - \rangle$ is a standard pairing function and write $\langle x, y, z \rangle$ for $\langle x, \langle y, z \rangle \rangle$ and similarly for $\langle x, y, z, w \rangle$.

If we use a script letter as the name of a poset, then we assume that the domain of the poset is named by the corresponding Roman letter and the ordering is $\leq$ with the script letter as a subscript. Thus, for example, a poset $\mathcal{P}$ will be assumed to be $(P, \leq_{\mathcal{P}})$. We denote the least element of $\mathcal{P}$, if any, by $0_{\mathcal{P}}$, and similarly for $1_{\mathcal{P}}$, and we use $\vee_{\mathcal{P}}$, $\wedge_{\mathcal{P}}$ to denote joins and meets in $\mathcal{P}$. We sometimes drop the $\mathcal{P}$ subscript when there is no risk of confusion. If $\mathcal{X}$ and $\mathcal{Y}$ are posets, we write $\mathcal{X} \subseteq \mathcal{Y}$ to mean that $X \subseteq Y$ and $\leq_{\mathcal{X}} = \leq_{\mathcal{Y}} \restriction X$. If $\mathcal{X}, \mathcal{Y}$ are posets with least element, $\mathcal{X} \subseteq_0 \mathcal{Y}$ means $\mathcal{X} \subseteq \mathcal{Y}$ and $0_{\mathcal{X}} = 0_{\mathcal{Y}}$. If $\mathcal{X}, \mathcal{Y}$ are bounded posets, $\mathcal{X} \subseteq_{0,1} \mathcal{Y}$ means $\mathcal{X} \subseteq_0 \mathcal{Y}$ and $1_{\mathcal{X}} = 1_{\mathcal{Y}}$.

Let $\mathcal{X}$ and $\mathcal{Y}$ be posets. A *poset embedding* of $\mathcal{X}$ into $\mathcal{Y}$ is a function $f : X \to Y$ such that, for all $x, y \in X$, $x \leq_{\mathcal{X}} y$ if and only if $f(x) \leq_{\mathcal{Y}} f(y)$. A poset embedding is necessarily one-to-one. If $\mathcal{X}$ is an upper semi- lattice, a *usl embedding* of $\mathcal{X}$ into $\mathcal{Y}$ is an injective function $f : X \to Y$ which preserves joins, i.e., for all $x, y \in X$, $f(x \vee_{\mathcal{X}} y) = f(x) \vee_{\mathcal{Y}} f(y)$. A usl embedding is necessarily a poset embedding. If $\mathcal{X}$ is a lattice, a *lattice embedding* of $\mathcal{X}$ into $\mathcal{Y}$ is a usl embedding that also preserves meets. A function $f : X \to Y$ is said to *preserve least element* (or *preserve* 0) if $f(0_{\mathcal{X}}) = 0_{\mathcal{Y}}$ when $\mathcal{X}$ and $\mathcal{Y}$ both have least elements. (So, if either $\mathcal{X}$ or $\mathcal{Y}$ fails to have a least element, every function from X to Y preserves least element.) The terms *preserve greatest element* (*preserve* 1) are defined similarly.

If $\mathcal{U}$ is an upper semi-lattice, a subset I of U is an *ideal* of U if I is nonempty, I is downwards closed (i.e., if $x \in U$, $y \in I$ and $x \leq_{\mathcal{U}} y$, then $x \in I$) and I is closed under join (i.e., if $x, y \in I$, then $x \vee y \in I$). If $\mathcal{U}$ has a least element and $S \subseteq U$, then there is a smallest ideal $I(S)$ of $\mathcal{U}$ which contains S. If $S \neq \emptyset$, an element x of U is in $I(S)$ if and only if $x \leq_{\mathcal{U}} \bigvee A$ for some nonempty finite subset A of S. A subset F of U is a *filter* of $\mathcal{U}$ if F is nonempty, F is upwards closed (i.e., if $x \in F$, $y \in U$ and $x \leq_{\mathcal{U}} y$, then $y \in F$) and closed under meet (i.e., if $x, y \in F$ and $x \wedge y$ exists, then $x \wedge y \in F$). A subset F of U is a *strong filter* of $\mathcal{U}$ if F is nonempty, upwards closed and for every $x, y \in F$, there is a $z \in F$ with $z \leq_{\mathcal{U}} x, y$. A strong filter of $\mathcal{U}$ is clearly a filter of $\mathcal{U}$. If $\mathcal{U}$ has a greatest element, then for any subset S

of U, there is a smallest filter $F(S)$ of $\mathcal{U}$ which contains S. If $\mathcal{U}$ is in fact a lattice and $S \neq \emptyset$, then an element x of U is in $F(S)$ if and only if $\bigwedge A \leq_{\mathcal{U}} x$ for some nonempty finite subset A of S.

If $\mathcal{U}$ is a bounded upper semi-lattice, we say that an element x of U is *cuppable* if there is a $y \neq 1_{\mathcal{U}}$ with $x \vee y = 1_{\mathcal{U}}$. We denote the set of cuppable elements of $\mathcal{U}$ by $\mathrm{CUP}_{\mathcal{U}}$ or just CUP if $\mathcal{U}$ is clear from the context. Dually, an element x of U is *cappable* if there is a $y \neq 0_{\mathcal{U}}$ such that $x \wedge y = 0_{\mathcal{U}}$. The notations $\mathrm{CAP}_{\mathcal{U}}$ and CAP are defined in the obvious way.

2. Lattice Embeddings

We now turn to the characterization of the finite lattices that can be embedded into $\mathcal{R}_{wtt}$ by maps that preserve least and greatest elements.

Theorem 1. *Let $\mathcal{L}$ be a finite lattice. Then, there is a lattice embedding of $\mathcal{L}$ into $\mathcal{R}_{wtt}$ that preserves 0 and 1 if and only if $\mathcal{L}$ is distributive and*

$$F(CUP_{\mathcal{L}}) \cap I(CAP_{\mathcal{L}}) = \emptyset. \tag{2.1}$$

The "only if" direction follows from results in the literature. First, a lemma due to Lachlan shows that all sublattices of $\mathcal{R}_{wtt}$ are distributive. A proof of this lemma is given in Stob [23].

Lemma 2. *The upper semi-lattice $\mathcal{R}_{wtt}$ is distributive. Hence, no nondistributive lattice can be lattice-embedded into $\mathcal{R}_{wtt}$.*

□

Next, we use some results on the distribution of the cuppable, the cappable, and the noncappable r.e. wtt-degrees. We will write CUP_{wtt} for $\mathrm{CUP}_{\mathcal{R}_{wtt}}$ and similarly for CAP_{wtt}. We also write NC_{wtt} for $\mathcal{R}_{wtt} - \mathrm{CAP}_{wtt}$. Part (a) of the following lemma is shown in Ambos- Spies [1] and Part (b) is shown in Ambos-Spies et al. [5].

Lemma 3. (a) *CAP_{wtt} is an ideal of $\mathcal{R}_{wtt}$ and NC_{wtt} is a strong filter of $\mathcal{R}_{wtt}$.*
(b) *$CUP_{wtt} \subseteq NC_{wtt}$.*

□

Now, to show that the embedding condition in Theorem 1 is necessary, let $\mathcal{L}$ be a finite lattice and let $f : L \to R_{wtt}$ be a lattice embedding that preserves 0 and 1. By Lemma 2, it suffices to show that (2.1) holds. Obviously, $f(\mathrm{CUP}_{\mathcal{L}}) \subseteq \mathrm{CUP}_{wtt}$ and $f(\mathrm{CAP}_{\mathcal{L}}) \subseteq \mathrm{CAP}_{wtt}$, so $f(F(\mathrm{CUP}_{\mathcal{L}}))$ and $f(I(\mathrm{CAP}_{\mathcal{L}}))$ are contained in the filter generated by CUP_{wtt} and the ideal generated by CAP_{wtt}, respectively. By Lemma 3, the former is contained in NC_{wtt} while the latter is CAP_{wtt}. Thus,

$$f(F(\mathrm{CUP}_{\mathcal{L}})) \subseteq \mathrm{NC}_{wtt} \text{ and } f(I(\mathrm{CAP}_{\mathcal{L}})) \subseteq \mathrm{CAP}_{wtt}$$

whence $F(\mathrm{CUP}_{\mathcal{L}}) \cap I(\mathrm{CAP}_{\mathcal{L}}) = \emptyset$.

In the remainder of this section, we show that the embedding condition of Theorem 1 is sufficient. We first need some more lattice-theoretic notations and results. If $\mathcal{L}$ is lattice, we denote the set of (nonzero) join-irreducible elements of L by $J_{\mathcal{L}}$ and for $a \in L$, we let

$$J(a) = \{j \in J_{\mathcal{L}} | j \leq_{\mathcal{L}} a\}.$$

Note that if L is finite, then $a = \bigvee J(a)$ for every $a \in L$, where $\bigvee \emptyset = 0_{\mathcal{L}}$ by convention. (This is easily shown by induction on $|\{b \in L | b \leq_{\mathcal{L}} a\}|$.)

The next lemma gives some simple properties of finite distributive lattices which we will need for our proof.

Lemma 4. *Let $\mathcal{L}$ be a finite distributive lattice.*

(a) *For every $j \in J_{\mathcal{L}}$ and $A \subseteq L$ such that $j \leq_{\mathcal{L}} \bigvee A$, there is an $a \in A$ with $j \leq_{\mathcal{L}} a$.*
(b) *For every $a, b \in L$ with $a \not\leq_{\mathcal{L}} b$, there is a $j \in J_{\mathcal{L}}$ such that $j \leq_{\mathcal{L}} a$ and $j \not\leq_{\mathcal{L}} b$.*
(c) *For every $a \in L$, there is a least $b \in L$ such that $a \vee b = 1_{\mathcal{L}}$.*
(d) *For every $a \in CUP$ and $b >_{\mathcal{L}} a$, there is a $c <_{\mathcal{L}} b$ with $a \vee c = b$.*
(e) *For every $a \in CUP$, there is a $j \in CUP \cap J(a)$.*
(f) *Let $CUP_{\min} = \{a \in CUP | \forall b <_{\mathcal{L}} a (b \notin CUP)\}$. Then, $CUP_{\min} \subseteq J_{\mathcal{L}}$ and $1_{\mathcal{L}} = \bigvee CUP_{\min}$.*

Proof. Part (a) is straightforward by distributivity. Part (b) follows from the observation that $a = \bigvee J(a)$.

For a proof of Part (c), for a contradiction, assume that the claim fails. Then, there are incomparable elements b_0 and b_1 that are minimal such that $a \vee b_0 = 1_{\mathcal{L}}$ and $a \vee b_1 = 1_{\mathcal{L}}$. But then, by distributivity, $a \vee (b_0 \wedge b_1) = 1_{\mathcal{L}}$, contrary to minimality of b_0 and b_1.

Part (d) is an immediate consequence of distributivity.

For a proof of Part (e), take $b \neq 1_{\mathcal{L}}$ with $a \vee b = 1_{\mathcal{L}}$ and fix a maximal element j of $J(a)$ such that $j \not\leq_{\mathcal{L}} b$. Then, $j \leq_{\mathcal{L}} a$, $1_{\mathcal{L}} = j \vee (\bigvee(J(a) - J(j)) \vee b)$ and $j \not\leq_{\mathcal{L}} \bigvee(J(a) - J(j)) \vee b$, by (a). So, $j \in J_{\mathcal{L}} \cap \mathrm{CUP}$.

