A Direct Construction of a Minimal Recursively Enumerable Truth-Table Degree*

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1 Introduction

Given a reducibility r between sets of natural numbers, one can form the structures $\mathcal{D}_r$ of all the r-degrees, $\mathcal{D}_r(\leq \mathbf{0}')$ of the r-degrees less than or equal to the r-degree of $\emptyset'$, and $\mathcal{R}_r$ of the recursively enumerable (r.e.) r-degrees. An obvious question about any of these structures is whether they contain minimal degrees. Besides being of intrinsic interest, a positive answer to this question is usually a first step towards showing the undecidability of the structure.

Lachlan has shown for both one-one (1) reducibility ([6]) and many-one (m) reducibility ([7]) that there is an r.e. degree which is minimal among the r.e. degrees, hence among all the degrees, since for these reducibilities the r.e. degrees are closed downwards. (See Rogers [14] and Odifreddi [11] and [12] for the definitions of and more information on reducibilities stronger than Turing reducibility.) Thus for these two reducibilities, there are minimal degrees in all three of the structures mentioned above.

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For Turing (T) reducibility, one of the earliest results is Spector's construction in [19] of a minimal degree. Since this construction uses total trees, it is not hard to see that it produces a set whose degrees with respect to both weak truth table (wtt) and truth table (tt) reducibility are also minimal. However, the minimal degree produced is not below $\mathbf{0}'$, so Spector's result does not produce minimal degrees in any of the structures $\mathcal{D}_r(\leq \mathbf{0}')$ or $\mathcal{R}_r$. (Note that by the density of the r.e. Turing degrees ([16]) and r.e. wtt-degrees ([8]), there are no minimal degrees in $\mathcal{R}_T$ and $\mathcal{R}_{wtt}$.) In [15], Sacks constructed a minimal degree in $\mathcal{D}_T(\leq \mathbf{0}')$ and later Shoenfield ([17]) gave a simpler oracle construction of this type of degree. Both of these constructions use partial trees and hence do not produce minimal degrees below $\mathbf{0}'$ in any of the reducibilities stronger than Turing reducibility.

In 1973, Degtev [1] showed that there is a minimal degree in $\mathcal{R}_{tt}$ and in 1975 Marchenkov [9] showed that there is an r.e. degree in $\mathcal{D}_{tt}(\leq \mathbf{0}')$ which is minimal. Both of these results were indirect in that they have the flavor of the 1978 result of Kobzev [5] who showed that the degree of a nonrecursive η-maximal semirecursive set (a type of set shown to exist by Degtev [1]) is minimal among all tt-degrees. In this paper we give a direct construction of a minimal r.e. tt-degree. The obvious next step after giving such a direct construction would be to extend it to embed other lattices as initial segments. If a large enough class of lattices (e.g., the finite distributive lattices) can be so embedded, then the theory of the r.e. tt-degrees is undecidable. We tried to embed the finite distributive lattices as initial segments using our methods, but found that we could not even embed the four element Boolean algebra. However, an extension of our construction has been used by Haught and Shore [3] to show the undecidability of the theory of the r.e. tt-degrees. They showed that for every finite partition lattice $\mathcal{L}_0$, if $\mathcal{L}$ is the lattice obtained from $\mathcal{L}_0$ by taking the dual and adding an extra element below all the others, then $\mathcal{L}$ is embeddable in $\mathcal{R}_{tt}$. The reason why we could not embed the four element Boolean algebra as an initial segment, and the reason why Haught and Shore had to use such unusual lattices in their proof, has been made clear by Harrington and Haught. They show in [2] that in every nontrivial finite initial segment of $\mathcal{R}_{tt}$, the elements other than $\mathbf{0}$ have a least element. Exactly which finite lattices can be obtained as initial segments of $\mathcal{R}_{tt}$ is an interesting open question.

Another interesting open question (Problem 16 in [11]) is whether $\mathcal{R}_{tt}$ and $\mathcal{D}_{tt}(\leq \mathbf{0}')$ are elementarily equivalent. In [10], Marchenkov shows that there is an incomplete r.e. tt-degree which bounds all minimal r.e. tt-degrees. One might hope that our construction could be used to show that the minimal tt-degrees below $\mathbf{0}'$ are not bounded away from $\mathbf{0}'$ and thus obtain an elementary difference in the two structures. However, Shore has remarked that Marchenkov's proof can be modified to show that there is an incomplete tt-degree below $\mathbf{0}'$ which bounds all minimal tt-degrees below $\mathbf{0}'$, so some other approach will be needed to show that the two structures are not elementarily equivalent.

In our construction of a recursively enumerable set A whose tt-degree is minimal, if we put a number x into A at stage s, then we also put into A at stage s all numbers y with $x \leq y \leq s$. This ensures that the set we construct is semirecursive. We do not know if our set is η-maximal or if it can be made so.

Our construction can be modified to show that there is a minimal degree in $\mathcal{D}_{wtt}(\leq \mathbf{0}')$. In [4], Haught and Shore extend the construction of a minimal wtt-degree below $\mathbf{0}'$ (in ways analogous to those in [3]) to show that the theory of $\mathcal{D}_{wtt}(\leq \mathbf{0}')$ is undecidable.

To summarize then, for $r = 1,m$, and tt, there are r.e. r-degrees which are minimal among all the r-degrees. For $r = wtt$ and T, there are no minimal r.e. degrees, but there are minimal r-degrees below $\mathbf{0}'$. In each case, the minimal degree construction is a first step towards showing undecidability.

We complete the introduction by giving the definitions and notations we will use.

Our notation is for the most part standard. (See, e.g., [18].) By "set" we mean set of natural numbers and by "number" we mean natural number. We identify a set with its characteristic function. A *string* is an element of $2^{<\omega}$. If $\sigma \in 2^n$, then the *length* of σ, denoted $|\sigma|$, is n. For two strings σ and τ, $\sigma \subseteq \tau$ means that σ is an initial segment of τ while $\sigma \preceq \tau$ means that σ comes before τ in lexicographic ordering where 0 is considered to be less than 1. We say that two strings σ and τ *split* a string ρ if σ and τ are incompatible extensions of ρ. We write $\{e\}$ for the eth Turing reduction.

A *truth table* is a pair $\alpha = (\langle x_0, \ldots, x_{n-1}\rangle, t)$ consisting of a strictly increasing sequence $\langle x_0, \ldots, x_{n-1}\rangle$ of numbers and a function $t : 2^n \to 2$ for some number n. For a truth table α as in the previous sentence, the *size* of α is n and the *length* of α, denoted $|\alpha|$, is $x_{n-1}+1$ if $n > 0$ and is 0 if $n = 0$. We take $\{\alpha_n\}_{n\in\omega}$ to be an effective enumeration of all truth tables such that $|\alpha_n| \leq n$.

If A is a set and $\alpha = (\langle x_0, \ldots, x_{n-1}\rangle, t)$ is a truth table, then we say that A *satisfies* α if $t(A(x_0), \ldots, A(x_{n-1})) = 1$, in which case we write $\alpha^A = 1$. Otherwise, we write $\alpha^A = 0$ and similarly for finite strings σ with $|\sigma| \geq |\alpha|$.

If A and B are sets, then we say that A is truth table (tt) reducible to B ($A \leq_{tt} B$) if for some recursive function f, for every x, $x \in A \leftrightarrow \alpha^B_{f(x)} = 1$. Then $\equiv_{tt}$ and the notion of truth table (tt)-degree are defined as usual. By a result of Nerode (see [14, p. 143]) $A \leq_{tt} B$ iff $A = \{e\}^B$ for some e such that $\{e\}^X$ is total for every X.

If $\{e\}(x)\downarrow$, we write $[e](x)$ for $\alpha_{\{e\}(x)}$. For a number e and set A, we define a partial function $[e]^A$ by

$$[e]^A(x) = \begin{cases} ([e](x))^A & \text{if } \{e\}(x)\downarrow \\ \uparrow & \text{otherwise.} \end{cases}$$

Then $A \leq_{tt} B$ iff $(\exists e)(A = [e]^B)$. If σ is a string, we define a partial function $[e]^\sigma$ by

$$[e]^\sigma(x) = \begin{cases} ([e](x))^\sigma & \text{if } \{e\}(x)\downarrow \text{ and } |\sigma| \geq |[e](x)| \\ \uparrow & \text{otherwise.} \end{cases}$$

We also define $[e]^A_s$ by

$$[e]^A_s(x) = \begin{cases} ([e](x))^A & \text{if } \{e\}_s(x)\downarrow \\ \uparrow & \text{otherwise.} \end{cases}$$

$[e]^\sigma_s$ is defined similarly.

