Embeddings of N_5 and the Contiguous Degrees

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Abstract

Downey and Lempp [8] have shown that the contiguous computably enumerable (c.e.) degrees, i.e. the c.e. Turing degrees containing only one c.e. weak truth-table degree, can be characterized by a local distributivity property. Here we extend their result by showing that a c.e. degree $\mathbf{a}$ is noncontiguous if and only if there is an embedding of the nonmodular 5-element lattice N_5 into the c.e. degrees which maps the top to the degree $\mathbf{a}$. In particular, this shows that local nondistributivity coincides with local nonmodularity in the computably enumerable degrees.

1 Introduction

Lachlan [13] has shown that the two five-element nondistributive lattices, the modular lattice M_3 and the nonmodular lattice N_5 (see Figure 1 below) can be embedded into the upper semilattice $(\mathbf{E}, \leq)$ of the computably enumerable degrees. These lattices capture nondistributivity and nonmodularity in the sense that every nondistributive lattice contains one of these lattices as a sublattice and that every nonmodular lattice contains a copy of the lattice N_5 as a sublattice.

However, Lachlan's result cannot be extended to show the embeddability of all finite lattices into $(\mathbf{E}, \leq)$. Lachlan and Soare [15] have shown that the top of any M_3 in $(\mathbf{E}, \leq)$ is nonbranching whence the lattice S_8 consisting of the lattice M_3 with a diamond (2-atom Boolean algebra) on the top cannot be embedded into $(\mathbf{E}, \leq)$. Moreover, Downey [5] has shown that there are initial segments of $(\mathbf{E}, \leq)$ which do not contain a copy of the lattice M_3. (For more recent results along this line see [4].) In contrast, the limitations for embeddings of the nonmodular lattice N_5 are much less severe; e.g. there are finite chains of the lattice N_5 of any length in $(\mathbf{E}, \leq)$ (see e.g. [1]) whence, in particular there are embeddings of N_5 with branching top, and every interval of $(\mathbf{E}, \leq)$ contains a copy of N_5 (see Ambos-Spies and Fejer [3]).

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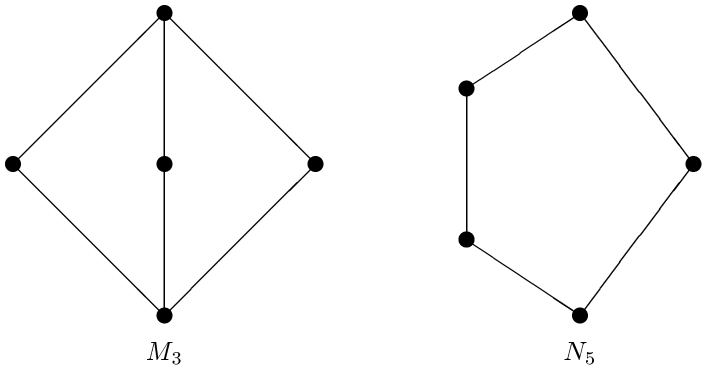

Figure 1

Examples of degrees which are neither the top of an M_3 nor an N_5 are the contiguous degrees, i.e., the c.e. degrees which contain only one c.e. *wtt*-degree (see Ladner and Sasso [16]). Lachlan (see [19]) has observed that every contiguous degree $\mathbf{a}$ is locally distributive, i.e.,

$$\begin{aligned}&\forall \mathbf{a}_0, \mathbf{a}_1, \mathbf{b}(\mathbf{b} \leq \mathbf{a} = \mathbf{a}_0 \vee \mathbf{a}_1 \Rightarrow \\ &\quad \exists \mathbf{b}_0, \mathbf{b}_1(\mathbf{b} = \mathbf{b}_0 \vee \mathbf{b}_1 \; \& \; \mathbf{b}_0 \leq \mathbf{a}_0 \; \& \; \mathbf{b}_1 \leq \mathbf{a}_1))\end{aligned}$$

This definition is justified by the fact that a lattice is distributive if and only if every element of it is locally distributive. A locally distributive degree cannot be the top of a lattice embedding M_3 or N_5.