The first part of (f) is immediate by (e). To show the second part, for a contradiction, assume that $\bigvee \mathrm{CUP}_{\min} <_{\mathcal{L}} 1_{\mathcal{L}}$. Then, by definition of $\mathrm{CUP}_{\min}$ and (c), there is a least $a \in L - \{1_{\mathcal{L}}\}$ such that $a \vee \bigvee \mathrm{CUP}_{\min} = 1_{\mathcal{L}}$ and $a \notin \mathrm{CUP}_{\min}$. So, there are $b <_{\mathcal{L}} a$ and $c <_{\mathcal{L}} 1_{\mathcal{L}}$ such that $b \in \mathrm{CUP}_{\min}$ and $b \vee c = 1_{\mathcal{L}}$. Hence, by (d), $a = b \vee c'$ for some $c' <_{\mathcal{L}} a$ and

$$1_{\mathcal{L}} = a \vee \bigvee \mathrm{CUP}_{\min} = (b \vee c') \vee \bigvee \mathrm{CUP}_{\min} = c' \vee (b \vee \bigvee \mathrm{CUP}_{\min}) = c' \vee \bigvee \mathrm{CUP}_{\min},$$

contrary to choice of a. □

For the remainder of this section, we fix a finite distributive lattice $\mathcal{L}$ such that Condition (2.1) holds. For each join-irreducible $j \in L$, let

$$J_j = \{j' \in J_{\mathcal{L}} | j' \not\geq_{\mathcal{L}} j\}$$

and let $\min_F$ be the least element of $F(\mathrm{CUP})$ and $\max_I$ be the greatest element of $I(\mathrm{CAP})$.

Since $I(\mathrm{CAP})$ is closed downwards, Condition (2.1) implies that $\min_F \not\leq_{\mathcal{L}} \max_I$ and hence, by Part (b) of Lemma 4, there is a join- irreducible element j_0 of L such that

$$j_0 \leq_{\mathcal{L}} \min_F \text{ and } j_0 \not\leq_{\mathcal{L}} \max_I . \tag{2.2}$$

Moreover, we may define two functions u,d from $J_{\mathcal{L}}$ to $J_{\mathcal{L}}$ such that for each $j \in J_{\mathcal{L}}$,

$$u(j) \geq_{\mathcal{L}} j, j_0 \text{ and } d(j) \leq_{\mathcal{L}} j, j_0. \tag{2.3}$$

(The existence of u and d is shown as follows. By Lemma 4 (a) and (f), for every $j \in J_{\mathcal{L}}$, there is a join-irreducible $j' \in$ CUP such that $j \leq_{\mathcal{L}} j'$. So, we may let $u(j)$ be any such j'. For the existence of $d(j)$, note that j_0 is noncappable, so there is some nonzero $a \in L$ with $a \leq_{\mathcal{L}} j, j_0$. Now, we can take $d(j)$ to be any element of $J(a)$.)

In the following, let $j_0, \dots, j_p$ be some ordering of $J_{\mathcal{L}}$ where j_0 is chosen as in (2.2).

We now turn to the construction of an embedding of $\mathcal{L}$ into $\mathcal{R}_{wtt}$. By a standard infinite injury tree argument, we construct disjoint r.e. sets A_j $(j \in J_{\mathcal{L}})$ such that for

$$A_J = \bigcup_{j \in J} A_j \qquad\qquad (J \subseteq J_{\mathcal{L}})$$

the function

$$f : L \to R_{wtt}$$

defined by

$$f(a) = \deg_{wtt}(A_{J(a)})$$

for $a \in L$ will be a lattice embedding of $\mathcal{L}$ into $\mathcal{R}_{wtt}$ that preserves least and greatest elements.

Note that $f(0_{\mathcal{L}}) = 0$ and, for any $a, b, c \in L$,

$$a \leq_{\mathcal{L}} b \Rightarrow f(a) \leq_{wtt} f(b) \tag{2.4}$$

and, since $A_{J(a) \cup J(b)} = A_{J(a \vee_{\mathcal{L}} b)}$,

$$c = a \vee_{\mathcal{L}} b \Rightarrow f(c) = f(a) \vee_{wtt} f(b).$$

So, it suffices to ensure that the function f has the following properties:

(2.5)	$f(1_{\mathcal{L}}) = 0'$	(greatest element)
(2.6)	$a \not\leq_{\mathcal{L}} b \Rightarrow f(a) \not\leq_{wtt} f(b)$	(nonordering)
(2.7)	$c = a \wedge_{\mathcal{L}} b \Rightarrow f(c) = f(a) \wedge_{wtt} f(b)$	(meets)

To satisfy these conditions, it suffices to ensure that the sets we will construct have the following properties:

$$K \leq_{wtt} A_{J_{\mathcal{L}}}, \tag{2.8}$$

for some wtt-complete r.e. set K,

$$A_j \not\leq_{wtt} A_{J_j} \tag{2.9}$$

for $j \in J_{\mathcal{L}}$, and

$$C \leq_{wtt} A_{J(a)}, A_{J(b)} \Rightarrow C \leq_{wtt} A_{J(a) \cap J(b)} \tag{2.10}$$

for any set C and $a, b \in L$.

(Note that $A_{J(a) \cap J(b)} = A_{J(a \wedge_{\mathcal{L}} b)}$, so (2.7) is a direct consequence of (2.10) and (2.4).)

Conditions (2.9) and (2.10) are broken up into the following diagonalization and meet requirements, respectively (for $j \in J_{\mathcal{L}}$, $a, b \in L$, $e = \langle e_0, e_1 \rangle$):

$$\begin{aligned} D_{j,e}: &\quad A_j \neq [e]^{A_{J_j}} \\ M_{a,b,e}: &\quad [e_0]^{A_{J(a)}} = [e_1]^{A_{J(b)}}, \text{total} \Rightarrow [e_0]^{A_{J(a)}} \leq_{wtt} A_{J(a)\cap J(b)}. \end{aligned}$$

Let $\langle D_n : n \geq 0 \rangle$ and $\langle M_n : n \geq 0 \rangle$ be recursive listings of the D and M requirements, respectively, and let $R_{2n} = M_n$ and $R_{2n+1} = D_n$. The strategies for both the D and M requirements will have two possible outcomes. Hence, the priority tree of the construction is $T = 2^{<\omega}$ and we assign requirement R_n to the n-th level of T so that any strategy α with $|\alpha| = 2n$ $(2n+1)$ is a strategy for M_n (D_n). We write R_α for the requirement for which α is a strategy. As usual, a strategy α will be allowed to act at α-stages, i.e., at stages at which its guess about the outcomes of the higher priority strategies seems to be correct.

The strategies for satisfying (2.8) and the D and M requirements are as follows.

Condition (2.8) is ensured by direct coding: Let K be a *wtt*-complete r.e. set such that $K \subseteq \omega^{[0]}$ and let $\{K[s] : s \geq 0\}$ be a recursive enumeration of K such that $K[0] = \emptyset$ and $|K[s+1] - K[s]| = 1$, say $k_s \in K[s+1] - K[s]$. We will ensure that for any s, either (by the activity of some strategy)

$$A_j[s]\restriction k_s + 1 \neq A_j[s+1]\restriction k_s + 1$$

for some $j \in J_{\mathcal{L}}$, or

$$\exists j \in J_{\mathcal{L}} \cap F(\mathrm{CUP})(k_s \in A_j[s+1] - A_j[s]). \tag{2.11}$$

Obviously, this implies (2.8).

For a meet requirement $M_n = M_{a,b,e}$, we have two different types of strategies depending on whether $J(a) \cap J(b) = \emptyset$ or not. If $J(a) \cap J(b) = \emptyset$, we call M_n a *minimal pair requirement*, and we call M_n a *proper meet requirement* otherwise.

For the proper meet requirements, we adapt Fejer's meet strategy (from [11]) to *wtt*-reductions. For the minimal pair requirements, we use the standard minimal pair technique, but impose some additional restraint.

For $M_n = M_{a,b,e}$ $(e = \langle e_0, e_1 \rangle)$, let

$$l_n[s] = \max\{x : \forall y < x([e_0]^{A_{J(a)}}(y)[s] \downarrow = [e_1]^{A_{J(b)}}(y)[s] \downarrow)\}$$

be the length of agreement between $[e_0]^{A_{J(a)}}$ and $[e_1]^{A_{J(b)}}$ at the end of stage s. Note that

$$[e_0]^{A_{J(a)}} = [e_1]^{A_{J(b)}}, \text{ total iff } \lim_s l_n[s] = \omega \text{ iff } \limsup_s l_n[s] = \omega. \tag{2.12}$$

(Notice that for Turing reductions (2.12) in general fails.) The behavior of a strategy α $(|\alpha| = 2n)$ for M_n depends on the hypothesis of M_n. If the hypothesis is true, i.e., $\lim_s l_n[s] = \omega$, then α is infinitary (outcome 0); otherwise, it is finitary (outcome 1).

Now, if M_n is a proper meet requirement, then a strategy α $(|\alpha| = 2n)$ for M_n works roughly as follows: If s is the first stage such that $x < l_n[s]$ (and α has highest priority to act), then α defines a set $\mathrm{COR}(\alpha,x)[s+1]$ of correction markers for x:

$$\mathrm{COR}(\alpha,x)[s+1] = \{\langle s+1, 2n, x, y\rangle : 0 \le y \le s\}, \tag{2.13}$$

i.e., $\mathrm{COR}(\alpha,x)[s+1]$ consists of $s+1$ numbers all greater than s and, by construction, none of them has been enumerated in any set under construction by the end of stage s. Now, after stage s, a correction marker for x will be put into $A_{J(a)\cap J(b)}$ only in order to let $A_{J(a)\cap J(b)}$ compute $[e_0]^{A_{J(a)}}(x) = [e_1]^{A_{J(b)}}(x)$ (if these computations are equal and defined): If there are stages t and u such that $s \le t \le u$, $[e_0]^{A_{J(a)}}(x)[t] \downarrow= [e_1]^{A_{J(b)}}(x)[t] \downarrow$ and u is the least stage such that $[e_0]^{A_{J(a)}}(x)[t] \downarrow\ne [e_0]^{A_{J(a)}}(x)[u] \downarrow$ and $[e_1]^{A_{J(b)}}(x)[t] \downarrow\ne [e_1]^{A_{J(b)}}(x)[u] \downarrow$, then the least marker in $\mathrm{COR}(\alpha,x)[s+1]$ not yet used is put into $A_{J(a)\cap J(b)}$ at stage $u+1$. Notice that, as $x < l_n[s]$, we have $\max\{\{e_0\}(x), \{e_1\}(x)\} < s$, whence this can happen at most s times. So, $\mathrm{COR}(\alpha,x)[s+1]$ contains sufficiently many markers for these corrections and, moreover, assuming that $[e_0]^{A_{J(a)}} = [e_1]^{A_{J(b)}}$ is total, for the greatest element y of $\mathrm{COR}(\alpha,x)[s+1]$ and any stage $v \ge s$, if $A_{J(a)\cap J(b)}[v]\restriction y = A_{J(a)\cap J(b)}\restriction y$ and $[e_0]^{A_{J(a)}}(x)[v] \downarrow= [e_1]^{A_{J(b)}}(x)[v] \downarrow$, then these computations are correct. Since y will be computable from x, this will imply that $[e_0]^{A_{J(a)}} \le_{wtt} A_{J(a)\cap J(b)}$.

Once appointed, the set $\mathrm{COR}(\alpha,x)[s+1]$ will not change during the construction unless the strategy α is initialized. (In this case, we might define a new copy of $\mathrm{COR}(\alpha,x)$ later.) We let $\mathrm{COR}(\alpha,x)[t]$ be the current copy of $\mathrm{COR}(\alpha,x)$ at the end of stage t (if there is one) and we let

$$\mathrm{cor}(\alpha,x)[t] = \min(\mathrm{COR}(\alpha,x)[t] - A_{J(a)\cap J(b)}[t]). \tag{2.14}$$

To satisfy a diagonalization requirement $D_n = D_{j,e}$, we use the Friedberg-Muchnik strategy: we pick a follower x, wait for $[e]^{A_{J_j}}(x) = 0$, put x into A_j and preserve the computation $[e]^{A_{J_j}}(x)$ by a restraint. So, the possible outcomes for a D_n-strategy are either that we wait forever for $[e]^{A_{J_j}}(x) \downarrow= 0$ (outcome 1) or that we ensure that $A_j(x) = 1 \ne 0 = [e]^{A_{J_j}}(x)$ (outcome 0).

The restraints for the diagonalization and meet strategies are as follows: First, by initialization of lower priority strategies, computations will be protected against followers of diagonalization strategies and correction markers of proper meet strategies of lower priority.