If σ and τ are strings, then say that σ and τ *e-split* if for $t = \min\{|\sigma|, |\tau|\}$, there is some x such that $[e]^\sigma_t(x)$ and $[e]^\tau_t(x)$ both converge and give different answers. Since we adopt the usual convention that if $\{e\}_s(x)\downarrow$ then $x < s$, given two strings σ and τ, we can effectively tell whether or not they e-split.

2 Construction of a Minimal R.E. tt-Degree

We will construct an r.e. set A. In order to make A have minimal tt-degree, we have for all $e \geq 0$ the requirements

$$P_e : A \neq \overline{W_e}$$

and

$$Q_e : \text{ If } [e]^A \text{ is total, then either } [e]^A \text{ is recursive or } A \leq_{tt} [e]^A.$$

If we meet all of these requirements, then A will in fact have minimal tt-degree.

We base our construction on the so-called "full-approximation" (no oracle) construction of a minimal Turing degree below $\mathbf{0}'$ and we assume familiarity with this method. (Posner [13] is an excellent introduction to this area.) Of course there must be some modifications to suit our current situation. First of all, we seek to build an r.e. set, something that does not happen with the usual full approximation construction. Second, we want our set to have minimal tt-degree and not just minimal Turing degree. As mentioned in the introduction, constructions using partial trees do not produce tt-reductions so we need a device to deal with this problem as well.

When thinking about trees, we envision 0 as being drawn to the left of 1. Since we are building A to be r.e., we only add elements to A, never remove them. Hence A can only move to the right as it is constructed. This means that there is no reason to keep track of any branchings to the left of A's current value. In fact, in our construction, rather than keep track of a full binary tree of branchings, we just keep track of a single infinite sequence of branchings. For each branching, one half of the branching (the "left" half) is contained in the current value of A, and the other half can be obtained from the left half by changing 0's to 1's. (Thus if we wish to move A to the right half of one of the branchings in order to meet one of the P_e requirements, we can do so by only adding elements to A.) Each branching extends the left half of the previous branching.

For a given e, if on the eth tree we are able eventually to find on each level of the tree an e-splitting to put up as the next branching and $[e]^A$ is total, then we need to show that $A \leq_{tt} [e]^A$. When we put up an e-splitting on some level of the eth tree, we have no way of knowing effectively whether diagonalization will ever move A to the right half of the splitting. If we are just trying to get a T-reduction of A to $[e]^A$ this is not a problem since we can ask the $[e]^A$ oracle if A is ever moved to the right side of the splitting, and if it does get moved, we can wait for this to happen. For a tt-reduction however, we must give an answer no matter what oracle we are given and hence we cannot have as part of our reduction procedure to wait for something that might never happen. Instead, we think of the right halves of the e-splittings we put up as being extended by 1's. If at stage s we move A to the right side of a splitting, then we also put into A every number from the length of the splitting up to the stage number s. Thus as we go through more stages, we are committing ourselves more and more as to what A will look like if we do eventually move A to the right side of the e-splitting. This allows us to get a tt-reduction of A to $[e]^A$. At the same time, when diagonalization occurs, we have committed ourselves to only a finite amount of information about A, so we still have room to work above this commitment to look for more e-splittings.

With this as motivation, we now formally define the kind of trees we will be using.

Let $\Omega = \{\emptyset, 0, 1, 00, 01, 000, 001, \ldots\} = \{\sigma \in 2^{<\omega} : \forall n(n < |\sigma| - 1 \rightarrow \sigma(n) = 0)\}$. A *tree* is a function $T : \Omega \rightarrow 2^{<\omega}$ such that for all k, $T(0^k0)$ and $T(0^k1)$ split $T(0^k)$ and $|T(0^k0)| = |T(0^k1)|$. We write $T(\sigma)(\tau)$ for the pair of strings $T(\sigma), T(\tau)$. $T(0^k0), T(0^k1)$ are called *T-partners*. If T is a tree and $q = T(0^k)$, then the *left successor* of q is $q_0 = T(0^k0)$ and the *right successor* of q is $q_1 = T(0^k1)$. $T(0^k)$ is the *predecessor* of both $T(0^k0)$ and $T(0^k1)$.

If T is a tree, then we adopt the convention that R denotes the range of T. Similarly, R_1 is the range of T_1 and so on. If T is a tree, then we define

$$R^- = \{T(0^k) : k \geq 0\}$$

and

$$R^* = R^- \cup \{\sigma \in 2^{<\omega} : \exists k, k', s[|\sigma| = |T(0^k)| \wedge \sigma = T(0^{k'}1)*1^s]\}.$$

Thus for any tree T, $R^- \subseteq R \subseteq R^*$.

If T is a tree and k a natural number, we define *level* k of T to consist of those strings in R^* which have the same length as $T(0^k)$. Thus there are $k+1$ strings in level k of T. These are $T(0^k)$ together with all strings obtained from strings $T(0^{k'}1)$ with $k' < k$ by adding enough 1's at the end to get the correct length. Note that any string on level k of a tree T must have length $\geq k$.

We can now state the computation lemmas which are used to construct a minimal tt-degree using trees.

Lemma 1 *Let $\langle T_t : t \geq 0\rangle$ be a recursive sequence of trees and let A be a set. Suppose that for each $t \geq 0$,*

1. *A contains one of the strings on level t of T_t.*
2. *There are no e-splittings on level t of T_t.*

Then if $[e]^A$ is total, it is recursive.

Proof: Suppose that $[e]^A$ is total. Given an x, find a $t \geq x$ such that $\{e\}_t(x)\downarrow$ and such that $t \geq |[e](x)|$. If σ is a string on level t of T_t, then $|\sigma| \geq t$, so $[e]^\sigma(x)\downarrow$. Since there are no e-splittings on level t of T_t, this answer must be the same for all strings σ on level t. Since A contains one of the strings on level t this answer is $[e]^A(x)$. ∎

Lemma 2 *Let $\langle T_t : t \geq 0\rangle$ be a recursive sequence of trees and let A be a set. Suppose that for each $t \geq 0$,*

1. *A contains one of the strings on level t of T_t.*
2. *For each $k < t$, $T_t(0^k0)(0^k1)$ e-split.*

Then if $[e]^A$ is total, $A \leq_{tt} [e]^A$.

Proof: For a given x, consider T_t with $t = x+1$. All of the strings on level t of T_t have length greater than x and since T_t e-splits up through level t, it is possible to distinguish between the strings on level t using a truth table. More precisely, for $0 \leq k < t$, let q_k be the extension of $T_t(0^k1)$ on level t of T_t, and let $q_t = T_t(0^t)$. Then $q_0, \ldots, q_t$ are all the strings on level t of T_t. By assumption, for each $k < t$, $T_t(0^k0)(0^k1)$ e-split, so for each k with $1 \leq k \leq t$, there are y_k, a_k, b_k with $[e]^{T_t(0^k)}(y_k) = a_k$, $[e]^{T_t(0^{k-1}1)}(y_k) = b_k$, and $a_k \neq b_k$. If $r = \max\{r' : 1 \leq r' \leq t \wedge [e]^A(y_{r'}) = a_{r'}\}$ (taking $\max\emptyset = 0$), then $A(x) = q_r(x)$. It follows that it is possible to write a truth table (of length $\max\{y_k : 1 \leq k \leq t\}$) such that $[e]^A$ satisfies the truth table if and only if $x \in A$. Since the sequence of trees is recursive, given x, the numbers y_k and a_k can be found effectively. Thus the truth table can be found effectively given x, and $A \leq_{tt} [e]^A$. ∎

We next define the appropriate notion of subtree for the kind of trees we are using. If T and T_1 are trees, then we call T_1 a *subtree* of T and write $T_1 \subseteq^* T$ if $R_1^- \subseteq R^-$ and $R_1 \subseteq R^*$. The following Lemma is an easy consequence of this definition.