Moreover, local modularity of a degree $\mathbf{a}$ can be defined correspondingly by

$$\begin{aligned}&\forall \mathbf{a}_0, \mathbf{a}, \mathbf{b}(\mathbf{a}_0 \leq \mathbf{b} \leq \mathbf{a} = \mathbf{a}_0 \vee \mathbf{a}_1 \Rightarrow \\ &\quad \exists \mathbf{b}_0, \mathbf{b}_1(\mathbf{b} = \mathbf{b}_0 \vee \mathbf{b}_1 \; \& \; \mathbf{b}_0 \leq \mathbf{a}_0 \; \& \; \mathbf{b}_1 \leq \mathbf{a}_1))\end{aligned}$$

Again, a lattice is modular if and only if all of its elements are locally modular, and a locally modular degree cannot be the top of a lattice embedding of N_5.

Recently, Downey and Lempp [8] have shown that the contiguous degrees coincide with the locally distributive degrees. Here we extend this result by showing that

Theorem 1.1 *Every noncontiguous c.e. degree is the top of a lattice embedding of the nonmodular lattice N_5 into $(\mathbf{E}, \leq)$.*

By Lachlan's observation and the simple algebraic facts above this immediately gives the following equivalences.

Corollary 1.2 *Let $\mathbf{a}$ be a computably enumerable degree. The following are equivalent.*

(i) **a** *is noncontiguous.*

(ii) **a** *is the top of an* N_5.

(iii) **a** *is not locally modular.*

(iv) **a** *is not locally distributive.*

In particular this shows that every top of an M_3 is also a top of an N_5. Moreover, since contiguous degrees are low$_2$ (see [11] and [12]), every non-low$_2$ c.e. degree is the top of an N_5 lattice.

Our notation is standard. See Soare [18] for unexplained notation. We use "computable" as a synonym for "recursive".

The outline of the paper is as follows. In Section 3 we give the proof of Theorem 1.1. We use marker systems for modeling reductions. In Section 2 we discuss such systems abstractly and give a characterization of noncontiguity in terms of the existence of a marker system with a certain property which is exactly what is needed for our main result. Finally, in Section 4 we discuss some open problems.

2 Marker Systems

In the proof of our main theorem we will use marker systems for modeling the use functions of Turing reductions to c.e. sets (see for instance Soare [18], IX, 4). In this section we will introduce this concept and summarize the facts we will need. In particular, we will give a characterization of the noncontiguous degrees in terms of markers.

Definition 2.1 *A* marker system *(or just a* marker *for short) is a binary computable partial function* $\lambda x, s.\ \gamma(x, s)$ *with computable domain satisfying the following conditions for all numbers* x, y, s, t.

(M1) *If* $\gamma(x, s) \neq \gamma(x, s+1)\downarrow$, *then* $\gamma(x, s+1) \geq s+1$ *and* $\gamma(x, s+1) > \gamma(x, t)$ *for all* $t \leq s$ *such that* $\gamma(x, t)\downarrow$.

(M2) *If* $\gamma(x, s)\downarrow \neq \gamma(x, s+1)$, $x < y$ *and* $\gamma(y, s)\downarrow$, *then* $\gamma(y, s) \neq \gamma(y, s+1)$.

(M3) *If* $x < y$ *and* $\gamma(y, s)\downarrow$ *then* $\gamma(x, s)\downarrow$ *and* $\gamma(x, s) < \gamma(y, s)$.

(M4) *If* $\gamma(x, s)\downarrow, \gamma(y, t)\downarrow, x \neq y$, *then* $\gamma(x, s) \neq \gamma(y, t)$.

(M5) *The limit* $\lim_s \gamma(x, s)$ *exists and is defined, i.e., there is a number* s_0 *such that* $\gamma(x, s_0)\downarrow$ *and* $\gamma(x, s) = \gamma(x, s_0)$ *for* $s \geq s_0$.