A second type of restraint is imposed by the diagonalization strategies to protect their computations against the coding requirement (2.11): When a diagonalization strategy α, $|\alpha| = 2n+1$, $D_n = D_{j,e}$ completes a diagonalization via follower x at stage $s+1$, it will impose restraint on $A_{J_{u(j)}}$ of length $s+1$ (and of priority α) to protect the computation $[e]^{A_{J_j}}(x) = 0$. (Note that, by (2.3), in general, this is more than restraining A_{J_j} which would be sufficient solely for the protection of $[e]^{A_{J_j}}(x)$. This stronger restraint will help the minimal pair strategies succeed.) This restraint applies to coding only, i.e., it gives a targeting procedure for coding numbers as in the Sacks splitting theorem.

A third type of restraint is imposed by minimal pair strategies α. This restraint is put on at α-expansionary stages and applies to the correction markers of proper meet strategies only. (Note that by (2.1), coding has no *direct* impact on the minimal pair strategies.) The goal of this restraint is to target the correction markers of the infinitary meet strategies β below α (i.e., $\beta 0 \subseteq \alpha$) into the side which has been (possibly) destroyed at the α-expansionary stage. Here, we will give α's restraint priority $\alpha 1$ to ensure that the restraint of another minimal pair strategy γ above the infinitary outcome of α (i.e., $\alpha 0 \subseteq \gamma$) has *higher* priority than that of α. The α restraint will be cancelled only if α (not $\alpha 1$) is initialized.

These minimal pair restraints are further supported by ensuring that any meet requirement $M_{a,b,e}$ with $j_0 \in J(a) \cap J(b)$ will correct above j_0 so that correction markers for such requirements cannot enter either side of the minimal pair strategies.

Finally, to have a better way of controlling the side effects on correction markers of K-coding or diagonalizations, at each stage of the construction we put numbers into exactly one set A_j.

We now turn to the formal construction:

α-stages s and, for meet strategies α, *α-expansionary stages s* and the *certified length function* $l_\alpha[s]$ of α at the end of stage s are defined by induction on s and $|\alpha|$ as follows:

$s = 0$. Stage 0 is an α-stage for all α and α-expansionary for all α with $|\alpha|$ even. For the latter, $l_\alpha[0] = 0$.

$s > 0$. Stage s is a λ-stage (where λ is the empty string). If $|\alpha|$ is even, i.e., R_α is a meet requirement $M_n = M_{a,b,e}$ ($e = \langle e_0, e_1 \rangle$), then

$$l_\alpha[s] = l_n[s]$$

if M_n is a minimal pair requirement and

(2.15)

$$\begin{aligned} l_\alpha[s] = \max\{x : \forall y < x(&[e_0]^{A_{J(a)}}(y)[s] \downarrow = [e_1]^{A_{J(b)}}(y)[s] \downarrow \\ &\wedge (\operatorname{cor}(\alpha, y)[t+1] \downarrow \to [[e_0]^{A_{J(a)}}(y)[s] = [e_0]^{A_{J(a)}}(y)[t] \vee \\ &[e_1]^{A_{J(b)}}(y)[s] = [e_1]^{A_{J(b)}}(y)[t] \vee \\ &A_{J(a)\cap J(b)}[t] \restriction \operatorname{cor}(\alpha, y)[t+1] + 1 \neq A_{J(a)\cap J(b)}[s] \restriction \operatorname{cor}(\alpha, y)[t+1] + 1 \\ &\vee \quad \alpha \text{ has been initialized at some stage } v \text{ with } t + 1 < v \leq s] \\ &\text{where } t \text{ is the greatest } \alpha\text{-expansionary stage} < s)\} \end{aligned}$$

if M_n is a proper meet requirement. Moreover, for α as above such that s is an α-stage, we say s is α-expansionary if

$$l_\alpha[s] > \max\{l_\alpha[t] : t < s \wedge t \text{ is an } \alpha\text{-stage}\};$$

s is an $\alpha 0$-stage if s is α-expansionary, and s is an $\alpha 1$-stage, otherwise. (The purpose of the seemingly complicated definition in (2.15) is to ensure that if s is an α-expansionary stage for some proper meet strategy α, then no correction for α is needed at stage $s+1$.) Finally, if s is an α-stage, with $|\alpha|$ odd, i.e., R_α is a diagonalization requirement $D_n = D_{j,e}$, then s is an $\alpha 0$- stage if there is a follower x of α at the end of stage s such that $x \in A_j[s]$; otherwise, s is an $\alpha 1$-stage.

We say that α is *accessible* at stage $s+1$ if s is an α-stage and $|\alpha| \leq s$ and we let $\delta[s]$ be the unique string of length s accessible at stage $s+1$.

A strategy α *requires attention* at stage $s+1$ if one of the following two cases, depending on the type of R_α, applies:

Case 1: R_α is a diagonalization requirement $D_n = D_{j,e}$ and one of the following holds:

$$\alpha \subseteq \delta[s] \text{ and } \alpha \text{ has no follower.} \tag{2.16}$$

$$\alpha \subseteq \delta[s] \text{ and there is an } \alpha\text{-follower } x \text{ such that } A_j(x) = [e]^{A_{J_j}}(x) = 0[s]. \tag{2.17}$$

$$\alpha \leq \delta[s] \text{ and } k_s \text{ is less than the current } \alpha \text{ restraint.} \tag{2.18}$$

Case 2: R_α is a proper meet requirement $M_n = M_{a,b,e}$ and there is a number y such that

(2.19)
$$\begin{aligned}
&\alpha \subseteq \delta[s] \wedge s > 0 \wedge \mathrm{cor}(\alpha, y)[t+1] \downarrow\leq k_s \wedge [e_0]^{A_{J(a)}}(y)[s] \downarrow\neq [e_0]^{A_{J(a)}}(y)[t] \\
&\wedge [e_1]^{A_{J(b)}}(y)[s] \downarrow\neq [e_1]^{A_{J(b)}}(y)[t] \\
&\wedge A_{J(a)\cap J(b)}[s]\restriction \mathrm{cor}(\alpha, y)[t+1] + 1 = A_{J(a)\cap J(b)}[t]\restriction \mathrm{cor}(\alpha, y)[t+1] + 1 \\
&\wedge \ \ \alpha \text{ has not been initialized at any stage } v \text{ with } t+1 < v \leq s, \\
&\text{where } t \text{ is the greatest } \alpha\text{-expansionary stage less than } s.
\end{aligned}$$

Note that in (2.18), we only require $\alpha \leq \delta[s]$, while in (2.16), (2.17) and (2.19) we require $\alpha \subseteq \delta[s]$.

If we initialize a strategy, we cancel all parameters associated with the strategy. Otherwise, a parameter of a strategy at some stage will be unchanged at the following stages unless we explicitly redefine it.

Construction.

Stage 0: Initialize all strategies.

Stage $s+1$: The stage consists of four steps. A number can enter a set only in Step 1.

Step 1: Fix α (if there is any) minimal such that α requires attention and distinguish the following cases.

Case 1: R_α is a diagonalization requirement $D_n = D_{j,e}$. Distinguish the following two subcases depending on the clause via which α requires attention. In either case, initialize all strategies β with $\alpha < \beta$.

Case 1.1: (2.16) or (2.18) holds. Put k_s into $A_{u(j)}$. If (2.16) holds, appoint $\langle s+1, 2n+1, 0\rangle$ as an α-follower.

Case 1.2: (2.17) holds. If $k_s < x$, put k_s into $A_{u(j)}$. Otherwise, put x into A_j and impose an α- restraint of length $s+1$ (and priority α) on $A_{J_{u(j)}}$.

Case 2: R_α is a proper meet requirement $M_n = M_{a,b,e}$. Fix the least y for which (2.19) holds and let $c = \mathrm{cor}(\alpha, y)[t+1]$. Distinguish the following two subcases. In either case, initialize every strategy β with $\alpha 1 \leq \beta$.

Case 2.1: $j_0 \in J(a) \cap J(b)$. Put c into A_{j_0}.

Case 2.2: Otherwise. Fix $\beta 1$ minimal such that β is a minimal pair strategy which imposes a restraint $> c$ on a set $A_{J_{d(j)}}$ with $d(j) \in J(a) \cap J(b)$. Put c into $A_{d(j)}$. If no such β exists, put c into the least $j \in J(a) \cap J(b)$.

Case 3: No α requires attention. Put k_s into A_{j_0} and initialize all strategies β with $\delta[s] \leq \beta$.

Step 2: Let j be the unique element of $J_{\mathcal{L}}$ such that a number enters A_j at step 1. Call $s+1$ a j-stage.

Step 3: Let α ($|\alpha| = 2n$, $M_n = M_{a,b,e}$) be any proper meet strategy that has not been initialized in Step 1 and such that $\alpha 0 \subseteq \delta[s]$. For any $x < l_\alpha[s]$ such that $\mathrm{COR}(\alpha, x)$ is not defined at the end of stage s, define $\mathrm{COR}(\alpha, x)[s+1]$ and $\mathrm{cor}(\alpha, x)[s+1]$ by (2.13) and (2.14), respectively.

Step 4: Let α ($|\alpha| = 2n$, $M_n = M_{a,b,e}$) be any minimal pair strategy that has not been initialized in Step 1 and such that $\alpha 0 \subseteq \delta[s]$. Cancel any previous α-restraint (if any) and impose a new α-restraint on $A_{J_{d(j)}}$ of length $s+1$ (for j as in step 2).

This completes the construction.

Verification. Let f be the true path of the construction, i.e., the leftmost path such that for every n

$$\exists^\infty s (f \restriction n \subseteq \delta[s])$$

and let s_n be a stage such that, for every $s \geq s_n$, $f \restriction n \leq \delta[s]$.

Note that elements of K, followers of diagonalization strategies and correction markers for proper meet requirements are all of different forms (namely $\langle 0, - \rangle$, $\langle s+1, 2n+1, 0 \rangle$, and $\langle s+1, 2n, -, - \rangle$, respectively). Moreover, for different (α, x), the sets $\mathrm{COR}(\alpha, x)$ are mutually disjoint (and so are different copies of $\mathrm{COR}(\alpha, x)$ defined at different stages following initialization of α). So if, in Step 1 of stage $s+1$, we say that we put a number x into A_j, then this number is not in any of the sets $A_{j'}[s]$, $j' \in J_{\mathcal{L}}$; in particular, $x \in A_j[s+1] - A_j[s]$. It follows that $\{A_j | j \in J_{\mathcal{L}}\}$ is a disjoint family of sets.

Note also that if α requires attention at stage $s+1$ by Case 2 and y and t are as in (2.19), then $l_\alpha[s] \leq y$. Since $\mathrm{cor}(\alpha, y)[t+1] \downarrow$, there must be an α-expansionary stage $t' < s$ with $y < l_\alpha[t']$. Thus, s is not α-expansionary.

Claim 1. $K \leq_{wtt} A_{J_{\mathcal{L}}}$.

Proof. Since at any stage $s+1$ a number $x \leq k_s$ enters one of the sets A_j, the claim follows by permitting. □

Claim 2. Let $\beta \subseteq f$.

(1) β is initialized only finitely often.
(2) If R_β is a diagonalization requirement, then β requires attention only finitely often and the restraint for β goes to a (finite) limit.
(3) If R_β is a proper meet requirement and $\beta 1 \subseteq f$, then β requires attention only finitely often.

Proof. Routine. □

Claim 3. For all $j \in J_{\mathcal{L}}$ and e, $D_{j,e}$ is met.