Lemma 3 *For all trees T_1, T_2, T_3,*

1. *If* $T_1 \subseteq^* T_2$, *then* $R_1^* \subseteq R_2^*$.

2. *If* $T_1 \subseteq^* T_2 \subseteq^* T_3$, *then* $T_1 \subseteq^* T_3$.

∎

We now consider the format of the construction and define the e-state of a string on the trees we will construct. At stage s, we will define a finite set A_s and trees $T_{-1,s}, T_{0,s}, T_{1,s}, \ldots$. We will have $A_s \subseteq A_{s+1}$ for all $s \geq 0$ and we will set $A = \bigcup_s A_s$, so, since the construction will be effective, A will be r.e.

We will also have $T_{e,s} \subseteq^* T_{e-1,s}$ for all $e \geq 0$. Thus for every e', e with $-1 \leq e' \leq e$, by the second part of Lemma 3, $T_{e,s} \subseteq^* T_{e',s}$ and hence by the first part of the Lemma, $R^*_{e,s} \subseteq R^*_{e',s}$. For $q \in R^*_{e,s}$,we define $U_{e,s}(q)$, *the* e*-state of* q *at stage* s as follows. First, for each $e \geq 0$ we define $S_{e,s} : R_{e,s} \to \{0,1\}$ by

$$\begin{aligned} S_{e,s}(T_{e,s}(\emptyset)) &= 1 \\ S_{e,s}(T_{e,s}(0^k0)) = S_{e,s}(T_{e,s}(0^k1)) &= \begin{cases} 1 & \text{if } T_{e,s}(0^k0)(0^k1)\ e\text{-split} \\ & \text{and have length} \leq s \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Next we extend $S_{e,s}$ to $R^*_{e,s}$ by defining for $\sigma \in R^*_{e,s}$

$$S_{e,s}(\sigma) = S_{e,s}(\tau) \text{ where } \tau \in R^-_{e,s} \text{ and } |\tau| = |\sigma|.$$

Note that the value of $S_{e,s}(\sigma)$ is the same for all σ on a given level of $R^*_{e,s}$. Finally, for $e \geq -1$, we define $U_{e,s} : R^*_{e,s} \to 2^{e+1}$ by

$$U_{e,s}(\sigma) = (S_{0,s}(\sigma), S_{1,s}(\sigma), \ldots, S_{e,s}(\sigma))$$

(This definition makes sense since for all e' with $0 \leq e' \leq e$, $R^*_{e,s} \subseteq R^*_{e',s}$.) Note that $U_{-1,s}(\sigma) = \emptyset$ for all $\sigma \in R^*_{-1,s}$ and that if $e' < e$ and $\sigma \in R^*_{e,s} \subseteq R^*_{e',s}$, then $U_{e',s}(\sigma) \subseteq U_{e,s}(\sigma)$. Also, for a given $e \geq -1$, the definition of $U_{e,s}$ does not depend on any of the trees $T_{e',s}$ with $e' > e$, so we may compute $U_{e,s}$ once $T_{-1,s}, \ldots, T_{e,s}$ are defined. If σ, τ are two strings in $R^*_{e,s}$ which have the same value under $U_{e,s}$, then we write $U_{e,s}(\sigma, \tau)$ for this value. In particular, suppose that σ and τ are two strings in $R^*_{e,s}$ of the same length. Then (since strings in different levels must have different lengths), σ and τ are on the same level of each $R_{e',s}$ with $e' \leq e$ and hence (since the value of $S_{e',s}$ depends only on the level of the argument), $U_{e,s}(\sigma) = U_{e,s}(\tau)$.

Before we can give the conditions that our construction will meet, there is one technicality we must discuss since it does not come up in the usual full approximation constructions. This is what we call "stretching". Suppose that, at stage s, string q is on $T_{e,s}$, the left and right successors of q are q_0 and q_1, and the successors of q_0 are q_{00} and q_{01}. If at the beginning of stage $s+1$ we move A over to q_{01} to diagonalize, it might seem that, while it is obvious that the splitting q_{00}, q_{01} cannot be left on the eth tree, there is no reason to take q, q_0, and q_1 off of this tree unless it is possible to improve e-state. However, although q_{00} and q_{01} are incompatible extensions of q_0, the first disagreement between these two strings need not be until well past the length of q_0. Thus if q_0, q_1 are left as the successors of q, then in looking for a splitting to put up as the successor of q_0, we might find a splitting one half of which is contained in q_{01} (and hence is contained in the new A) and the other half of which is to the right of q_{01}. A further diagonalization could then push A to the right half of this new splitting and hence off of $T^*_{e,s}$. The way to avoid this undesirable situation is to not leave q_0, q_1 as the successors of q but instead to make the new successors of q be extensions of q_0, q_1 which have been stretched out long enough to prevent A from going off $T^*_{e,s}$. This stretching adds complication to the construction, but does have the advantage of making it correct.

Before we give the construction, we will list conditions which the construction will satisfy, and then we will derive consequences from these conditions. The intuitive meanings of the conditions will be given after we list them. For all $e \geq -1$ and $s \geq 0$ the following will be true.

C1) For all $q \in R^-_{e,s}$, $q \subset A_s$.

C2) If $e \geq 0$, $q \in R^-_{e,s}$, q_0, q_1 are the $T_{e,s}$ successors of q, and r_0, r_1 are the $T_{e-1,s}$ successors of q, then
$$U_{e-1,s}(q_0, q_1) = U_{e-1,s}(r_0, r_1).$$

C3) If $e \geq 0$, $q \in R^-_{e,s}$, q_0, q_1 are the $T_{e,s}$-successors of q and there exist r_0, r_1 with

a) $r_0, r_1 \in R^*_{e-1,s}$,

b) $r_0 \in R^-_{e-1,s}$,

c) $|r_0| = |r_1| \leq s$,

d) $r_0, r_1 \supseteq q$,

e) r_0, r_1 e-split,

f) $U_{e-1,s}(r_0, r_1) = U_{e-1,s}(q_0, q_1)$,

then q_0, q_1 e-split.

C4) If $s > 0$, $q \in R^-_{e,s} \cap R^-_{e,s-1}$, q_0, q_1 are the $T_{e,s-1}$- successors of q, q_{00}, q_{01} are the $T_{e,s-1}$-successors of q_0, r_0, r_1 are the $T_{e,s}$-successors of q, and $q_0 \subset A_s$, then $U_{e,s}(r_0, r_1) \succeq U_{e,s-1}(q_0, q_1)$. Further, if $U_{e,s}(r_0, r_1) = U_{e,s-1}(q_0, q_1)$, then $q_0 \subseteq r_0, q_1 \subseteq r_1$ and the following hold.

a) If $q_{00} \subset A_s$, then $q_0 = r_0, q_1 = r_1$.

b) If $q_{00} \not\subset A_s$ and $U_{e,s-1}(q_{00}) = U_{e,s-1}(q_0)$, then $|r_0| \geq s$.

C5) For all x, if x enters A at stage s and $x \leq y \leq s$, then y enters A at stage s.

Condition C1 simply states that the left half of every branching on all of the trees built at stage s is on A_s. Condition C2 says that if q is in $R^-_{e,s}$, then in searching for the successors of q on $T_{e,s}$, we must look only among those strings on $T_{e-1,s}$ which have the same $e-1$-state as the successors of q on $T_{e-1,s}$ do. This is the usual nesting of e-states strategy used in the full-approximation construction to make sure that the construction settles down. Condition C3 says that if there are strings on $T_{e-1,s}$ which are suitable for use as successors of some string q on $T_{e,s}$ and which e-split, then the successors of q on $T_{e,s}$ do e-split. Condition C4 is somewhat complicated due to stretching, but it says that if a string q is on both the trees $T_{e,s}$ and $T_{e,s-1}$, then, unless diagonalization interferes, the e-state of the successors of q on $T_{e,s}$ is at least as high as the e-state of the successors of q on $T_{e,s-1}$ and if the e-state has not improved then the successors have not changed (unless stretching interferes in which case the new successors of q are extensions of the old ones). The final condition C5 ensures that when diagonalization occurs, A remains on the $T^*_{e,s}$ for those e which are less than or equal to the priority of the diagonalizing requirement.