The marker system γ *is* total *if the function* γ *is total.*

If $\gamma(x,s) = y$ then we call y the *position* of $\gamma(x)$ *at stage* s, and we let $\gamma^*(x) = \lim_s \gamma(x,s)$ denote the final position of $\gamma(x)$. We say that $\gamma(x)$ is *lifted at stage* $s+1$ if $\gamma(x,s)\downarrow \neq \gamma(x,s+1)$, and we say that $\gamma(x)$ is *put down at stage* $s+1$ if $\gamma(x,s) \neq \gamma(x,s+1)\downarrow$. Note that, by (M1), if $\gamma(x)$ is put down at stage $s+1$ then $\gamma(x,s+1)$ is greater than s and greater than all previous positions of $\gamma(x)$ (if any). Hence, in particular, $\gamma^*(x) = \sup_s \gamma(x,s)$. Condition (M3) enforces that at any stage the marker system is defined on an initial segment of ω and there the positions are strictly increasing in the argument. This implies that if $\gamma(x,s)\downarrow$ then $\gamma(x,s) \geq x$.

Definition 2.2 *Let A be a c.e. set and let $\{A_s\}_{s\in\omega}$ be a computable enumeration of A. Then A* controls *the marker γ* via *$\{A_s\}_{s\in\omega}$ if, for all numbers x and s,*

(2.1) $\quad \gamma(x,s)\downarrow \neq \gamma(x,s+1) \ \Rightarrow\ A_s\restriction\gamma(x,s)+1 \ \neq\ A\restriction\gamma(x,s)+1.$

A controls γ *if A controls γ via some computable enumeration.*

Lemma 2.3 *If the c.e. set A controls the marker γ then $\gamma^* \leq_T A$.*

Proof. Fix a computable enumeration $\{A_s\}_{s\in\omega}$ of A such that A controls γ via $\{A_s\}_{s\in\omega}$, and let s_x be the least stage s such that $\gamma(x,s)\downarrow$ and $A_s\restriction\gamma(x,s)+1 = A\restriction\gamma(x,s)+1$. Note that s_x exists by (M5) and s_x can be computed by A. By (2.1), $\gamma^*(x) = \gamma(x,s_x)$ whence $\gamma^* \leq_T A$. □

Definition 2.4 *Let A be a Δ^0_2-set and let $\{A_s\}_{s\in\omega}$ be a computable approximation of A. The marker γ* computes *A* via *$\{A_s\}_{s\in\omega}$ if $A_{\gamma^*(x)}(x) = A(x)$ for all numbers x. γ* computes *A if γ computes A via some computable approximation.*

Note that for a c.e. set A and a computable enumeration $\{A_s\}_{s\in\omega}$ of A, a marker γ computes A via $\{A_s\}_{s\in\omega}$ if and only if

(2.2) $\quad A_s(x) \neq A_{s+1}(x) \ \Rightarrow\ \gamma^*(x) \geq s+1$

for all numbers x and s.

Lemma 2.5 *If the marker γ computes the Δ^0_2-set A then $A \leq_T \gamma^*$.*

Proof. Immediate. □

Lemma 2.6 *Let γ be a marker system which computes a Δ^0_2-set A and which is controlled by a c.e. set B. Then $A \leq_T B$.*

Proof. By Lemmas 2.3 and 2.5. □

The converse of Lemma 2.6 is true in a strong form.

Lemma 2.7 *Let $\{A_s\}_{s\in\omega}$ be a computable approximation of a Δ^0_2 set A, let $\{B_s\}_{s\in\omega}$ be a computable enumeration of a c.e. set B and suppose that $A \leq_T B$. Then there is a total marker system γ which computes A via $\{A_s\}_{s\in\omega}$ and is controlled by B via $\{B_s\}_{s\in\omega}$.*