Proof. Fix $\alpha \subseteq f$ such that $|\alpha| = 2n+1$ and $D_n = D_{j,e}$. By Claim 2, let t be the last stage at which α was initialized. Again by Claim 2, α requires attention only finitely often. Since there are infinitely many stages s with $\alpha \subseteq \delta[s]$, we may conclude that there is a least stage $u \geq t$ such that a follower x for α is appointed at stage $u+1$ and that this follower is permanent. Now, distinguish two cases:
Case 1: $x \in A_j$.
Fix $v > u$ such that x enters A_j at stage $v+1$. Then, by (2.17), $[e]^{A_{J_j}}(x)[v] = 0$, and, at stage $v+1$, all strategies β with $\alpha < \beta$ are initialized, and α imposes a restraint on $A_{J_{u(j)}}$ of length $v+1$. So it suffices to show that

$$A_{J_j}[v]\restriction v+1 = A_{J_j}\restriction v+1.$$

We will show more, namely

$$\forall y < v+1(y \in A_{J_{\mathcal{L}}} - A_{J_{\mathcal{L}}}[v] \rightarrow y \in A_j \cup A_{u(j)}). \tag{2.20}$$

For a proof of (2.20), we first note that, since α is not initialized after v, no diagonalization strategy β with $\beta < \alpha$ and no proper meet strategy β with $\beta 1 \leq \alpha$ receives attention after stage v. It follows that for no $w \geq v$ do we have $\delta[w] < \alpha$. Moreover, since at stage $v+1$, all strategies β with $\beta > \alpha$ are initialized, we may conclude that the only numbers $y < v+1$ which can enter $A_{J_{\mathcal{L}}}$ after stage v are elements of K, followers of α, or correction markers of proper meet strategies β with $\beta 0 \subseteq \alpha$. Now, no number from K enters $A_{J_{\mathcal{L}}}$ at stage $v+1$ and, by α's restraint which is permanent from stage $v+1$ on, elements y of K with $y < v+1$ that enter $A_{J_{\mathcal{L}}}$ later enter $A_{u(j)}$. Since x is permanent, it is α's only follower after v and it enters A_j. This leaves the correction markers. For a contradiction, assume that $s+1$ is the least stage $> v+1$ at which a correction marker $y < v+1$ of a meet strategy β with $\beta 0 \subseteq \alpha$ – say $|\beta| = 2m$ and $M_m = M_{a,b,e}$ – enters a set $A_{j'}$ with $j' \neq j, u(j)$. Let s' be the greatest β-expansionary stage less than s. Then, by construction, there must be numbers $z_a, z_b < y < v+1$ that entered $A_{J(a)}$ and $A_{J(b)}$, respectively, after stage s' and before stage $s+1$, but no number $< y$ entered $A_{J(a) \cap J(b)}$ at such a stage. Since, by $\beta 0 \subseteq \alpha$, v is β-expansionary, i.e., $v \leq s'$, it follows from minimality of s that z_a and z_b entered A_j or $A_{u(j)}$. Since $j \leq_{\mathcal{L}} u(j)$, this implies $z_a \in A_{J(a) \cap J(b)}$ or $z_b \in A_{J(a) \cap J(b)}$, a contradiction.
Case 2: $x \notin A_j$.
It suffices to show that we don't have $[e]^{A_{J_j}}(x) = 0$. For a contradiction, assume that $[e]^{A_{J_j}}(x) = 0$. Then, α will require attention infinitely often via (2.17) and x. This contradicts Claim 2. □

Claim 4. Let $M_n = M_{a,b,e}$ with $e = \langle e_0, e_1 \rangle$ and assume that

$$[e_0]^{A_{J(a)}} = [e_1]^{A_{J(b)}} = g \text{ is total.} \tag{2.21}$$

Then, $(f\restriction 2n)0 \subseteq f$.

Proof. Let $\alpha = f\restriction 2n$. Then, there are infinitely many α-stages and, by (2.21) and (2.12),

$$\lim_s l_n[s] = \infty. \tag{2.22}$$

So, if M_n is a minimal pair requirement, there are infinitely many α-expansionary stages, whence, $\alpha 0 \subseteq f$. So, without loss of generality, we may assume that M_n is a proper meet requirement and, for a contradiction, that $\alpha 1 \subseteq f$. Then, by Claim 2, we may choose t such that no stage $> t$ is α-expansionary and at no such stage does α require attention. By (2.22), choose an α-stage $s > t$ such that

$$l_n[s] > \max\{l_n[u] : u < s \wedge u \text{ is an } \alpha\text{-stage}\}$$

and such that

$$K_s \restriction N = K \restriction N$$

where N is the largest number ever assigned to be a correction marker for α. (There are only finitely many numbers ever assigned to be correction markers for α since such markers are only assigned at α-expansionary stages.) By definition, either s is α-expansionary or α requires attention at stage $s+1$. But this contradicts the choice of t. $\square$

Claim 5. Let u, t, α, j be given such that

$$R_\alpha \text{ is a minimal pair requirement.} \tag{2.23}$$

$$\alpha 0 \subseteq \delta[u] \tag{2.24}$$

$$u+1 \text{ is a } j\text{-stage.} \tag{2.25}$$

$$u < t. \tag{2.26}$$

$$\alpha \text{ is not initialized at any stage } v \text{ with } u < v \leq t. \tag{2.27}$$

$$\text{No stage } v \text{ with } u < v < t \text{ is } \alpha\text{-expansionary.} \tag{2.28}$$

Then

$$\forall j' \in J_{\mathcal{L}}(j' \not\geq_{\mathcal{L}} j_0 \wedge j' \neq d(j) \to A_{j'}[u+1]\restriction u+1 = A_{j'}[t]\restriction u+1) \tag{2.29}$$

and

$$d(j) \neq j_0 \to A_{d(j)}[u+1]\restriction \min\{y, u+1\} = A_{d(j)}[t]\restriction \min\{y, u+1\}. \tag{2.30}$$

where y is the unique number that enters A_j at stage $u+1$.

Proof. The proof is by induction on u. Fix a, b, e such that $R_\alpha = M_{a,b,e}$. By (2.25) and our choice of y,

$$A_j[u+1] - A_j[u] = \{y\} \text{ and } A_{j'}[u] = A_{j'}[u+1] \text{ for } j' \neq j.$$

Moreover, at stage $u+1$, α imposes a target restraint for correction markers on $A_{J_{d(j)}}$ of length $u+1$, all strategies β with $\alpha 0 <_L \beta$ are initialized at stage $u+1$, and, since α is not initialized at any stage v with $u < v \leq t$, no diagonalization strategy β with $\beta \leq \alpha$ and no proper meet strategy β with $\beta 1 \leq \alpha$ acts at such a stage. So, the only strategies that can act at stage $u+1$ are strategies β with $\alpha 0 \subseteq \beta$ and the only strategies that can act at a stage $v+1$ with $u+1 < v+1 \leq t$ are proper meet strategies β with $\beta 0 \subseteq \alpha$, diagonalization strategies γ with $\alpha 0 \subseteq \gamma$ that are receiving attention via (2.18), and strategies η with $\alpha 0 <_L \eta$.

Now, for a contradiction, assume that (2.29) or (2.30) fails. Fix v with $t \geq v+1 > u+1$ minimal witnessing the failure of (2.29) or (2.30) and pick the unique j' and z such that

$$j' \not\geq_{\mathcal{L}} j_0 \wedge j' \neq d(j) \wedge z \leq u \wedge z \in A_{j'}[v+1] - A_{j'}[v] \tag{2.31}$$

or

$$j' = d(j) \wedge d(j) \neq j_0 \wedge z < \min\{y, u+1\} \wedge z \in A_{j'}[v+1] - A_{j'}[v]. \tag{2.32}$$

Since $j' \not\geq_{\mathcal{L}} j_0$, z is not a coding number. Moreover, since at stage $v+1$ only diagonalization strategies γ with $\alpha 0 <_L \gamma$, i.e., strategies which have been initialized at stage $u+1$, can become active via (2.17), z is too small to be a follower. So, z is a correction marker of a proper meet strategy β with $\beta 0 \subseteq \alpha$. (The correction markers of β's with $\alpha 0 <_L \beta$ are too big for z and no other proper meet strategies may act.)

Fix $c, d, e' = \langle e'_0, e'_1 \rangle$ such that $R_\beta = M_{c,d,e'}$ and let u' be the greatest β-expansionary stage $< v$. Note that, as $\beta 0 \subseteq \alpha$, $u' \geq u$. Let $z = \mathrm{cor}(\alpha, w)[u'+1]$. Then, $[e'_0]^{A_{J(c)}}(w)[u'] \downarrow = [e'_1]^{A_{J(d)}}(w)[u'] \downarrow$ and the use in both computations is $\leq z$. By the end of stage v, both of these computations have been destroyed, so we may take $m_c, m_d < z \leq u$ with $m_c \in A_{J(c)}[v] - A_{J(c)}[u']$ and $m_d \in A_{J(d)}[v] - A_{J(d)}[u']$. By minimality of v and the fact that $u' \geq u$, m_c either entered some $A_{j''}$ with $j'' \geq_{\mathcal{L}} j_0$, or m_c entered $A_{d(j)}$, or $m_c = y$ and m_c entered A_j at stage $u+1$. The same is true for m_d. We now consider six cases and derive a contradiction in each one. Often, the contradiction will be "self-correction", namely, showing that either m_c or m_d is in $A_{J(c) \cap J(d)}$. This is a contradiction because then β does not require attention at stage $v+1$ through w.

Case 1: $j_0 \in J(c) \cap J(d)$.
By Case 2.1 of the construction, z will enter A_{j_0}, a contradiction.

Case 2: $j_0 \notin J(c) \cup J(d)$.
Since $J(c)$ and $J(d)$ are downwards closed subsets of $J_{\mathcal{L}}$, no j'' with $j'' \geq_{\mathcal{L}} j_0$ is in $J(c)$ or $J(d)$. Thus, m_c enters either A_j or $A_{d(j)}$ and the same holds for m_d. Since $d(j) \leq_{\mathcal{L}} j$, $d(j) \in J(c) \cap J(d)$, so if either m_c or m_d enters $A_{d(j)}$, we get self-correction. If m_c, m_d both enter A_j (so $m_c = m_d = y$), then $j \in J(c) \cap J(d)$, and we again have self-correction.

Case 3: $j \in J(c) \cap J(d)$.
By Case 1 and symmetry, w.l.o.g. $j_0 \notin J(c)$ and, hence, no $j'' \geq_{\mathcal{L}} j_0$ is in $J(c)$. Thus, m_c enters either A_j or $A_{d(j)}$. Since $d(j) \leq_{\mathcal{L}} j$ and $j \in J(d)$, m_c also enters $A_{J(d)}$ and we have self-correction.

Case 4: $j \notin J(c) \cup J(d)$.
By Cases 1 and 2, w.l.o.g. j_0 is in exactly one of the sets $J(c), J(d)$, say $j_0 \in J(c) - J(d)$. Then, m_d enters $A_{d(j)}$. Since $d(j) \leq_{\mathcal{L}} j_0$, m_d is also in $A_{J(c)}$, and we have self- correction.

Case 5: $j, j_0 \in J(c) - J(d)$ or $j, j_0 \in J(d) - J(c)$.
By symmetry, we need only consider the former possibility. As in Case 4, m_d must enter $A_{d(j)}$ and hence also enters $A_{J(c)}$.

Case 6: Otherwise.
By symmetry, we may assume that

$$j_0 \in J(c) - J(d) \text{ and } j \in J(d) - J(c).$$

Since, for j'' with $j'' \geq_{\mathcal{L}} j_0$, $j'' \notin J(d)$, it follows from minimality of $v+1$ that only numbers $\geq \min\{y, u+1\}$ have entered $A_{J(d)}$ since the last β-expansionary stage u' and before stage $v+1$. Thus, $z \geq \min\{y, u+1\}$, so $\min\{y, u+1\} \leq z \leq u$. This implies $\min\{y, u+1\} = y$ and hence $z \geq y$. Thus, (2.32) fails.

Now, since α is not initialized after stage u and before stage $t+1$ and since u is the greatest α-expansionary stage less than t, the target restraint of α of length $u+1 > z$ imposed on $A_{J_{d(j)}}$ at stage $u+1$ is still in force at stage $v+1$. Moreover, by case assumption, $d(j) \in J(c) \cap J(d)$. So, the fact that z does not enter $A_{d(j)}$ implies that there is a minimal pair strategy α' with $\alpha' 1 < \alpha 1$ and with valid restraint of length $l > z$ on some $A_{d(j'')}$ with $j'' \neq j$.