We now prove some lemmas which follow from the just-given conditions which our construction will meet. The first lemma says that as you go away from the root of $T_{e,s}$, the e-state gets lower.

Lemma 4 *Suppose that $s \geq 0, e \geq -1$ and condition C3 holds for all e' with $0 \leq e' \leq e$. Then for all $q, r \in R^-_{e,s}$, $q \subseteq r \rightarrow U_{e,s}(q) \succeq U_{e,s}(r)$.*

Proof: We use induction on e. For $e = -1$ the result is trivial ($U_{e,s}(q) = \emptyset = U_{e,s}(r)$).

Suppose that $e \geq 0$, that the result holds for $e-1$, that condition C3 holds for all e' with $0 \leq e' \leq e$, and that $q, r \in R^-_{e,s}$ with $q \subseteq r$. Since $T_{e,s} \subseteq^* T_{e-1,s}$, $R^-_{e,s} \subseteq R^-_{e-1,s}$ and thus by induction hypothesis, $U_{e-1,s}(q) \succeq U_{e-1,s}(r)$. Suppose for a contradiction that $U_{e,s}(q) \prec U_{e,s}(r)$. Then $U_{e-1,s}(q) = U_{e-1,s}(r)$, $S_{e,s}(q) = 0$, and $S_{e,s}(r) = 1$. Since $S_{e,s}(q) = 0$, $q \neq T_{e,s}(\emptyset)$. Let q^- be the $T_{e,s}$-predecessor of q, and let q', r' be the $T_{e,s}$-partners of q, r respectively. Then by definition of $S_{e,s}$, r and r' e-split and have length less than or equal to s. Since $q \subseteq r$, $|q| \leq |r| \leq s$, so since $S_{e,s}(q) = 0$, q and q' must not e-split. This contradicts condition C3 because $q^- \in R^-_{e,s}$, q, q' are the $T_{e,s}$-successors of q^-, r, r' are in $R_{e,s} \subseteq R^*_{e-1,s}$, $r \in R^-_{e,s} \subseteq R^-_{e-1,s}$, $|r| = |r'| \leq s$, $r, r' \supseteq q^-$, r, r' e-split, and $U_{e-1,s}(r, r') = U_{e-1,s}(q, q')$. Thus $U_{e,s}(q) \succeq U_{e,s}(r)$. ∎

Our next lemma is a somewhat technical extension of condition C4. It states that if a string q is on both $T_{e,s}$ and $T_{e,s-1}$ and the e-state of the successors of q does not change from stage $s-1$ to s (call this e-state u), then the whole part of $T_{e,s-1}$ which is above q and has e-state u at stage $s-1$ is still on $T_{e,s}$ and has e-state u, unless diagonalization or stretching interfere.

Lemma 5 *Suppose that $s > 0$, $e \geq -1$, conditions C3 and C4 hold for all e', s' with $s' \in \{s, s-1\}$ and $-1 \leq e' \leq e$, $q \in R^-_{e,s} \cap R^-_{e,s-1}$, q_0, q_1 are the $T_{e,s}$ successors of q, $r \in R^-_{e,s-1}$, $q \subset r \subset A_s$, and $U_{e,s-1}(r) = U_{e,s}(q_0, q_1) = u$. Let r_0, r', r^- be the $T_{e,s-1}$ left successor, partner, and predecessor of r, respectively.*

1. *If $r_0 \subset A_s$, then $r, r' \in R_{e,s}$ and $U_{e,s}(r) = U_{e,s-1}(r)$.*
2. *If $r_0 \not\subset A_s$, then $r^- \in R^-_{e,s}$, and if p, p' are the $T_{e,s}$ successors of r^-, then $U_{e,s}(p, p') = U_{e,s-1}(r)$ and $r \subseteq p, r' \subseteq p'$. Further, if $U_{e,s-1}(r_0) = U_{e,s-1}(r)$, then $|p| \geq s$.*

Proof: 1) Let $q = p_0 \subset p_1 \subset \ldots \subset p_t = r$ be all strings between q and r on $T_{e,s-1}$ (so $t > 0$). Let p'_i be the $T_{e,s-1}$-partner of p_i, $1 \leq i \leq t$. We show by induction on i, $0 \leq i \leq t-1$, that

1. $p_i \in R^-_{e,s}$,
2. the $T_{e,s}$-successors of p_i are p_{i+1}, p'_{i+1},
3. $U_{e,s}(p_{i+1}, p'_{i+1}) = u = U_{e,s-1}(p_{i+1}, p'_{i+1})$.

For $i = 0$ we have $p_i = p_0 = q_0 \in R^-_{e,s}$ by hypothesis. Also

$$\begin{array}{rcll} u & = & U_{e,s}(q_0, q_1) & \text{by definition of } u \\ & \succeq & U_{e,s-1}(p_1, p'_1) & \text{by C4 for } T_{e,s} \text{ since } p_1 \subset r \subset A_s \\ & \succeq & U_{e,s-1}(r) & \text{by Lemma 4 for } T_{e,s-1} \text{ since } p_1 \subseteq r \in R^-_{e,s-1} \\ & = & u & \text{by hypothesis} \end{array}$$

Thus by C4 for $T_{e,s}$ (since $p_1 \subseteq r \subset r_0 \subset A_s$), $q_0 = p_1, q_1 = p_1'$ and hence $U_{e,s}(p_1, p_1') = U_{e,s}(q_0, q_1) = u = U_{e,s-1}(p_1, p_1')$.

Now suppose that the result holds for $i-1$ with $0 < i \leq t-1$. Then by induction hypothesis, $p_i \in R^-_{e,s}$. Let s_0, s_1 be the $T_{e,s}$-successors of p_i. We have

$$\begin{array}{rcll} u & = & U_{e,s}(p_i, p_i') & \text{by induction hypothesis} \\ & \succeq & U_{e,s}(s_0, s_1) & \text{by Lemma 4 for } T_{e,s} \\ & \succeq & U_{e,s-1}(p_{i+1}, p_{i+1}') & \text{by C4 for } T_{e,s} \text{ since } p_{i+1} \subseteq r \subset A_s \\ & \succeq & U_{e,s-1}(r) & \text{by Lemma 4 for } T_{e,s-1} \\ & = & u & \text{by hypothesis} \end{array}$$

Thus $u = U_{e,s}(s_0, s_1) = u_{e,s-1}(p_{i+1}, p_{i+1}')$, so by C4 for $T_{e,s}$ (since $p_{i+1} \subseteq r \subset r_0 \subset A_s$), $s_0 = p_{i+1}, s_1 = p_{i+1}'$ and $U_{e,s}(p_{i+1}, p_{i+1}') = U_{e,s}(s_0, s_1) = u = u_{e,s-1}(p_{i+1}, p_{i+1}')$.

Taking $i = t-1$ gives the result.

2) If $r^- = q$, then $r^- \in R^-_{e,s}$ and $u = U_{e,s}(p, p') = U_{e,s-1}(r, r')$, so the result follows from C4.

If $r^- \neq q$, then $u = U_{e,s-1}(r) \preceq U_{e,s-1}(r^-) \preceq U_{e,s-1}(p_1, p_1') \preceq U_{e,s}(q_0, q_1) = u$ (where p_1, p_1' are the $T_{e,s-1}$-successors of q), so $U_{e,s-1}(r^-) = u$. Thus by part (1), $r^- \in R^-_{e,s}$ and $U_{e,s}(r^-) = U_{e,s-1}(r^-) = u$. We now have

$$\begin{array}{rcll} u & = & U_{e,s}(r^-) & \\ & \succeq & U_{e,s}(p, p') & \text{by Lemma 4 for } T_{e,s} \\ & \succeq & U_{e,s-1}(r, r') & \text{by C4 for } T_{e,s} \text{ since } r \subset A_s \\ & = & u & \text{by hypothesis} \end{array}$$

Thus $U_{e,s}(p, p') = U_{e,s-1}(r, r')$, so the result follows from C4 for $T_{e,s}$. ∎

The next lemma is just an extension of the previous lemma to all of $T^*_{e,s-1}$.