Proof. Fix a Turing reduction $A = \{e\}^B$ such that

(2.3) $\{e\}_s^{B_s}(x)\downarrow \Rightarrow e, x, u(B_s; e, x, s) < s$

(2.4) $\{e\}_s^{B_s}(x)\downarrow \;\&\; B_s\restriction u(B_s; e, x, s) \neq B_{s+1}\restriction u(B_s; e, x, s) \Rightarrow \{e\}_{s+1}^{B_{s+1}}(x)\uparrow$

hold for all numbers s, x ((2.4) is the usual "hat trick"). Let

$$l(s) = \mu x(A_s(x) \neq \{e\}_s^{B_s}(x))$$

be the length function of this reduction and let

$$\begin{aligned} m(s) &= \max\{z : \exists t \leq s[z \leq l(t)\ \& \\ &\qquad \forall y < z(B_t\restriction u(B_t; e, y, t) = B_s\restriction u(B_t; e, y, t))]\} \end{aligned}$$

be the modified length function. Note that $m(s) \geq l(s)$ for all s, $\lim_s l(s) = \lim_s m(s) = \omega$ and, by (2.3), $l(0) = m(0) = 0$. Now, the desired marker system γ is defined by

$$\gamma(x,s) = \begin{cases} \langle x, s, \mu t(\forall y \leq \langle x, s\rangle (l(\langle y, t\rangle) > x))\rangle & \text{if } s = 0 \text{ or} \\ & \text{if } s > 0, m(s) \leq x \\ & \text{and } m(s) < m(s-1) \\ \gamma(x, s-1) & \text{otherwise} \end{cases}$$

where we assume that $\langle ., . \rangle$ is a computable pairing function which is strictly increasing in both arguments and that $\langle x, y, z\rangle = \langle\langle x, y\rangle, z\rangle$. It is easily shown that γ is a total marker system in the sense of Definition 2.1. Moreover, by (2.3) and (2.4),

(2.5) $\forall y \leq x(\gamma(x, s) > u(B_s; e, y, s))$.

To show that γ computes A via $\{A_s\}_{s\in\omega}$, for a contradiction assume that, for some x, $A_{\gamma^*(x)}(x) \neq A(x)$. Pick s minimal such that $\gamma(x, s) = \gamma^*(x)$. Then, by definition of γ, $l(\gamma(x, s)) > x$, so $\{e\}_{\gamma(x,s)}^{B_{\gamma(x,s)}}(x) \neq A(x)$. Hence, by (2.4), there is a stage $t > \gamma(x, s)$ with $\{e\}_t^{B_t}(x)\uparrow$ whence $m(t) \leq x$. So if we take $u > \gamma(x, s)$ minimal with $m(u) \leq x$ then $\gamma(x)$ is put down at stage u. But this implies that $\gamma^*(x) > \gamma(x, s)$ contrary to assumption.

Finally, to show that B controls γ via $\{B_s\}_{s\in\omega}$, assume that $\gamma(x, s) \neq \gamma(x, s + 1)$. We have to show that $B_s\restriction\gamma(x, s) + 1 \neq B\restriction\gamma(x, s) + 1$. By definition of γ and by assumption, $m(s + 1) \leq x$ and $m(s + 1) < m(s)$, whence $B\restriction u(B_s; e, y, s) \neq B_s\restriction u(B_s; e, y, s)$ for some $y \leq x$. So the claim follows from (2.5). $\square$

In modeling a Turing reduction $A = \{e\}^B$ by a marker system γ, the fact that the use of the reduction might not be computably bounded is reflected by the fact that there is no computable bound on the part $B\restriction u(x)$ of B required for computing the final position $\gamma^*(x)$ of γ at x.

In a situation where A actually is not wtt-reducible to B, this implies that if we determine marker positions $\gamma(x, f(x))$ by some computable function $f(x)$, then for infinitely many numbers x, these positions will not be the final ones and x will enter A only after B has settled down on the initial segment $B\restriction\gamma(x, f(x))$.

Lemma 2.8 *Let A and B be c.e., let $\{A_s\}_{s\in\omega}$ and $\{B_s\}_{s\in\omega}$ be computable enumerations of A and B, respectively, and suppose that $A \leq_T B$ but $A \not\leq_{wtt} B$. Then there is a total marker system γ which computes A via $\{A_s\}_{s\in\omega}$ and which is controlled by B via $\{B_s\}_{s\in\omega}$ such that the following holds. For all computable functions f and g and for any infinite computable set S there is a function $h : \omega \to S$ such that $h(n) > g(n)$ for all numbers n and*

$$(2.6)\quad \overset{\infty}{\exists} n(n \in A - A_{h(n)+1} \;\&\; B_{h(n)}\restriction\gamma(n, f(n)) + 1 = B\restriction\gamma(n, f(n)) + 1).$$