Now, $\alpha' 1 < \alpha 1$ implies that either $\alpha' <_L \alpha$, $\alpha 0 \subseteq \alpha'$, or $\alpha' 1 \subseteq \alpha$. If $\alpha 0 \subseteq \alpha'$, then every α'-expansionary stage is an α-expansionary stage and hence there are no such stages $> u$ and $< v+1$. If either of the other two possibilities holds, then at any α'-expansionary stage, α would be initialized. Thus, if u'' is the greatest α'-expansionary stage $< v+1$, we have $u'' \leq u$ and hence, by $j'' \neq j$, $u'' < u$.

So, (2.23)-(2.28) hold for u'', v, α' and j'' in place of u, t, α and j. Since $y < l = u'' + 1$, it follows, by inductive hypothesis, from (2.29) (for u'', v, α' and j'' in place of u, t, α, and j) that y enters $A_{d(j'')}$ at stage $u+1$, whence $j = d(j'')$. So, $j \leq_{\mathcal{L}} j_0$, contrary to case assumption. □

Claim 6. For all $a, b \in L$ and $e = \langle e_0, e_1 \rangle$, $M_{a,b,e}$ is met.

Proof. Fix n and α such that $M_n = M_{a,b,e}$ and α is the strategy for M_n on the true path f. Moreover, w.l.o.g. assume that (2.21) holds so that, by Claim 4, $\alpha 0 \subseteq f$. By Claim 2, fix s_α minimal such that α is not initialized after this stage. By $\alpha 0 \subseteq f$, there are infinitely many stages s with $\alpha 0 \subseteq \delta[s]$. So, for each x, we can let $s_{\alpha,x}$ be the least stage $\geq s_\alpha$ such that $\alpha 0 \subseteq \delta[s_{\alpha,x}]$ and $l_\alpha[s_{\alpha,x}] > x$. Now, distinguish the following two cases depending on whether M_n is a proper meet or minimal pair requirement.

Case 1: $J(a) \cap J(b) \neq \emptyset$.
By choice of $s_{\alpha,x}$,

$$[e_0]^{A_{J(a)}}(x)[s_{\alpha,x}] = [e_1]^{A_{J(b)}}(x)[s_{\alpha,x}] \downarrow$$

and the set

$$\mathrm{COR}(\alpha, x)[s_{\alpha,x} + 1] = \{\langle s_{\alpha,x} + 1, 2n, x, y \rangle : 0 \leq y \leq s\}$$

of correction markers defined at stage $s_{\alpha,x} + 1$ will be permanent. So for any two consecutive α-expansionary stages u and v with $s_{\alpha,x} \leq u < v$,

$$[e_0]^{A_{J(a)}}(x)[u] = [e_1]^{A_{J(b)}}(x)[u] = [e_0]^{A_{J(a)}}(x)[v] = [e_1]^{A_{J(b)}}(x)[v]$$

unless

$$A_{J(a) \cap J(b)}[u] \restriction m(x) \neq A_{J(a) \cap J(b)}[v] \restriction m(x)$$

where

$$m(x) = \max(\mathrm{COR}(\alpha, x)[s_{\alpha,x} + 1]) + 1.$$

So,

$$g(x) = [e_0]^{A_{J(a)}}(x) = [e_0]^{A_{J(a)}}(x)[t_x],$$

where

$$t_x = \mu s > s_{\alpha,x}(A_{J(a)\cap J(b)}[s]\restriction m(x) = A_{J(a)\cap J(b)}\restriction m(x) \text{ and } s \text{ is } \alpha\text{-expansionary}),$$

whence, $g \leq_{wtt} A_{J(a)\cap J(b)}$.

Case 2: $J(a) \cap J(b) = \emptyset$.
We will show that

$$g(x) = [e_0]^{A_{J(a)}}(x)[s_{\alpha,x}]. \tag{2.33}$$

Obviously, this will imply that g is recursive.

To prove (2.33), it suffices to show that for any two consecutive α-expansionary stages u and v with $v > u \geq s_{\alpha,x}$,

$$[e_0]^{A_{J(a)}}(x)[u] = [e_0]^{A_{J(a)}}(x)[v].$$

Since, by choice of v and u,

$$[e_0]^{A_{J(a)}}(x)[u] = [e_1]^{A_{J(b)}}(x)[u] \text{ and } [e_0]^{A_{J(a)}}(x)[v] = [e_1]^{A_{J(b)}}(x)[v]$$

this will follow from

$$A_{J(a)}[u]\restriction u+1 = A_{J(a)}[v]\restriction u+1 \text{ or } A_{J(b)}[u]\restriction u+1 = A_{J(b)}[v]\restriction u+1.$$

But this is immediate by Claim 5. (Since $J(a)\cap J(b) = \emptyset$, if $u+1$ is a j-stage, then for some $c \in \{a, b\}$, $d(j) \notin J(c)$. If $j' \geq_{\mathcal{L}} j_0$, then $j' \geq_{\mathcal{L}} d(j)$, so $j' \notin J(c)$. Thus, by (2.29) and the fact that $j \notin J(c)$ (so no number enters $A_{J(c)}$ at stage $u+1$), $A_{J(c)}[u]\restriction u+1 = A_{J(c)}[v]\restriction u+1$.) □

3. The Two-Quantifier Decision Procedure

In this section, we show how the characterization given in the last section of the finite lattices lattice-embeddable into $\mathcal{R}_{wtt}$ preserving 0 and 1, together with an already known extension-of-embeddings result, can be used to give a decision procedure for the two-quantifier theory of $\mathcal{R}_{wtt}$ in the language $\{\leq, 0, 1\}$. We will in fact formulate general conditions under which a distributive upper semi-lattice has a decidable two-quantifier theory and we will use these general conditions to show, using results already in the literature, that several complexity-theoretic structures also have decidable two-quantifier theories.

In order to give our decision procedure, we must first develop some algebraic background.

Lemma 5. *Let $\mathcal{U} = (U, \leq_{\mathcal{U}})$ be an upper semi-lattice and let S be a finite subset of U. Then, the closure of S in U under join is finite.*

Proof. Let $S' = \{\bigvee F | \emptyset \neq F \subseteq S\}$. Then S' is closed under join, since $(\bigvee F) \vee (\bigvee F') = \bigvee(F \cup F')$, and contains S, so it follows that S' is the closure of S under join, and S' is finite. □

The following result is known in lattice theory. (It is for instance Exercise 2 on page 146 of [8] and follows from Theorem 1 on page 80 of [13].)

Lemma 6. *Let $\mathcal{L}$ be a distributive lattice and let S be a finite subset of L which generates L under join and meet. Then, $|L| \leq 2^{2^{|S|}}$.*

□

If $\mathcal{U} = (U, \leq_{\mathcal{U}})$ is an upper semi-lattice, we let $\mathrm{ID}_{\mathcal{U}}$ denote the set of ideals of $\mathcal{U}$ and $\mathcal{I}_{\mathcal{U}}$ denote the structure $(\mathrm{ID}_{\mathcal{U}}, \subseteq)$. For each $a \in U$, $\downarrow a$ denotes $\{b \in U | b \leq_{\mathcal{U}} a\}$, which is easily seen to be in $\mathrm{ID}_{\mathcal{U}}$. The ideal $\downarrow a$ is called the *principal ideal generated by a*.

Part (a) of the following lemma, in the case that $\mathcal{U}$ is a distributive lattice, goes back to Stone [24]. (See for example Theorem 7 on page 141 of [8].) It is no harder to show the result when $\mathcal{U}$ is just a distributive upper semi-lattice, and this is done in the proof of Proposition VI.1.11 in Odifreddi [19]. The other two parts are easy to show.

Lemma 7. *Let $\mathcal{U}$ be a distributive upper semi-lattice with least element. Then:*

(a) *$\mathcal{I}_{\mathcal{U}}$ is a distributive lattice. If $I, J \in ID_{\mathcal{U}}$, then $I \wedge_{\mathcal{I}_{\mathcal{U}}} J = I \cap J$ and $I \vee_{\mathcal{I}_{\mathcal{U}}} J = \{a \vee_{\mathcal{U}} b | a \in I$ and $b \in J\}$.*
(b) *$\mathcal{I}_{\mathcal{U}}$ has least element $\{0_{\mathcal{U}}\}$ and greatest element U.*
(c) *The mapping $\rho : U \to \mathcal{I}_{\mathcal{U}}$ given by $\rho(a) = \downarrow a$ is a usl embedding which preserves 0 and 1.*

□

Lemma 8. *Let $\mathcal{U}$ be an upper semi-lattice with least and greatest elements and let I be an ideal of $\mathcal{U}$.*

(a) *If I is cuppable (in $\mathcal{I}_{\mathcal{U}}$), then some element of I is cuppable (in $\mathcal{U}$).*
(b) *If I is cappable (in $\mathcal{I}_{\mathcal{U}}$), then every element of I is cappable (in $\mathcal{U}$).*

Proof. First suppose that $I \in \mathrm{ID}_{\mathcal{U}}$ is cuppable in $\mathcal{I}_{\mathcal{U}}$, say $I \vee_{\mathcal{I}_{\mathcal{U}}} J = U$ with $J \in \mathrm{ID}_{\mathcal{U}}$, $J \neq U$. Then $1_{\mathcal{U}} \in I \vee_{\mathcal{I}_{\mathcal{U}}} J$, so $1_{\mathcal{U}} = a \vee_{\mathcal{U}} b$ for some $a \in I, b \in J$. Since $b \in J$, $b \neq 1_{\mathcal{U}}$, so $a \in I$ is cuppable.

Now, suppose that $I \in \mathrm{ID}_{\mathcal{U}}$ is cappable. Then, there is $J \in \mathrm{ID}_{\mathcal{U}}$, $J \neq \{0_{\mathcal{U}}\}$, with $I \cap J = \{0_{\mathcal{U}}\}$. Take $b \in J$, $b \neq 0_{\mathcal{U}}$. For every $a \in I$, if $c \in U$ and $c \leq_{\mathcal{U}} a, b$, then, since I and J are closed downwards, $c \in I \cap J$, so $c = 0_{\mathcal{U}}$. Thus, $a \wedge_{\mathcal{U}} b = 0_{\mathcal{U}}$, which means that a is cappable. □

Lemma 9. *In $\mathcal{I}_{\mathcal{R}_{wtt}}$, $I(CAP) \cap F(CUP) = \emptyset$.*

Proof. Let $\mathcal{I} = \mathcal{I}_{\mathcal{R}_{wtt}}$ and suppose, for a contradiction, that I is an ideal of $\mathcal{R}_{wtt}$ that is in $I(CAP_{\mathcal{I}}) \cap F(CUP_{\mathcal{I}})$. Since $I \in I(CAP_{\mathcal{I}})$, there are ideals $J_1, \ldots, J_k$ $(k > 0)$ of $\mathcal{R}_{wtt}$ that are cappable in $\mathcal{I}$ such that $I \subseteq J_1 \vee_{\mathcal{I}} \cdots \vee_{\mathcal{I}} J_k$. Hence, every element of I can be expressed as $a_1 \vee_{wtt} \cdots \vee_{wtt} a_k$ with $a_i \in J_i$ for $1 \leq i \leq k$. By Lemma 8(b), each a_i is cappable in $\mathcal{R}_{wtt}$. Since, by Lemma 3, the cappable elements of $\mathcal{R}_{wtt}$ are an ideal, it follows that each element of I is cappable.