Lemma 6 *Suppose that $s > 0$, $e \geq -1$, conditions C3 and C4 hold for all e', s' with $s' \in \{s, s-1\}$ and $-1 \leq e' \leq e$, $q \in R^-_{e,s} \cap R^-_{e,s-1}$, q_0, q_1 are the $T_{e,s}$ successors of q, $r \in R^-_{e,s-1}$, $q \subset r \subset A_s$, $U_{e,s-1}(r) = U_{e,s}(q_0, q_1) = u$, $p^* \in R^*_{e,s-1}$, $|p^*| = |r|$, and $q \subseteq p^*$. Let r_0 be the $T_{e,s-1}$ left successor of r.*

1. *If $r_0 \subset A_s$, then $p^* \in R^*_{e,s}$ and $U_{e,s}(p^*) = U_{e,s-1}(p^*)$.*
2. *If $r_0 \not\subset A_s$, then some extension p_1 of p^* is in $R^*_{e,s}$ with $U_{e,s}(p_1) = U_{e,s-1}(p^*)$, and if $U_{e,s-1}(r_0) = U_{e,s-1}(r)$, then we may take $|p_1| \geq s$.*

Proof: 1) If $p^* = r$, then this is just a restatement of Lemma 5. Suppose that $p^* \neq r$. Then since $q \subset p^*$, for some $s_0 \in R^-_{e,s-1}$, $q \subset s_0 \subseteq r$ and $p^* = s_0' * 1^t$ for some t where s_0' is the $T_{e,s-1}$-partner of s_0. By the usual argument, if p_1, p_1' are the $T_{e,s-1}$-successors of q, then $U_{e,s-1}(r) \preceq U_{e,s-1}(s_0) \preceq U_{e,s-1}(p_1, p_1') \preceq U_{e,s}(q_0, q_1) = U_{e,s}(r)$ and $s_0 \subseteq r \subset r_0 \subset A_s$, so by part 1 of Lemma 5 applied to s_0, $s_0' \in R_{e,s}$. By the same Lemma applied to r, $r \in R^-_{e,s}$. Thus $p^* \in R^*_{e,s}$. Finally $U_{e,s}(p^*) = U_{e,s}(r) = U_{e,s-1}(r) = U_{e,s-1}(p^*)$.

We omit the proof of (2) which follows from the second part of Lemma 5 in much the same way that the argument just given used the first part of that Lemma. ∎

We now give the construction and along the way show that all of the conditions C1 to C5 are met. At any stage during the construction, a requirement P_e may receive attention. At most one P_e receives attention at a given stage. P_e can receive attention at stage s only if it has never received attention at any earlier stage, so each P_e receives attention at most once.

The construction has infinitely many stages. At stage $s \geq 0$, we construct a sequence of trees $\langle T_{-1,s}, T_{0,s}, T_{1,s}, \ldots\rangle$ with $T_{e,s} \subseteq^* T_{e-1,s}$ for all $e \geq 0$.

Stage s.

(1) If $s > 0$, find the least $e \leq s-1$, if any, such that

a) P_e has not received attention at any stage $s' < s$.

b) $x \in W_e$ where x is minimal so that $T_{e,s-1}(0)(x) \neq T_{e,s-1}(1)(x)$.

If any such e exists, take the least such and put the corresponding x into A_s. In addition, for any y with $x \leq y \leq s$, put y into A_s. (Thus condition C5 is met for s.) We say that P_e receives attention at stage s.

(2) We now construct the trees $T_{e,s}$ by induction on $e \geq -1$. We assume that each of the conditions C1 to C5 is met by all trees $T_{e',s'}$ with $s' < s$ and we will verify that the trees constructed at stage s also meet these conditions.

First we define $T_{-1,s}$. Let $\overline{A_s} = \{a_0^s < a_1^s < \ldots\}$. We define

$$T_{-1,s}(\emptyset) = \emptyset$$

and for $k \geq 0$

$$\begin{aligned} T_{-1,s}(0^k 0) &= A_s \lceil a_{k+1}^s \\ T_{-1,s}(0^k 1) &= \begin{cases} T_{-1,s}(0^k 0)(y) & \text{if } y < a_{k+1}^s,\ y \neq a_k^s \\ 1 & \text{if } y = a_k^s. \end{cases} \end{aligned}$$

It is easy to see that $T_{-1,s}$ is a tree, and that C1 is met, while C2 and C3 are vacuously true. Condition C4 is partly vacuous and the nontrivial part follows from the way $T_{-1,s}$ is defined. (We need to assume of course that when $s > 0$, $T_{-1,s-1}$ was defined the same way.) We leave this verification to the reader. (The fact that when stretching occurs the stretched string has length greater than or equal to s follows from C5.)

Note that since we will have each later $T_{e,s}$ a subtree of $T_{e-1,s}$, we have by Lemma 3 that $T_{e,s} \subseteq^* T_{-1,s}$ and hence C1 will automatically be met for $T_{e,s}$ since it is met by $T_{-1,s}$.

Now suppose that $e \geq 0$ and that for all e' with $-1 \leq e' < e$ we have defined $T_{e',s}$ satisfying conditions C1 to C4. We define $T_{e,s}$ inductively.

First we set

$$T_{e,s}(\emptyset) = T_{e-1,s}(0).$$

Then, assuming that we have defined $T_{e,s}(0^k) \in R^-_{e-1,s}$, say $T_{e,s}(0^k) = q$, we define $T_{e,s}(0^k 0)(0^k 1)$. Let u be the $e-1$ state at stage s of the $T_{e-1,s}$-successors of q.

Case 1. $s > 0$, $q \in R^-_{e,s-1}$, u is the $e-1$-state at stage $s-1$ of the $T_{e-1,s-1}$-successors of q, and, if q_0, q_1 are the $T_{e,s-1}$-successors of q and q_{00} is the $T_{e,s-1}$ left successor of q_0, $q_{00} \subset A_s$.

Then by C2 for $T_{e,s-1}$, $U_{e-1,s-1}(q_0, q_1) = u$, and hence by Lemmas 5 and 6 for $T_{e-1,s}$, $q_0 \in R^-_{e-1,s}$, $q_1 \in R^*_{e-1,s}$, and $U_{e-1,s}(q_0, q_1) = u$. Thus q_0, q_1 are still available at stage s to be used as the successors of q. We do this unless we can improve the e-state. That is, if $S_{e,s-1}(q_0, q_1) = 0$ and there are strings $r_0, r_1 \in R_{e-1,s}$ such that $r_0 \in R^-_{e-1,s}$, $|r_0| = |r_1| \leq s$, $r_0, r_1 \supseteq q$, r_0, r_1 e-split, and $U_{e-1,s}(r_0) = U_{e-1,s}(r_1) = u$, then we take the first such strings in $R^*_{e-1,s}$ (in some effective ordering of pairs of strings) and let $T_{e,s}(0^k 0)(0^k 1)$ be this pair. Otherwise, we let $T_{e,s}(0^k 0)(0^k 1) = (q_0, q_1)$.

It is now easy to see that $T_{e,s}(0^k)(0^k1)$ split $T(0^k)$ and have the same length, that $T_{e,s}(0^k0) \in R^-_{e-1,s}$, $T_{e,s}(0^k1) \in R^*_{e-1,s}$ and that C2, C3, C4 all remain true up through level $k+1$ of $T_{e,s}$.

Case 2. Same as Case 1 except that $q_0 \subset A_s$, $q_{00} \not\subset A_s$.

Then by C2 for $T_{e,s-1}$, $U_{e-1,s-1}(q_0, q_1) = u$. The $T_{e-1,s-1}$ left successor of q_0 may or may not be $\subset A_s$, but in either case one part or the other of Lemma 6 says that some extension p_1 of q_1 is on $R^*_{e-1,s}$ and $U_{e-1,s}(p_1) = U_{e-1,s-1}(q_1) = u$. Then we use this p_1 to define $T_{e,s}(0^k0)(0^k1)$ unless we can improve the e-state. That is, if $S_{e,s-1}(q_0, q_1) = 0$ and there are strings $r_0, r_1 \in R^*_{e-1,s}$ meeting the conditions as given in Case 1, then we define $T_{e,s}(0^k0)(0^k1)$ to be the first such pair. Otherwise, we define $T_{e,s}(0^k0)(0^k1)$ as follows. We will take $T_{e,s}(0^k1)$ to be an extension p_1 of q_1 such that $p_1 \in R^*_{e-1,s}$ and $U_{e-1,s}(p_1) = u$ (such p_1 exists as noted above) and then we will take $T_{e,s}(0^k0)$ to be the element p in $R^-_{e-1,s}$ with $|p| = |p_1|$ (so $q_0 \subseteq T_{e,s}(0^k0)$ and $U_{e-1,s}(T_{e,s}(0^k0)) = U_{e-1,s}(p_1) = u$). To choose p_1, take the shortest possible p_1 with length $\geq s$, if such p_1 exists; otherwise, take the shortest possible p_1.