Proof. Let γ be the marker system provided by Lemma 2.7, fix f, g and S as above and, for a contradiction, assume that the required function h does not exist. Then

$$\overset{\infty}{\forall} n\, \forall m \in S(m > g(n) \Rightarrow$$
$$[A_{m+1}(n) = A(n) \vee B_m\restriction\gamma(n, f(n)) + 1 \neq B\restriction\gamma(n, f(n)) + 1])$$

So, if we let

$$s(n) = \mu s(s \in S \;\&\; s > g(n) \;\&\; B_s\restriction\gamma(n, f(n)) + 1 = B\restriction\gamma(n, f(n)) + 1$$

then $A_{s(n)+1}(n) = A(n)$ for almost all numbers n. Since $\lambda n.\gamma(n, f(n))$ is computable this implies $A \leq_{wtt} B$ contrary to assumption. $\square$

If in Lemma 2.8 we choose $g(n)$ to be bigger than $\gamma(n, f(n))$ then, for a number n as in (2.6), the number $\gamma(n)$ has to be lifted after stage $f(n)$ and before stage $h(n) + 1$ (to a position $> f(n)$). So no marker $< f(n)$ will have to enter B after stage $h(n)$ (for the sake of B controlling γ) while n will enter A later. In the proof of our main theorem we will use this observation as follows: We will argue that certain computations will be cleared of all markers which have not yet reached their final position when we get permission (by A) to diagonalize against these computations. The technical fact we will use there is more explicitly stated in the following characterization of the noncontiguous degrees.

Theorem 2.9 *For any c.e. degree* **a** *the following are equivalent.*

(i) **a** *is not contiguous.*

(ii) *There is a c.e. set $A \in \mathbf{a}$, a computable enumeration $\{A_s\}_{s\in\omega}$ of A, and a total marker system γ which computes A via $\{A_s\}_{s\in\omega}$ and which is controlled by A via $\{A_s\}_{s\in\omega}$ such that the following holds. For any infinite computable strictly increasing sequence $\{s_n\}_{n\in\omega}$ and any computable function g with $g(n) \geq n$ for all n, there is a function $f : \omega \to \omega$ with $f(n) > g(n)$ such that for infinitely many numbers n*

(2.7) $\gamma(n, s_{g(n)}) < \gamma(n, s_{f(n)})$

(2.8) $\forall m\, (\gamma(m, s_{f(n)}) \leq s_{g(n)} \Rightarrow \gamma(m, s_{f(n)}) = \gamma^*(m))$

(2.9) $n \in A - A_{s_{f(n)}+1}$.

Proof. For a proof of the implication (i) $\Rightarrow$ (ii) let $\mathbf{a}$ be a noncontiguous c.e. degree. Then we may fix a c.e. set $A \in \mathbf{a}$ such that, for the even part $B = A \cap \{2n : n \geq 0\}$ of A, $B =_T A$ but $A \not\leq_{wtt} B$. Let $\{A_s\}_{s\in\omega}$ be a computable enumeration of A and let $\{B_s\}_{s\in\omega}$ be the corresponding computable enumeration of B, i.e., $B_s = A_s \cap \{2n : n \geq 0\}$. Moreover, fix a marker system γ as in Lemma 2.8. By choice of $\{B_s\}_{s\in\omega}$, control of γ by $\{B_s\}_{s\in\omega}$ implies control by $\{A_s\}_{s\in\omega}$. So pick a sequence $\{s_n\}_{n\in\omega}$ and a function g as in (ii). We obtain the desired function $f(n)$ as follows. Let

$$F(n) = s_{g(n)}$$

$$G(n) = \max(F(n), \gamma(n, F(n)))$$

and

$$S = \{s_n : n \in \omega\}.$$

Apply Lemma 2.8 to the functions F, G (in place of f, g) and the set S to obtain h and let f be given by

$$h(n) = s_{f(n)}.$$

Then $h(n) > G(n)$ implies $f(n) > g(n)$, so it suffices to show that for any number n as in (2.6) conditions (2.7) to (2.9) are satisfied.