Since $I \in F(CUP_{\mathcal{I}})$, there are ideals $K_1, \ldots, K_r$ $(r > 0)$ of $\mathcal{R}_{wtt}$ that are cuppable in $\mathcal{I}$ such that $K_1 \cap \cdots \cap K_r \subseteq I$. By Lemma 8(a), each K_i, $1 \leq i \leq r$, contains an element cuppable in $\mathcal{R}_{wtt}$, say b_i. Since, by Lemma 3, each cuppable element of $\mathcal{R}_{wtt}$ is noncappable, each b_i is noncappable. We claim that $K_1 \cap \cdots \cap K_r$ contains a noncappable element. Indeed, set $c_1 = b_1$ and suppose that $1 \leq j < r$ and we have c_j a noncappable element of $\mathcal{R}_{wtt}$ with $c_j \in K_\ell$ for $1 \leq \ell \leq j$. Since, again by Lemma 3, the noncappable elements of $\mathcal{R}_{wtt}$ from a strong filter, there is a noncappable element c_{j+1} of $\mathcal{R}_{wtt}$ with $c_{j+1} \leq_{wtt} c_j, b_{j+1}$. Since the K_i's are all ideals of $\mathcal{R}_{wtt}$, $c_{j+1} \in K_\ell$ for $1 \leq \ell \leq j+1$. The element c_r constructed by this process is noncappable and is in $K_1 \cap \cdots \cap K_r \subseteq I$. This contradicts the conclusion of the previous paragraph that every element of I is cappable. $\square$

Lemma 10. *Let $\mathcal{U}$ be an upper semi-lattice with least and greatest elements such that the diamond lattice can be lattice-embedded into $\mathcal{I}_{\mathcal{U}}$ preserving 0 and 1. Then, the diamond lattice can be lattice-embedded into $\mathcal{U}$ preserving 0 and 1.*

Proof. If the diamond lattice can be lattice-embedded into $\mathcal{I}_{\mathcal{U}}$ preserving 0 and 1, there are ideals I and J of $\mathcal{U}$, neither equal to U, such that $I \vee_{\mathcal{I}_{\mathcal{U}}} J = U$ and $I \cap J = \{0_{\mathcal{U}}\}$. There must be $x \in I, y \in J$ with $x \vee_{\mathcal{U}} y = 1_{\mathcal{U}}$ and, since I, J are not equal to U, x, y are not equal to $1_{\mathcal{U}}$. If $z \in \mathcal{U}$ is such that $z \leq_{\mathcal{U}} x, y$, then $z \in I \cap J$, so $z = 0_{\mathcal{U}}$. Thus, $x \wedge_{\mathcal{U}} y = 0_{\mathcal{U}}$. It follows that the diamond lattice can be lattice-embedded into $\mathcal{U}$ preserving 0 and 1. $\square$

The following pullback lemma is due to Ershov [10]. It is Proposition VI.1.12 of Odifreddi [19].

Lemma 11. *If $\mathcal{U}$ is a distributive upper semi-lattice with least element, S is a nonempty finite subset of U closed under join, and $\mathcal{L} = (L, \leq_{\mathcal{I}_{\mathcal{U}}} \restriction L)$ is the sublattice of $\mathcal{I}_{\mathcal{U}}$ generated by $\rho(S)$ (where ρ is the canonical embedding of Lemma 7(c)), then there is a sub-upper semi-lattice $\mathcal{L}'$ of $\mathcal{U}$ such that $S \subseteq L'$ and $\mathcal{L}'$ is isomorphic to $\mathcal{L}$ by an isomorphism that extends $\rho \restriction S$.*

$\square$

An important step in determining the two-quantifier theory of a poset $\mathcal{U}$ is to solve an extension-of-embeddings problem appropriate for $\mathcal{U}$. When $\mathcal{U}$ has distinct least and greatest elements, the appropriate extension-of-embeddings problem for $\mathcal{U}$ is the 0,1-extension-of-embeddings problem, which we now describe. An *instance of the* $0,1$*-extension-of-embeddings problem* is a pair $(\mathcal{X}, \mathcal{Y})$ of finite bounded posets such that $\mathcal{X} \subseteq_{0,1} \mathcal{Y}$ and $0_{\mathcal{X}} \neq 1_{\mathcal{X}}$. If $\mathcal{U}$ is a bounded poset, a *positive instance of the* $0,1$*-extension-of-embeddings problem for* $\mathcal{U}$ is an instance $(\mathcal{X}, \mathcal{Y})$ of the 0,1-extension-of-embeddings problem such that every partial-order embedding of $\mathcal{X}$ into $\mathcal{U}$ preserving 0 and 1 can be extended to a partial-order embedding of $\mathcal{Y}$ into $\mathcal{U}$. In Fejer-Shore [12], it is shown that an instance $(\mathcal{X}, \mathcal{Y})$ of the 0,1-extension-of-embeddings problem is a positive instance for $\mathcal{R}_{wtt}$ if and only if the following condition is met:

(3.1) There are no subsets A and B of X such that, in $\mathcal{X}$, every upper bound for A is greater than or equal to every lower bound for B, but, in $\mathcal{Y}$, there is an upper bound z for A and a lower bound z' for B such that $z' \not\leq_{\mathcal{Y}} z$.

If $(\mathcal{X}, \mathcal{Y})$ is an instance of the 0,1-extension-of-embeddings problem and $\mathcal{X}$ is a lattice, we claim that condition (3.1) is equivalent to the following condition:

(3.2) For all $x_1, x_2 \in X$, and $y \in Y$, if $x_1, x_2 \leq_{\mathcal{Y}} y$, then $x_1 \vee_{\mathcal{X}} x_2 \leq_{\mathcal{Y}} y$, and, if $x_1, x_2 \geq_{\mathcal{Y}} y$, then $x_1 \wedge_{\mathcal{X}} x_2 \geq_{\mathcal{Y}} y$.

To see this, first suppose that (3.1) holds. If $x_1, x_2 \in X, y \in Y$ and $x_1, x_2 \leq_{\mathcal{Y}} y$, take $A = \{x_1, x_2\}, B = \{x_1 \vee_{\mathcal{X}} x_2\}$. Then, in $\mathcal{X}$, every upper bound for A is greater than or equal to every lower bound for B, and, in $\mathcal{Y}$, y is an upper bound for A and $x_1 \vee_{\mathcal{X}} x_2$ is a lower bound for B, so, by (3.1), $x_1 \vee_{\mathcal{X}} x_2 \leq_{\mathcal{Y}} y$. The other half of (3.2) is shown similarly.

Conversely, suppose that condition (3.2) holds and that A and B are subsets of $\mathcal{X}$ such that, in $\mathcal{X}$, every upper bound for A is greater than or equal to every lower bound for B. Then, $\bigvee_{\mathcal{X}} A \geq_{\mathcal{X}} \bigwedge_{\mathcal{X}} B$. If, in $\mathcal{Y}$, z is an upper bound for A and z' is a lower bound for B, then, by (3.2), $z \geq_{\mathcal{Y}} \bigvee_{\mathcal{X}} A \geq_{\mathcal{Y}} \bigwedge_{\mathcal{X}} B \geq_{\mathcal{Y}} z'$. Thus, (3.1) holds.

From the point of view of two-quantifier decision procedures, the important facts about condition (3.2) are that it is effective and that the following lemma holds.

Lemma 12. *Let $(\mathcal{X}, \mathcal{Y})$ be a pair of finite partial orders such that $\mathcal{X} \subseteq \mathcal{Y}$ and $\mathcal{X}$ is a lattice, let $\mathcal{U}$ be a poset and let f be a lattice embedding of $\mathcal{X}$ into $\mathcal{U}$. Then, if there is a poset embedding f' of $\mathcal{Y}$ into $\mathcal{U}$ that extends f, $(\mathcal{X}, \mathcal{Y})$ satisfies* (3.2).

Proof. The result is immediate. $\square$

With these lemmas out of the way, we can state a general result which gives a decision method for the two-quantifier theory of many of the bounded distributive upper semi-lattices that occur in recursion and complexity theory.

Theorem 13. *Let $\mathcal{U}$ be a bounded, distributive upper semi-lattice such that*

- *if $\mathcal{X}$ is a finite lattice that can be lattice-embedded into $\mathcal{U}$ preserving* 0 *and* 1 *and $(\mathcal{X}, \mathcal{Y})$ is an instance of the* 0,1*-extension-of-embeddings problem satisfying* (3.2)*, then $(\mathcal{X}, \mathcal{Y})$ is a positive instance of the* 0,1*-extension-of-embeddings problem for $\mathcal{U}$,*
- *if a finite lattice can be lattice-embedded into $\mathcal{I}_{\mathcal{U}}$ preserving* 0 *and* 1*, then it can be lattice-embedded into $\mathcal{U}$ preserving* 0 *and* 1*,*

and let $\varphi = \forall x_1 \cdots \forall x_n \exists y_1 \cdots \exists y_m \psi$ be a sentence over the language $\{\leq, 0, 1\}$ with ψ quantifier-free and $x_1, \ldots, x_n, y_1, \ldots, y_m$ all distinct. Then, $\mathcal{U} \models \varphi$ if and only if the following condition is met:

for every finite lattice $\mathcal{L}$ that can be lattice- embedded into $\mathcal{U}$ preserving 0 *and* 1 *and each n-tuple $\vec{a} = (a_1, \ldots, a_n)$ of elements of L such that $\{a_1, \ldots, a_n, 0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ generates L under join and meet, there exists a bounded poset $\mathcal{P}$ and an m-tuple $\vec{b} = (b_1, \ldots, b_m)$ of elements of P such that:* (3.3)

- *$|P - L| \leq m$,*
- *$\mathcal{L} \subseteq_{0,1} \mathcal{P}$,*
- *$(\mathcal{L}, \mathcal{P})$ satisfies* (3.2)*, and*
- *$\mathcal{P} \models \psi[\vec{a}, \vec{b}]$.*

Proof. First suppose that $\mathcal{U} \models \varphi$. We show that (3.3) holds. Let $\mathcal{L}$ be a finite lattice that can be lattice-embedded into $\mathcal{U}$ preserving 0 and 1, and let $\vec{a} = (a_1, \ldots, a_n)$ be an n-tuple of elements of L. Fix a lattice embedding f of $\mathcal{L}$ into $\mathcal{U}$ that preserves 0 and 1. Since $\mathcal{U} \models \varphi$, there is an m-tuple $\vec{b}' = (b'_1, \ldots, b'_m)$ of elements of $\mathcal{U}$ such that $\mathcal{U} \models \psi[f(\vec{a}), \vec{b}']$. Let $P' = f(L) \cup \{b'_1, \ldots, b'_m\}$ and define $\mathcal{P}' = (P', \leq_{\mathcal{U}} \restriction P')$. Since f preserves 0 and 1, $0_{\mathcal{U}}, 1_{\mathcal{U}} \in P'$, so $\mathcal{P}'$ is a substructure of $\mathcal{U}$ when $\mathcal{U}$ is considered as a structure for the language $\{\leq, 0, 1\}$. Since ψ is quantifier-free, $\mathcal{P}' \models \psi[f(\vec{a}), \vec{b'}]$. We now construct $\mathcal{P}$ to be an isomorphic copy of $\mathcal{P}'$ which contains $\mathcal{L}$ in the same way that $\mathcal{P}'$ contains $f(L)$. To be precise, let T be a set of objects not in L of cardinality $|\{b'_i | 1 \leq i \leq m \text{ and } b'_i \notin f(L)\}|$ and let $g : T \to \{b'_i | 1 \leq i \leq m \text{ and } b'_i \notin f(L)\}$ be a bijection. Let $P = L \cup T$ and define $f' : P \to P'$ by

$$f'(z) = \begin{cases} f(z) & \text{if } z \in L \\ g(z) & \text{if } z \in T. \end{cases}$$

Then, f' is a bijection. Define $\mathcal{P} = (P, \leq_{\mathcal{P}})$ where $z \leq_{\mathcal{P}} w$ if and only if $f'(z) \leq_{\mathcal{U}} f'(w)$ and let $\vec{b} = ((f')^{-1}(b'_1), \ldots, (f')^{-1}(b'_m))$. We show that $\mathcal{P}$ and $\vec{b}$ are as desired. We have $\mathcal{P}$ a bounded poset and $\vec{b}$ an m-tuple of elements of P. We also have $L \subseteq P$ and $|P - L| = |T| \leq m$. If $x, y \in L$, then $x \leq_{\mathcal{P}} y$ is equivalent to $f'(x) \leq_{\mathcal{U}} f'(y)$, which in turn is equivalent to $f(x) \leq_{\mathcal{U}} f(y)$ and, since f is a lattice embedding, this last is equivalent to $x \leq_{\mathcal{L}} y$. Thus, $\mathcal{L} \subseteq \mathcal{P}$. Since f preserves 0 and 1, $0_{\mathcal{P}} = 0_{\mathcal{L}}$ and $1_{\mathcal{P}} = 1_{\mathcal{L}}$, so $\mathcal{L} \subseteq_{0,1} \mathcal{P}$. Also, the lattice embedding f of $\mathcal{L}$ into $\mathcal{U}$ can be extended to a poset embedding f' of $\mathcal{P}$ into $\mathcal{U}$, so, by Lemma 12, $(\mathcal{L}, \mathcal{P})$ satisfies (3.2). Finally, $\mathcal{P}$ is isomorphic to $\mathcal{P}'$ via f', $\mathcal{P}' \models \psi[f(\vec{a}), \vec{b'}]$, $f'(a_i) = f(a_i)$ (since $a_i \in L$) for $1 \leq i \leq n$, and $f'(b_i) = f'((f')^{-1}(b'_i)) = b'_i$ for $1 \leq i \leq m$, so $\mathcal{P} \models \psi[\vec{a}, \vec{b}]$, as desired.