We now verify one part of condition C4. The other needed verifications are easy. Suppose that $U_{e,s-1}(q_{00}) = U_{e,s-1}(q_0)$. We must show that there is an extension p_1 of q_1 on $R^*_{e-1,s}$ with $|p_1| \geq s$ and $U_{e-1,s}(p_1) = u$. Take $r \in R^-_{e-1,s-1}$ such that $q_0 \subseteq r \subset q_{00}$, $r \subset A_s$, and $r_0 \not\subset A_s$ where r_0 is the $T_{e-1,s-1}$ left successor of r. Since $U_{e,s-1}(q_{00}) = U_{e,s-1}(q_0)$, we have $U_{e-1,s-1}(q_{00}) = U_{e-1,s-1}(q_0) = u$. Thus by Lemma 4, $u = U_{e-1,s-1}(q_{00}) \preceq U_{e-1,s-1}(r_0) \preceq U_{e-1,s-1}(r) \preceq U_{e-1,s-1}(q_0) = u$. Hence $U_{e-1,s-1}(r) = U_{e-1,s-1}(r_0) = u$. Let p^* be the extension of q_1 by 1's with length $|r|$, so $p^* \in R^*_{e-1,s-1}$. By Lemma 6, there is an extension p_1 of p^* in $R^*_{e-1,s}$ with $U_{e-1,s}(p_1) = U_{e-1,s-1}(p^*) = u$ and $|p_1| \geq s$, as desired.

Case 3. Neither Case 1 nor Case 2 holds.

In this case either there is no $T_{e,s-1}$ since $s = 0$ or $q \notin R_{e,s-1}$ or q is in $R_{e,s-1}$ but the successors of q on $T_{e,s-1}$ do not have the right $e-1$ state as dictated to make C2 true. In this case we will ignore what happened at stage $s-1$ and just do the obvious thing. That is, if there are strings r_0, r_1 satisfying the properties given in Case 1, then we define $T_{e,s}(0^k0)(0^k1)$ to be the first such pair of strings. If no such pair exists, then we just define $T_{e,s}(0^k0)(0^k1)$ to be the $T_{e-1,s}$-successors of q.

We must verify that condition C4 is met; the other verifications are routine. Suppose that $s > 0$, $q \in R^-_{e,s-1}$, and with q_0, q_1, q_{00} as in the statement of C4 suppose that $q_0 \subset A_s$. Let p_0, p_1 be the $T_{e-1,s-1}$-successors of q, and let $u' = U_{e-1,s-1}(p_0, p_1)$. Since neither Case 1 nor Case 2 applies, we must have $u' \neq u$. Now $q \in R^-_{e-1,s} \cap R^-_{e-1,s-1}$ and since $T_{e,s-1} \subseteq^* T_{e-1,s-1}$, $p_0 \subseteq q_0 \subset A_s$. Thus by C4 for $T_{e-1,s}$, $u \succeq u'$. But $u \neq u'$, so $u \succ u'$. By C2 for $T_{e,s-1}$, $U_{e-1,s-1}(q_0, q_1) = u'$, and by construction, $U_{e-1,s}(r_0, r_1) = u$. Thus $U_{e,s}(r_0, r_1) \succ U_{e,s-1}(q_0, q_1)$, so C4 holds.

This completes the construction.

The next Lemma is the key to showing that the construction settles down. It states that if $T_{e,s}(0^k)$ has settled down by stage s_0 and u is the greatest e-state that the successors of $T_{e,s}(0^k)$ have at any stage later than s_0, then any splitting with e-state u put onto a tree $T_{e,s}$ with $s \geq s_0$ above $T_{e,s}(0^k)$ is permanent unless diagonalization or stretching interfere.

Lemma 7 *Suppose that for all $s \geq s_0$, $T_{e,s}(0^k) = T_{e,s_0}(0^k)$. Let*

$$u = \max\{U_{e,s}(T_{e,s}(0^k0)(0^k1)) : s \geq s_0\}.$$

Suppose that $k' \geq k$, $s_1 \geq s_0$, $U_{e,s_1}(T_{e,s_1}(0^{k'}0)) = u$, and $T_{e,s_1}(0^{k'}0) \subset A_{s_1+1}$. Then

$$U_{e,s_1+1}(T_{e,s_1+1}(0^{k'}0)(0^{k'}1)) = u.$$

(Hence by C4, $T_{e,s_1+1}(0^{k'}0) \supseteq T_{e,s_1}(0^{k'}0)$, and $T_{e,s_1+1}(0^{k'}1) \supseteq T_{e,s_1}(0^{k'}1)$. Further, if $T_{e,s_1}(0^{k'}00) \subset A_{s_1+1}$, then $T_{e,s_1+1}(0^{k'}0)(0^{k'}1) = T_{e,s_1}(0^{k'}0)(0^{k'}1)$, while if $T_{e,s_1}(0^{k'}00) \not\subset A_{s_1+1}$ and $U_{e,s_1}(T_{e,s_1}(0^{k'}00)) = u$, then $|T_{e,s_1+1}(0^{k'}0)| \geq s_1 + 1$.)

Proof: We use induction on k'. First suppose that $k' = k$. Let $q = T_{e,s_1}(0^k) = T_{e,s_1+1}(0^k)$. Then

$$\begin{array}{rcll} u & = & U_{e,s_1}(T_{e,s_1}(0^k0)) & \text{by hypothesis, since } k = k' \\ & \preceq & U_{e,s_1+1}(T_{e,s_1+1}(0^k0)) & \text{by C4 since } T_{e,s_1}(0^k0) \subset A_{s_1+1} \\ & \preceq & u & \text{by definition of } u. \end{array}$$

Thus $u = U_{e,s_1+1}(T_{e,s_1+1}(0^k0)(0^k1))$, as desired.

Now suppose that the result holds for some $k' \geq k$ and that for some $s_1 \geq s_0$, $U_{e,s_1}(T_{e,s_1}(0^{k'+1}0)) = u$ and $T_{e,s_1}(0^{k'+1}0) \subset A_{s_1+1}$. Then

$$\begin{array}{rcll} u & = & U_{e,s_1}(T_{e,s_1}(0^{k'+1}0)) & \text{by hypothesis} \\ & \preceq & U_{e,s_1}(T_{e,s_1}(0^{k'}0)) & \text{by Lemma 4} \\ & \preceq & U_{e,s_1}(T_{e,s_1}(0^{k}0)) & \text{by Lemma 4} \\ & \preceq & u & \text{by definition of } u \end{array}$$

Thus $u = U_{e,s_1}(T_{e,s_1}(0^{k'}0))$ and $T_{e,s_1}(0^{k'}00) = T_{e,s_1}(0^{k'+1}0) \subset A_{s_1+1}$. So by induction hypothesis, $U_{e,s_1+1}(T_{e,s_1+1}(0^{k'}0)(0^{k'}1)) = u$ and $T_{e,s_1+1}(0^{k'}0)(0^{k'}1) = T_{e,s_1}(0^{k'}0)(0^{k'}1)$.