Fix such an n. (2.9) is immediate. For a proof of (2.7) assume that $\gamma(n, s_{g(n)}) = \gamma(n, s_{f(n)})$. Then, since B controls γ, by the second part of (2.6), $\gamma^*(n) = \gamma(n, s_{g(n)})$. On the other hand, since $h(n) > G(n)$ it follows by definition of G and f that $\gamma(n, s_{g(n)}) < s_{f(n)}$. Since (2.9) holds for n this contradicts γ computing A via $\{A_s\}_{s\in\omega}$. For a proof of (2.8) fix m such that $\gamma(m, s_{f(n)}) \leq s_{g(n)}$. Then $\gamma(m, s_{f(n)}) = \gamma(m, s_{g(n)})$. So, by (2.7), $m < n$ whence $\gamma(m, s_{f(n)}) < \gamma(n, s_{g(n)})$. Hence, by the second part of (2.6), $B_{s_{f(n)}} \restriction \gamma(m, s_{f(n)}) + 1 = B \restriction \gamma(m, s_{f(n)}) + 1$. By B controlling γ this implies $\gamma(m, s_{f(n)}) = \gamma^*(m)$.

This completes the proof of the direction (i) $\Rightarrow$ (ii).

For a proof of the other direction fix $\mathbf{a}$, A, $\{A_s\}_{s\in\omega}$ and γ as in (ii), and let

$$\widehat{A} = \{\langle n, s, \gamma(n,s)\rangle : \gamma(n,s) \neq \gamma^*(n)\}.$$

Note that $\langle n, s, \gamma(n,s)\rangle \in \widehat{A}$ if and only if $\gamma(n,s) < \gamma(n,t)$ for some $t > s$, whence, for

$$\widehat{A}_s = \{\langle n, t, \gamma(n,t)\rangle : \ n, t < s \ \& \ \gamma(n,t) < \gamma(n,s)\},$$

$\{\widehat{A}_s\}_{s\in\omega}$ is a computable enumeration of $\widehat{A}$. Moreover, $\widehat{A} =_T \gamma^*$ whence by A controlling γ and by γ computing $A, \widehat{A} =_T A$. Hence it suffices to show that $A \not\leq_{wtt} \widehat{A}$.

For a contradiction assume that $A \leq_{wtt} \widehat{A}$ via $\{e\}$. Then, for the length function l of this reduction defined by $l(s) = \mu x(A_s(x) \neq \{e\}_s^{\widehat{A}_s}(x))$, $\lim_s l(s) = \omega$, and there is a recursive, nondecreasing function u such that $\sup_s u(\widehat{A}_s; e, x, s) < u(x)$. Now define an infinite strictly increasing computable sequence $S = \{s_n\}_{n\in\omega}$ by letting

$$s_n = \mu s > s_{n-1}(l(s) > n \text{ and } u(n) < s)$$

(where we assume that $s_{-1} = 0$) and define $g(n) = n$. Let f be the corresponding function given by the assumption (ii), and fix a number n satisfying (2.7) – (2.9). We will show that $A(n) \neq \{e\}^{\widehat{A}}(n)$ contrary to assumption.

Since $f(n) > n$, $l(s_{f(n)}) > n$ by choice of S, whence

$$(2.10) \quad A_{s_{f(n)}}(n) = \{e\}_{s_{f(n)}}^{\widehat{A}_{s_{f(n)}}}(n) = 0$$

by (2.9). On the other hand, again by choice of S, $u(n) < s_n = s_{g(n)} < s_{f(n)}$ whence by (2.8) and definition of $\{\widehat{A}_s\}_{s\in\omega}$ and $\widehat{A}$, we have $\widehat{A}\restriction u(n)+1 = \widehat{A}_{s_{f(n)}}\restriction u(n)+1$. This implies, by choice of u and by (2.10), $\{e\}^{\widehat{A}}(n) = 0$. Since $A(n) = 1$ by (2.9) this gives the desired contradiction. $\square$

Note that in the proof of the (ii) $\Rightarrow$ (i) direction of the previous theorem, we only needed a weak version of (ii), namely with $g(n) = n$ and only asserting the existence of one n rather than infinitely many. Thus, this weaker condition is also equivalent to noncontiguity.