Now, suppose that (3.3) holds. We show that $\mathcal{U} \models \varphi$, i.e., that for every n-tuple $\vec{a}' = (a'_1, \ldots, a'_n)$ of elements of $\mathcal{U}$, there is an m-tuple $\vec{b'} = (b'_1, \ldots, b'_m)$ of elements of $\mathcal{U}$ such that $\mathcal{U} \models \psi[\vec{a}', \vec{b'}]$. Let $\vec{a}' = (a'_1, \ldots, a'_n)$ be an n-tuple of elements of $\mathcal{U}$ and let S be the closure in $\mathcal{U}$ of $\{a'_1, \ldots, a'_n, 0_{\mathcal{U}}, 1_{\mathcal{U}}\}$ under join. By Lemma 5, S is finite. Let ρ be the canonical embedding of $\mathcal{U}$ into $\mathcal{I}_{\mathcal{U}}$ and let $\mathcal{L} = (L, \subseteq)$ be the sublattice of $\mathcal{I}_{\mathcal{U}}$ generated by $\rho(S)$. By Lemma 6 and the fact that $\mathcal{I}_{\mathcal{U}}$ is distributive, $\mathcal{L}$ is finite. Since S contains $0_{\mathcal{U}}$ and $1_{\mathcal{U}}$ and ρ preserves 0 and 1, L contains $0_{\mathcal{I}_{\mathcal{U}}}$ and $1_{\mathcal{I}_{\mathcal{U}}}$, so the identity map is a lattice embedding of $\mathcal{L}$ into $\mathcal{I}_{\mathcal{U}}$ preserving 0 and 1. By the second hypothesis on $\mathcal{U}$, $\mathcal{L}$ can be lattice-embedded into $\mathcal{U}$ preserving 0 and 1. Define an n- tuple $\vec{a} = (a_1, \ldots, a_n)$ of elements of L by $a_i = \rho(a'_i)$. Since $\{a'_1, \ldots, a'_n, 0_{\mathcal{U}}, 1_{\mathcal{U}}\}$ generates S under join, ρ preserves joins, 0, and 1, and $\rho(S)$ generates L under join and meet, it follows easily that $\{a_1, \ldots, a_n, 0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ generates L under join and meet. Thus, by (3.3), there is a bounded poset $\mathcal{P} = (P, \leq_{\mathcal{P}})$ and an m-tuple of elements $\vec{b} = (b_1, \ldots, b_m)$ of P such that $|P - L| \leq m$, $\mathcal{L} \subseteq_{0,1} \mathcal{P}$, $(\mathcal{L}, \mathcal{P})$ meets condition (3.2) and $\mathcal{P} \models \psi[\vec{a}, \vec{b}]$. Then, $(\mathcal{L}, \mathcal{P})$ is an instance of the 0,1-extension-of- embeddings problem satisfying condition (3.2), so, $(\mathcal{L}, \mathcal{P})$ is a positive instance for $\mathcal{U}$, by the first hypothesis on $\mathcal{U}$. By Lemma 11, there is a subset L' of U which contains S and a function $\rho' : L' \to L$ which extends $\rho \upharpoonright S$ and is an isomorphism of $(L', \leq_{\mathcal{U}} \upharpoonright L')$ with $\mathcal{L}$. Since $(\rho')^{-1} : L \to U$ is a poset embedding (although not necessarily a lattice embedding) of $\mathcal{L}$ into $\mathcal{U}$ which preserves 0 and 1 and $(\mathcal{L}, \mathcal{P})$ is a positive instance of the 0,1-extension-of-embeddings problem for $\mathcal{U}$, there is a poset embedding $\rho'' : P \to U$ of $\mathcal{P}$ into $\mathcal{U}$ which extends $(\rho')^{-1}$. Let $\vec{b'} = (\rho''(b_1), \ldots, \rho''(b_m))$. Since ψ is quantifier-free and ρ'' is a poset embedding which preserves 0 and 1, $\mathcal{U} \models \psi[\rho''(\vec{a}), \rho''(\vec{b})]$. By definition, $\rho''(\vec{b}) = \vec{b'}$. For $1 \leq i \leq n$, $a_i = \rho(a'_i)$ and $a'_i \in S$. Since ρ' extends $\rho \upharpoonright S$, $a_i = \rho'(a'_i)$ and $a_i \in L'$. Since ρ'' extends $(\rho')^{-1}$, $\rho''(a_i) = (\rho')^{-1}(\rho'(a'_i)) = a'_i$. Thus, $\rho''(\vec{a}) = \vec{a}'$ and $\mathcal{U} \models \psi[\vec{a}', \vec{b'}]$, as desired. $\square$

Corollary 14. *If $\mathcal{U}$ is a bounded distributive upper semi-lattice such that*

- *the set of finite lattices that can be lattice-embedded into $\mathcal{U}$ preserving* 0 *and* 1 *is decidable,*
- *if $\mathcal{X}$ is a finite lattice that can be lattice-embedded into $\mathcal{U}$ preserving* 0 *and* 1 *and $(\mathcal{X}, \mathcal{Y})$ is an instance of the* 0,1*-extension-of-embeddings problem satisfying* (3.2)*, then $(\mathcal{X}, \mathcal{Y})$ is a positive instance of the* 0,1*-extension-of-embeddings problem for $\mathcal{U}$,*
- *if a finite lattice can be lattice-embedded into $\mathcal{I}_{\mathcal{U}}$ preserving* 0 *and* 1*, then it can be lattice-embedded into $\mathcal{U}$ preserving* 0 *and* 1*,*

then the two-quantifier theory of $\mathcal{U}$ in the language $\{\leq, 0, 1\}$ is decidable.

Proof. If $\mathcal{U}$ is such an upper semi-lattice, Theorem 13 applies to it. Any $\forall\exists$ sentence of the language $\{\leq, 0, 1\}$ can be effectively translated into one of the form required in Theorem 13 by dropping redundant quantifiers. Thus, we need only to verify that condition (3.3) can be tested effectively. Since $\mathcal{U}$ is distributive, every lattice which is lattice-embeddable into $\mathcal{U}$ must be distributive. By Lemma 6, if $\mathcal{L}$ is a distributive lattice generated under join and meet by $n+2$ elements, then we get a recursive bound on the size of $\mathcal{L}$. Thus, there are only finitely many $\mathcal{L}$ to check, and, since we are assuming that the class of finite lattices lattice-embeddable into $\mathcal{U}$ preserving 0 and 1 is decidable, we can effectively find all the $\mathcal{L}$ and $\vec{a}$ we need to check. For each such $\mathcal{L}$ and $\vec{a}$, the test for the existence of the required $\mathcal{P}$ is effective, since $|P - L| \leq m$ and condition (3.2) can be checked effectively. $\square$

Theorem 15. *The two-quantifier theory of $\mathcal{R}_{wtt}$ in the language $\{\leq, 0, 1\}$ is decidable.*

Proof. We want to apply Corollary 14. We have $\mathcal{R}_{wtt}$ a bounded distributive upper semi-lattice. Theorem 1 shows that the set of finite lattices that are lattice-embeddable into $\mathcal{R}_{wtt}$ preserving 0 and 1 is decidable and it is shown in [12] that the positive instances of the 0,1-extension-of-embeddings problem for $\mathcal{R}_{wtt}$ are exactly those satisfying (3.1). As discussed previously, this gives the second condition of Corollary 14. Thus, we only need to show that $\mathcal{R}_{wtt}$ meets the third condition. Let $\mathcal{I}$ be $\mathcal{I}_{\mathcal{R}_{wtt}}$, let $\mathcal{L}$ be a finite lattice that can be lattice-embedded into $\mathcal{I}$ preserving 0 and 1 and let f be such an embedding of $\mathcal{L}$. Then, f maps $\mathrm{CAP}_{\mathcal{L}}$ into $\mathrm{CAP}_{\mathcal{I}}$ and $\mathrm{CUP}_{\mathcal{L}}$ into $\mathrm{CUP}_{\mathcal{I}}$, so f maps $I(\mathrm{CAP}_{\mathcal{L}})$ into $I(\mathrm{CAP}_{\mathcal{I}})$ and $F(\mathrm{CUP}_{\mathcal{L}})$ into $F(\mathrm{CUP}_{\mathcal{I}})$. Since, by Lemma 9, $I(\mathrm{CAP}_{\mathcal{I}}) \cap F(\mathrm{CUP}_{\mathcal{I}}) = \emptyset$, $I(\mathrm{CAP}_{\mathcal{L}}) \cap F(\mathrm{CUP}_{\mathcal{L}}) = \emptyset$. In addition, since $\mathcal{I}$ is distributive, $\mathcal{L}$ is distributive. Thus, by Theorem 1, $\mathcal{L}$ can be lattice-embedded into $\mathcal{R}_{wtt}$ preserving 0 and 1, as desired. $\square$

We are now going to apply Corollary 14 to some complexity-theoretic structures $\mathcal{U}$, specifically, to ideals of the *pm*-degrees of the recursive sets. In [2], it is shown that the upper semi-lattice of the *pm*-degrees of the recursive sets is distributive. In [21], the extension-of-embeddings problem for the structure of the *pm*-degrees of the recursive sets is taken up. There, Shore and Slaman show that if $\mathcal{X}$ and $\mathcal{Y}$ are finite posets with least element, $\mathcal{X} \subseteq_0 \mathcal{Y}$, $\mathcal{X}$ is a lattice and $(\mathcal{X}, \mathcal{Y})$ satisfies (3.2), then any poset embedding of $\mathcal{X}$ into the *pm*-degrees of the recursive sets preserving 0 can be extended to a poset embedding of $\mathcal{Y}$ into this structure.

Theorem 16. *The structures of the* pm*-degrees of the exponential-time computable sets and the* pm*-degrees of the exponential-space computable sets have decidable two-quantifier theories in the language* $\{\leq, 0, 1\}$.

Proof. Let $\mathcal{U}$ stand for either of these structures. As discussed above, $\mathcal{U}$ is a distributive upper semi-lattice. The existence of complete problems for the exponential-time and exponential-space computable sets under pm-reducibility is well-known. (See for instance Exercise 21 on page 96 of [7] for a complete exponential-time computable set and page 353 of [14] for a complete exponential-space computable set.) Thus, $\mathcal{U}$ is a bounded distributive upper semi-lattice. It follows from results in the literature that the finite lattices that are lattice-embeddable into $\mathcal{U}$ preserving 0 and 1 are exactly those finite lattices $\mathcal{L}$ such that $\mathcal{L}$ is distributive, $\mathcal{L}$ has more than one element, and the diamond lattice cannot be lattice-embedded into $\mathcal{L}$ preserving 0 and 1. Indeed, in [3], it is shown that any such lattice can be lattice-embedded into $\mathcal{U}$ preserving 0 and 1, while, in [4], it is shown that the diamond lattice cannot be lattice-embedded into $\mathcal{U}$ preserving 0 and 1, which implies that no finite lattice not in the given class can be lattice-embedded into $\mathcal{U}$ preserving 0 and 1.