We now have

$$\begin{array}{rcll} u & = & U_{e,s_1+1}(T_{e,s_1+1}(0^{k'}0)) & \\ & \succeq & U_{e,s_1+1}(T_{e,s_1+1}(0^{k'+1}0)) & \text{by Lemma 4} \\ & \succeq & U_{e,s_1}(T_{e,s_1}(0^{k'+1}0)) & \text{by C4 since } T_{e,s_1}(0^{k'+1}0) \subset A_{s_1+1} \text{ and} \\ & & & T_{e,s_1+1}(0^{k'}0) = T_{e,s_1}(0^{k'}0) \\ & = & u. & \end{array}$$

Thus $u = U_{e,s_1+1}(T_{e,s_1+1}(0^{k'+1}0)(0^{k'+1}1))$ as desired. ∎

Lemma 8 *For all $e \geq -1$ and $s, k \geq 0$, $T_{e,s}(0^k) \subseteq T_{e+k,s}(\emptyset)$.*

Proof: By induction on k. For $k = 0$ the result is trivial. Suppose that the result is true for k. Then $T_{e+k+1,s}(\emptyset) = T_{e+k,s}(0) \supset T_{e+k,s}(\emptyset) \supseteq T_{e,s}(0^k)$, so $T_{e+k+1,s}(\emptyset) \supset T_{e,s}(0^k)$. But $T_{e+k+1,s} \subseteq^* T_{e,s}$, so $T_{e+k+1,s}(\emptyset) \in R^-_{e,s}$. Thus $T_{e+k+1,s}(\emptyset) \supseteq T_{e,s}(0^{k+1})$. ∎

Lemma 9 *For all $e \geq 1$ and all $\sigma \in \Omega$, $\lim_s T_{e,s}(\sigma)$, $\lim_s U_{e,s}(T_{e,s}(\sigma))$ both exist.*

Proof: By induction on e. Suppose that the result is true for all $e' < e$. We proceed by induction on $|\sigma|$ to show that the desired limits exist. First suppose $\sigma = \emptyset$. If $e = -1$, then $T_{e,s}(\sigma) = \emptyset$ and $U_{e,s}(T_{e,s}(\sigma)) = \emptyset$ for all s, so the desired limits certainly exist. If $e > -1$, then for all s we have $T_{e,s}(\emptyset) = T_{e-1,s}(0)$ and $U_{e,s}(T_{e,s}(\emptyset)) = U_{e-1,s}(T_{e-1,s}(0))*1$ so by induction hypothesis the limits exist.

Now suppose that $\lim_s T_{e,s}(0^k)$ exists for some $k \geq 0$. Take s_0 such that

1. For all $s \geq s_0$, $T_{e,s}(0^k) = T_{e,s_0}(0^k)$.

2. For all $s \geq s_0$, for all $n \leq e+k+1$, P_e does not receive attention at stage s.

Note that by (2) and Lemma 8, for all $s \geq s_0$, $T_{e,s}(0^k00) \subset A_{s+1}$. Let $u = \max_{s \geq s_0} U_{e,s}(T_{e,s}(0^k0)(0^k1))$ and take $s_1 \geq s_0$ with $U_{e,s_1}(T_{e,s_1}(0^k0)(0^k1)) = u$. Then by induction on s, using Lemma 7 and the fact noted above, for all $s \geq s_1$,

$$T_{e,s}(0^k0)(0^k1) = T_{e,s_1}(0^k0)(0^k1)$$

and

$$U_{e,s}(T_{e,s}(0^k0)(0^k1)) = u.$$

∎

For all e, σ, let $T_e(\sigma) = \lim_s T_{e,s}(\sigma)$, so T_e is a tree. Define U_e by $U_e(T_e(\sigma)) = \lim_s U_{e,s}(T_{e,s}(\sigma))$. By Lemma 4, for all $q, r \in R_e^-$, if $q \subseteq r$, then $U_e(q) \succeq U_e(r)$. Thus, for fixed e, as k increases, $U_e(T_e(0^k))$ decreases, so has a limit, say u_e. Then $u_e \subset u_{e+1}$ for all $e \geq -1$.

For all $e, s \geq 0$, $T_{e,s} \subseteq^* T_{-1,s}$ and hence for any k, if

$$x = \mu y(T_{e,s}(0^k0)(y) \neq T_{e,s}(0^k1)(y))$$

then for all z

$$x \leq z \leq |T_{e,s}(0^k1)| \rightarrow T_{e,s}(0^k1)(z) = 1.$$

Lemma 10 *For all $e \geq 0$, P_e is met.*

Proof: If P_e ever receives attention, then for some x, $W_e(x) = 1 = A(x)$. If P_e never receives attention, let

$$x = \mu y(T_e(0)(y) \neq T_e(1)(y)).$$

Then $x \notin W_e$ (else P_e would eventually receive attention), so $W_e(x) = 0 = T_e(0)(x) = A(x)$. ∎

Lemma 11 *For all $e \geq 0$, Q_e is met.*

Proof: Fix $e \geq 0$. We wish to use Lemmas 1 and 2 to show that Q_e is met and hence we need to define a sequence $\langle T'_{e,t} : t \geq 0\rangle$ satisfying the hypotheses of one of these two Lemmas. Take $\overline{k} \geq 1$ so that $U_e(T_e(0^{\overline{k}-1})) = u_e$. Then for all $k \geq \overline{k}$ we also have $U_e(T_e(0^k)) = u_e$. Fix t_0 such that for all $s \geq t_0$,

$$\begin{aligned}
&T_{e,s}(0^{\overline{k}-1}) = T_e(0^{\overline{k}-1}),\\
&T_{e,s}(0^{\overline{k}}) = T_e(0^{\overline{k}}),\\
&U_{e,s}(T_{e,s}(0^{\overline{k}-1})) = u_e = U_{e,s}(T_{e,s}(0^{\overline{k}})).
\end{aligned}$$

Given $t \geq 0$, let $k = \overline{k} + t$ and find $s_0 \geq t_0$ such that

$$\begin{aligned}
&s_0 \geq |T_{e,s_0}(0^k)|,\\
&U_{e,s_0}(T_{e,s_0}(0^k)) = u_e.
\end{aligned}$$

Define $T'_{e,t}$ to be the subtree of T_{e,s_0} above $T_{e,s_0}(0^{\overline{k}})$. Once the values of u_e, $\overline{k}$, and t_0 are fixed, for a given t one can effectively find k and s_0, so the sequence $\langle T'_{e,t} : t \geq 0\rangle$ is recursive.

Note that level t of $T'_{e,t}$ consists of the leftmost $t+1$ strings (in the usual left to right ordering) in level $\overline{k}+t = k$ of T_{e,s_0}, i.e., those strings in level k of T_{e,s_0} which extend $T_{e,s_0}(0^{\overline{k}})$. Let these strings be $q_{\overline{k}}, \ldots, q_k$ where $q_k = T_{e,s_0}(0^k)$ and for $\overline{k} \leq k' < k$, $q_{k'}$ is $T_{e,s_0}(0^{k'}1)$ extended by enough 1's to have the right length.

Suppose that $u_e(e) = 1$. Then we claim that there are no e-splittings on level t of $T'_{e,t}$, i.e., no e-splittings among $q_{\overline{k}}, \ldots, q_k$. For if there were an e-splitting among these strings then since the strings are all of the same length and we are considering tt-reductions, there will be an e-splitting involving $q_k = T_{e,s_0}(0^k)$. Further, the strings on level k of T_{e,s_0} have length less than or equal to s_0 and have the same $e-1$-state at stage s_0 as $T_{e,s_0}(0^{\overline{k}})(0^{\overline{k}-1}1)$, the T_{e,s_0} successors of $T_{e,s_0}(0^{\overline{k}-1})$ (namely, u_{e-1}). Thus by C3, $T_{e,s_0}(0^{\overline{k}})(0^{\overline{k}-1}1)$ must e-split and they have length $\leq s_0$, so they cannot have e-state u_e at stage s_0, contradicting our choice of $s_0 \geq t_0$.

Now suppose that $u_e(e) = 0$. Then by Lemma 4, for each k' with $\overline{k} \leq k' < k$,

$$U_{e,s_0}(T_{e,s_0}(0^{k'}0)(0^{k'}1)) = u_e$$

and hence $T_{e,s_0}(0^{k'}0)(0^{k'}1)$ e-split. That is, for $0 \leq k' < t$, $T'_{e,t}(0^{k'}0)(0^{k'}1)$ e-split, as demanded by Lemma 2.