3 Proof of the Main Theorem

In this section we present the proof of Theorem 1.1. The proof combines ideas of Downey and Lempp's characterization of the contiguous degrees in [8] with the proof of Lachlan's splitting theorem [14] presented by Ambos-Spies, Ding and Fejer in [2] and the embedding of the lattice N_5 (Lachlan [13]).

Given a noncontiguous c.e. degree $\mathbf{a}$ we fix a c.e. set $A \in \mathbf{a}$, a computable enumeration $\{A_s\}_{s\in\omega}$ of A and a total marker system ε as in Theorem 2.9. We will construct c.e. sets $B_i, \widehat{B}_i$ and C $(i = 0, 1)$ which will satisfy the following conditions.

(3.1) $A \leq_T B_0 \oplus B_1$.

(3.2) $B_0 \not\leq_T C$ and $B_1 \not\leq_T C$.

(3.3) $B_0, B_1 \leq_T A$.

(3.4) For any set D, if $D \leq_T B_0 \oplus \widehat{B}_0 \oplus C$ and $D \leq_T B_1 \oplus \widehat{B}_1 \oplus C$ then $D \leq_T C$.

(3.5) $C \leq_T A$.

(3.6) $\widehat{B}_0 \not\leq_T B_0 \oplus C$ or $\widehat{B}_1 \not\leq_T B_1 \oplus C$.

(3.7) $\widehat{B}_0, \widehat{B}_1 \leq_T A$.

Then, by letting $\mathbf{c} = deg_T(C)$, $\mathbf{b}_i = deg_T(B_i \oplus C)$ and $\widehat{\mathbf{b}}_i = deg_T(B_i \oplus \widehat{B}_i \oplus C)$ and by fixing $j \leq 1$ according to (3.6) so that $\widehat{B}_j \not\leq_T B_j \oplus C$, the degrees $\mathbf{a}$, $\mathbf{b}_0$, $\mathbf{b}_1$, $\widehat{\mathbf{b}}_j$ and $\mathbf{c}$ give an embedding of the lattice N_5 in $\mathbf{E}$ as shown in Figure 2 below.

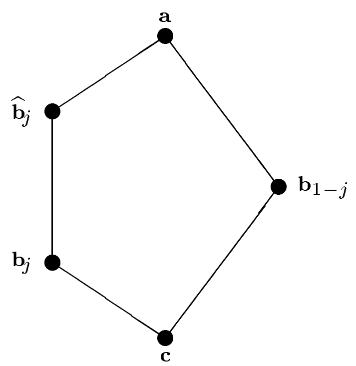

Figure 2

Before we will formally define computable enumerations $\{B_{i,s}\}_{s\in\omega}$, $\{\widehat{B}_{i,s}\}_{s\in\omega}$ and $\{C_s\}_{s\in\omega}$ of the required sets B_i, $\widehat{B}_i$ and C, respectively, we informally discuss the strategies for satisfying the conditions (3.1) – (3.7) and introduce some notation required for the construction.

Ignoring condition (3.6) for the moment, the remaining conditions assert that $\{\mathbf{c}, \mathbf{b}_0, \mathbf{b}_1, \mathbf{a}\}$ and $\{\mathbf{c}, \widehat{\mathbf{b}}_0, \widehat{\mathbf{b}}_1, \mathbf{a}\}$ both are lattice embeddings of the diamond lattice, i.e., the 2-atom Boolean algebra, into the c.e. degrees mapping the top of the lattice to the given degree $\mathbf{a}$. Lachlan's splitting theorem [14] asserts that such embeddings exist for any nonzero c.e. degree $\mathbf{a}$. Here we use a proof of Lachlan's result due to Ambos-Spies, Ding and Fejer [2].