If $(\mathcal{X}, \mathcal{Y})$ is an instance of the 0,1-extension-of-embeddings problem with $\mathcal{X}$ a lattice, $(\mathcal{X}, \mathcal{Y})$ satisfies (3.2) and f is a poset embedding of $\mathcal{X}$ into $\mathcal{U}$, then, by the result of Shore and Slaman mentioned above, there is a poset embedding f' of $\mathcal{Y}$ into the pm-degrees of the recursive sets that extends f. Since $1_{\mathcal{Y}} = 1_{\mathcal{X}}$, f' is actually an embedding of $\mathcal{Y}$ into $\mathcal{U}$, so $(\mathcal{X}, \mathcal{Y})$ is a positive instance of the 0,1-extension-of-embeddings problem for $\mathcal{U}$.

Thus, we have the first two conditions on $\mathcal{U}$ needed to apply Corollary 14 and all that is left is to show that if $\mathcal{L}$ is a finite lattice that can be lattice-embedded into $\mathcal{I}_{\mathcal{U}}$ preserving 0 and 1, then $\mathcal{L}$ can be lattice-embedded into $\mathcal{U}$ preserving 0 and 1, i.e, $\mathcal{L}$ is distributive, $\mathcal{L}$ has at least two elements and the diamond lattice cannot be lattice-embedded into $\mathcal{L}$ preserving 0 and 1. The first two of these conclusions are immediate. If the diamond lattice could be lattice-embedded into $\mathcal{L}$ preserving 0 and 1, then it could be lattice-embedded into $\mathcal{I}_{\mathcal{U}}$ preserving 0 and 1, and then, by Lemma 10, the diamond lattice could be lattice-embedded into $\mathcal{U}$ preserving 0 and 1, contradicting the characterization of embeddable lattices given earlier. □

Many structures in recursion and complexity theory do not have greatest elements. A version of Theorem 13 and Corollary 14 can be obtained for such structures as well, if we consider a slightly different extension-of-embeddings problem. An *instance of the* 0*-extension-of-embeddings problem* is a pair $(\mathcal{X}, \mathcal{Y})$ of finite posets with least element such that $\mathcal{X} \subseteq_0 \mathcal{Y}$. If $\mathcal{U}$ is a poset with least element, a *positive instance of the* 0*-extension-of-embeddings problem for* $\mathcal{U}$ is an instance $(\mathcal{X}, \mathcal{Y})$ of the problem such that every poset embedding of $\mathcal{X}$ into $\mathcal{U}$ that preserves 0 can be extended to a poset embedding of $\mathcal{Y}$ into $\mathcal{U}$.

Theorem 17. *Let* $\mathcal{U}$ *be a distributive upper semi-lattice with least element such that*

- *if* $\mathcal{X}$ *is a finite lattice that can be lattice-embedded into* $\mathcal{U}$ *preserving* 0 *and* $(\mathcal{X}, \mathcal{Y})$ *is an instance of the* 0*-extension-of-embeddings problem satisfying* (3.2)*, then* $(\mathcal{X}, \mathcal{Y})$ *is a positive instance of the* 0*-extension-of-embeddings problem for* $\mathcal{U}$*,*

- *if a finite lattice can be lattice-embedded into $\mathcal{I}_{\mathcal{U}}$ preserving* 0*, then it can be lattice-embedded into $\mathcal{U}$ preserving* 0*,*

and let $\varphi = \forall x_1 \cdots \forall x_n \exists y_1 \cdots \exists y_m \psi$ be a sentence over the language $\{\leq, 0\}$ with ψ quantifier-free and $x_1, \ldots, x_n, y_1, \ldots, y_m$ all distinct. Then, $\mathcal{U} \models \varphi$ if and only if the following condition is met:

> *for every finite lattice $\mathcal{L}$ that can be lattice- embedded into $\mathcal{U}$ preserving* 0 *and each n-tuple $\vec{a} = (a_1, \ldots, a_n)$ of elements of L such that $\{a_1, \ldots, a_n, 0_{\mathcal{L}}\}$ generates L under join and meet, there exists a poset $\mathcal{P}$ with least element and an m-tuple $\vec{b} = (b_1, \ldots, b_m)$ of elements of P such that:*
>
> - $|P - L| \leq m$,
> - $\mathcal{L} \subseteq_0 \mathcal{P}$,
> - $(\mathcal{L}, \mathcal{P})$ *satisfies* (3.2)*, and*
> - $\mathcal{P} \models \psi[\vec{a}, \vec{b}]$.

Proof. The proof is a slight modification of that of Theorem 13. □

Corollary 18. *If $\mathcal{U}$ is a distributive upper semi-lattice with least element such that*

- *the set of finite lattices that can be lattice-embedded into $\mathcal{U}$ preserving* 0 *is decidable,*
- *if $\mathcal{X}$ is a finite lattice that can be lattice-embedded into $\mathcal{U}$ preserving* 0 *and $(\mathcal{X}, \mathcal{Y})$ is an instance of the* 0*-extension-of-embeddings problem satisfying* (3.2)*, then $(\mathcal{X}, \mathcal{Y})$ is a positive instance of that problem for $\mathcal{U}$, and*
- *if a finite lattice can be lattice-embedded into $\mathcal{I}_{\mathcal{U}}$ preserving* 0*, then it can be lattice-embedded into $\mathcal{U}$ preserving* 0*,*

then the two-quantifier theory of $\mathcal{U}$ in the language $\{\leq, 0\}$ is decidable.

Proof. As for Corollary 14. □

Our next theorem answers a question raised by Shore and Slaman in [21]. The solution involves no new complexity-theoretic facts, but just the algebraic analysis that goes into Corollary 18.

Theorem 19. *Let $\mathcal{U}$ be an ideal of the pm-degrees of the recursive sets that has no greatest element (e.g., the pm-degrees of the elementary recursive sets, the primitive recursive sets, or all the recursive sets). Then, the two-quantifier theory of $\mathcal{U}$ in the language $\{\leq, 0\}$ is decidable.*

Proof. As mentioned previously, any such $\mathcal{U}$ is a distributive upper semi-lattice with least element. In [2], it is shown that every finite distributive lattice can be lattice-embedded into $\mathcal{U}$ preserving 0.

Let $\mathcal{X}$ be a finite lattice and let $(\mathcal{X}, \mathcal{Y})$ be an instance of the 0-extension-of-embeddings problem satisfying (3.2). By the Shore-Slaman result mentioned above, any poset embedding f of $\mathcal{X}$ into $\mathcal{U}$ can be extended to a poset embedding f' of $\mathcal{Y}$ into the *pm*-degrees of the recursive sets. However, since $\mathcal{Y}$ can add elements above $1_{\mathcal{X}}$, there is no guarantee that f' is an embedding of $\mathcal{Y}$ into $\mathcal{U}$. Thus, we consider partial orders $\mathcal{X}^*, \mathcal{Y}^*$, obtained from $\mathcal{X}, \mathcal{Y}$, respectively, by adding (the same) new element 1^* as a new greatest element. Then, it is easily checked that $\mathcal{X}^* \subseteq_0 \mathcal{Y}^*$, $\mathcal{X}^*$ is a lattice, and $(\mathcal{X}^*, \mathcal{Y}^*)$ satisfies (3.2). If f is a poset embedding of $\mathcal{X}$ into $\mathcal{U}$,

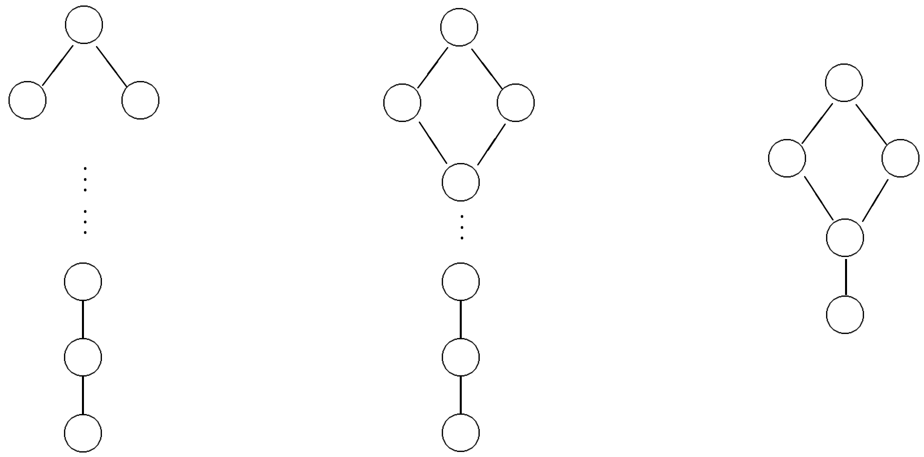

FIGURE 1. (a) A bounded distributive upper semi-lattice $\mathcal{U}$. (b) The lattice $\mathcal{I}_{\mathcal{U}} = \mathcal{Q}_U$. (c) A lattice that can be lattice-embedded into $\mathcal{I}_{\mathcal{U}}$ preserving 0 and 1, but cannot be lattice-embedded into $\mathcal{U}$.

then, since $\mathcal{U}$ is an upper semi-lattice without greatest element, there must be an element z of U with $z >_{\mathcal{U}} f(1_{\mathcal{X}})$. Thus, we may extend f to a poset embedding f^* of $\mathcal{X}^*$ into $\mathcal{U}$. By the Shore-Slaman result, there is a poset embedding f'^* of $\mathcal{Y}^*$ into the pm-degrees of the recursive sets that extends f^*. Then f'^* in fact embeds $\mathcal{Y}^*$ into $\mathcal{U}$, so f', the restriction of f'^* to Y, is a poset embedding of $\mathcal{Y}$ into $\mathcal{U}$ that extends f. Thus, $(\mathcal{X},\mathcal{Y})$ is a positive instance of the 0-extension-of- embeddings problem for $\mathcal{U}$.

The theorem follows immediately from Corollary 18. □

We close with some remarks about the third condition on a usl $\mathcal{U}$ given in Corollaries 14 and 18. If $\mathcal{U}$ is an upper semi-lattice, then an ideal of $\mathcal{U}$ is called *quasi-principal* if it is the intersection of finitely many principal ideals. If $\mathcal{U}$ is a distributive upper semi-lattice with least element, then it is not hard to show that $\mathcal{Q}_{\mathcal{U}}$, the set of all quasi-principal ideals of $\mathcal{U}$ ordered by set inclusion, is a sublattice of $\mathcal{I}_{\mathcal{U}}$. The canonical embedding ρ of $\mathcal{U}$ into $\mathcal{I}_{\mathcal{U}}$ actually maps $\mathcal{U}$ into $\mathcal{Q}_{\mathcal{U}}$. It follows that the third condition of Corollaries 14 and 18 can be weakened by replacing $\mathcal{I}_{\mathcal{U}}$ with $\mathcal{Q}_U$. For the particular structures we have considered, the weakened condition is no easier to show than the original condition, but use of the weakened condition in other situations could conceivably be advantageous.

The ease with which we have been able to show the third condition of Corollary 14 for the structures $\mathcal{U}$ we have considered might tempt one to conjecture that for any bounded distributive upper semi-lattice $\mathcal{U}$, if $\mathcal{L}$ is a finite lattice that can be lattice-embedded into $\mathcal{I}_{\mathcal{U}}$ preserving 0 and 1, then $\mathcal{L}$ can be lattice-embedded into $\mathcal{U}$ preserving 0 and 1. This conjecture is false. For instance, let $\mathcal{U}$ consist of a copy of ω with an exact pair above it, plus a greatest element. (See Figure 1a.)

It is easily checked that $\mathcal{U}$ is a distributive upper semi-lattice. The lattices $\mathcal{I}_{\mathcal{U}}$ and $\mathcal{Q}_U$ are the same. They contain only one element besides the principal ideals, namely, the ideal consisting of the copy of ω. (See Figure 1b.) The lattice given in Figure 1c can be lattice-embedded into $\mathcal{I}_{\mathcal{U}}$ preserving 0 and 1, but it cannot be lattice-embedded into $\mathcal{U}$. It would be interesting to have some general conditions which apply to recursion and complexity-theoretic structures and guarantee that they satisfy the third condition of Corollary 14.

Similar considerations apply to the third condition of Corollary 18.

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