Thus, we will be done if we can show that one of the strings on level t of $T'_{e,t}$ (i.e., one of $q_{\overline{k}}, \ldots, q_k$) is on A. If $t = 0$ (so $k = \overline{k}$) then this fact is immediate since level 0 of $T'_{e,0}$ is $T_{e,s_0}(0^{\overline{k}}) = T_e(0^{\overline{k}}) \subset A$. Thus we may assume that $t > 0$. By induction on $s \geq s_0$, we show that there is some $q_{k'}$ such that $A_s \supset q_{k'}$. In fact we show that for every $s \geq s_0$ there is a k_s, $\overline{k} \leq k_s \leq k-1$ such that

1. $T_{e,s}(0^{k_s}0) \supseteq q_{k_s+1}$, $T_{e,s}(0^{k_s}1) \supseteq T_{e,s_0}(0^{k_s}1)$,
2. $(\forall k')(\overline{k} \leq k' \leq k_s - 1 \to T_{e,s}(0^{k'}0)(0^{k'}1) = T_{e,s_0}(0^{k'}0)(0^{k'}1))$,
3. $(\forall k')(\overline{k} \leq k' \leq k_s \to U_{e,s}(T_{e,s}(0^{k'}0)(0^{k'}1)) = u_e)$.

The intuition is that at stage s the eth tree is the same up through level k_s as it was at stage s_0 and at level k_s+1 the only change has been stretching through the corresponding q_{k_s+1}.

We may take $q_{s_0} = k-1$. (Then (3) follows from Lemma 4.) Suppose that $s \geq s_0$ and we have k_s. Let k_{s+1} be the largest number $\leq k_s$ such that $T_{e,s}(0^{k_{s+1}}0) \subset A_{s+1}$. Since $T_{e,s}(0^{\overline{k}}) = T_{e,s+1}(0^{\overline{k}}) \subset A_{s+1}$, $\overline{k} - 1 \leq k_{s+1}$. (We will see later that $\overline{k} \leq k_{s+1}$.)

Suppose that $\overline{k} \leq k' \leq k_{s+1}$. Then by induction hypothesis (since $k_{s+1} \leq k_s$), $U_{e,s}(T_{e,s}(0^{k'}0)(0^{k'}1)) = u_e$ and

$$T_{e,s}(0^{k'}0) \subseteq T_{e,s}(0^{k_{s+1}}0) \subset A_{s+1},$$

so by Lemma 7

$$U_{e,s+1}(T_{e,s+1}(0^{k'}0)(0^{k'}1)) = u_e.$$

and

$$T_{e,s+1}(0^{k'}0) \supseteq T_{e,s}(0^{k'}0),$$

$$T_{e,s+1}(0^{k'}1) \supseteq T_{e,s}(0^{k'}1).$$

If, in addition, $k' \leq k_{s+1} - 1$, then

$$T_{e,s}(0^{k'}00) \subseteq T_{e,s}(0^{k_{s+1}}0) \subset A_{s+1}$$

so by Lemma 7,

$$T_{e,s+1}(0^{k'}0)(0^{k'}1) = T_{e,s}(0^{k'}0)(0^{k'}1).$$

Thus (2) and (3) hold for k_{s+1}.

To see that (1) holds, first suppose that $k_{s+1} = k_s$. Then

$$\begin{array}{rcll} T_{e,s+1}(0^{k_{s+1}}0) & \supseteq & T_{e,s}(0^{k_{s+1}}0) & \text{as established above} \\ & = & T_{e,s}(0^{k_s}0) & \\ & \supseteq & q_{k_s+1} & \text{by induction hypothesis} \\ & = & q_{k_{s+1}+1} & \end{array}$$

and

$$\begin{array}{rcl} T_{e,s+1}(0^{k_{s+1}}1) & \supseteq & T_{e,s}(0^{k_{s+1}}1) \\ & = & T_{e,s}(0^{k_s}1) \\ & \supseteq & T_{e,s_0}(0^{k_s}1) \\ & = & T_{e,s_0}(0^{k_{s+1}}1) \end{array}$$

so (1) holds and $\overline{k} \leq k_s = k_{s+1}$, so we are done.

Hence, for the rest of the proof, we suppose that $k_{s+1} < k_s$. Then $T_{e,s}(0^{k_{s+1}+1}0) \not\subset A_{s+1}$, and, by induction hypothesis, since $\overline{k} \leq k_{s+1}+1 \leq k_s$, we have $U_{e,s}(T_{e,s}(0^{k_{s+1}+1}0)) = u_e$. Thus, by Lemma 7,

$$|T_{e,s+1}(0^{k_{s+1}}0)| \geq s+1 > s_0 \geq |q_{k_{s+1}+1}|.$$

If we now show that $q_{k_{s+1}+1} \subset A_{s+1}$, then we will have $T_{e,s+1}(0^{k_{s+1}}0) \supseteq q_{k_{s+1}+1}$.

To see that $q_{k_{s+1}+1} \subset A_{s+1}$, first note that since $T_{e,s}(0^{k_{s+1}+1}0) \not\subset A_{s+1}$, some $P_{e'}$ receives attention at stage $s+1$. If $e' < e$, then $T_{e',s}(0) \subseteq T_{e,s}(\emptyset)$ and a number $< |T_{e,s}(\emptyset)|$ enters A at stage $s+1$, contradicting $T_{e,s}(0^{\overline{k}}) \subset A_{s+1}$. Thus $e \leq e'$ and hence $T_{e',s}(0) \in R^-_{e,s}$, $T_{e',s}(1) \in R^*_{e,s}$. Further, since $T_{e,s}(0^{k_{s+1}+1}) \subset A_{s+1}$, we have $T_{e,s}(0^{k_{s+1}+1}) \subseteq T_{e',s}(0), T_{e',s}(1)$ and since $T_{e,s}(0^{k_{s+1}+1}0) \not\subset A_{s+1}$, $T_{e,s}(0^{k_{s+1}+1}0) \not\subseteq T_{e',s}(1)$. Thus,

$$\begin{array}{rcl} T_{e,s}(0^{k_{s+1}+1}0) & \subseteq & T_{e',s}(0) \\ T_{e,s}(0^{k_{s+1}+1}1) & \subseteq & T_{e',s}(1). \end{array}$$

Hence,

$$\mu y(T_{e',s}(0)(y) \neq T_{e',s}(1)(y)) = \mu y(T_{e,s}(0^{k_{s+1}+1}0) \neq T_{e,s}(0^{k_{s+1}+1}1))$$

and if we let x_0 be this number, then x_0 as well as any number z with $x_0 \leq z \leq s+1$ is put into A_{s+1}.

First suppose that $k_{s+1} + 1 < k_s$. Then by induction hypothesis,

$$T_{e,s}(0^{k_{s+1}+1}0)(0^{k_{s+1}+1}1) = T_{e,s_0}(0^{k_{s+1}+1}0)(0^{k_{s+1}+1}1).$$

Since $s+1 > s_0 \geq |q_{k_{s+1}+1}|$, adding to A_{s+1} all numbers z, $x_0 \leq z \leq s+1$ and no others causes $q_{k_{s+1}+1} \subset A_{s+1}$, as desired.

Now suppose that $k_{s+1}+1 = k_s$. Then by induction hypothesis,

$$T_{e,s}(0^{k_{s+1}+1}0) \supseteq q_{k_s+1} \supseteq T_{e,s_0}(0^{k_s}0)$$

and

$$T_{e,s}(0^{k_{s+1}+1}1) \supseteq T_{e,s_0}(0^{k_s}1)$$

so $x_0 = \mu y(T_{e,s_0}(0^{k_s}0)(y) \neq T_{e,s_0}(0^{k_s}1)(y))$. Thus $q_{k_{s+1}+1} = q_{k_s} \subset A_{s+1}$.

Thus we now have $T_{e,s+1}(0^{k_{s+1}}0) \supseteq q_{k_{s+1}+1}$ and hence $k_{s+1} \neq \overline{k}-1$ (else $T_{e,s+1}(0^{\overline{k}}) \neq T_{e,s_0}(0^{\overline{k}})$), so $\overline{k} \leq k_{s+1}$.

Finally,

$$\begin{array}{rcll} T_{e,s+1}(0^{k_{s+1}}1) & \supseteq & T_{e,s}(0^{k_{s+1}}1) & \text{by Lemma 7} \\ & = & T_{e,s_0}(0^{k_{s+1}}1) & \text{by induction hypothesis,} \\ & & & \text{since } \overline{k} \leq k_{s+1} \leq k_s - 1. \end{array}$$

∎

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