To satisfy (3.1), we code A into the even parts of B_0 and B_1. As in the Sacks splitting theorem the side used for the coding will be determined by the restraints imposed by some of the strategies introduced below. Since the marker ε computes A, by Lemma 2.6, it suffices to ensure that B_0 and B_1 together control ε. For this sake, whenever $\varepsilon(x)$ is lifted at stage $s+1$ we put $2\varepsilon(x,s)$ for the least such x into either B_0 or B_1 at stage $s+1$ (and these marker positions will be the only even numbers entering B_0 or B_1). Hence, for

$$B_E = \{x : 2x \in B_0 \cup B_1\},$$

B_E controls ε via the enumeration

$$B_{E,s} = \{x : 2x \in (B_{0,s} \cup B_{1,s})\}$$

and, obviously, $B_E \leq_T B_0 \oplus B_1$. Hence (3.1) will hold by Lemma 2.6. Moreover, ε computes the part B_E of B_0 and B_1 used for this coding, whence, by A controlling ε and by Lemma 2.6, $B_E \leq_T A$. So this coding strategy for satisfying (3.1) is compatible with (3.3).

In order to satisfy (3.2), we meet, for each $e \in \omega$ and $i \leq 1$, the requirement

$$P_{e,i} : \; B_i \neq \{e\}^C$$

using the standard Friedberg-Muchnik technique, i.e. we will appoint a follower x, then wait for a stage s such that $\{e\}_s^{C_s}(x) = 0$ and preserve this computation (by restraining C) – we will say that x is *realized at stage* $s+1$ –, and finally put x into B_i thereby *completing the attack* on $P_{e,i}$ via x. Followers for the P-requirements will be odd in order to avoid conflicts with the strategy for satisfying (3.1).

Condition (3.3) is met as follows. The only numbers which will enter B_0 or B_1 will be either even numbers reporting the movement of an ε-marker (as described in the strategy for meeting (3.1) above) or (odd) followers for some P-requirement. As observed above, the former numbers can be recognized by A. In order to allow A to control the P-followers, we combine the Friedberg-Muchnik strategy for the P-requirements with the permitting method in the standard way: A realized follower x may enter A at stage $s+1$ only if $A_{s+1} \upharpoonright x+1 \neq A_s \upharpoonright x+1$. The P-strategy will still succeed if we appoint a new follower when the attack via x waits for permission by A to become completed.

To satisfy (3.4), we meet for each $e = \langle e_0, e_1 \rangle$ in ω, the requirement

$$N_e : \text{ If } D = \{e_0\}^{B_0 \oplus \widehat{B}_0 \oplus C} = \{e_1\}^{B_1 \oplus \widehat{B}_1 \oplus C} \text{ then } D \leq_T C.$$

using the following strategy introduced by Fejer in [10].

We will define a partial computable function γ such that, if the premise of the requirement N_e holds, then γ will be a partial marker system which computes D and which is controlled by C, whence $D \leq_T C$ by Lemma 2.6.

This is achieved as follows: Let

$$l_e(s) = max\{x : \forall y < x(\{e_0\}_s^{B_{0,s} \oplus \widehat{B}_{0,s} \oplus C_s}(y) = \{e_1\}_s^{B_{1,s} \oplus \widehat{B}_{1,s} \oplus C_s}(y)\downarrow)\}$$

denote the length of agreement between $\{e_0\}^{B_0 \oplus \widehat{B}_0 \oplus C}$ and $\{e_1\}^{B_1 \oplus \widehat{B}_1 \oplus C}$ at the end of stage s of the construction and call a stage s *e-expansionary* if $l_e(s) > l_e(t)$ for all $t < s$. Note that if the hypothesis of N_e holds then $\lim_s l_e(s) = \omega$ whence there are infinitely many e-expansionary stages.

Now the rules for putting down and lifting the marker γ have the following goal: If $\gamma(x)$ reaches its final value at stage $s+1$ then $\{e_0\}^{B_0 \oplus \widehat{B}_0 \oplus C}$ will give the correct value for $D(x)$ at stage $s+1$. This is achieved as follows. The marker $\gamma(x)$ is put down at stage $s+1$ if $\gamma(x,s)\uparrow, s$ is e-expansionary and $x < l_e(s)$. Note that if $\gamma(x)$ is put down at stage $s+1$, then the computations $\{e_0\}_s^{B_{0,s} \oplus \widehat{B}_{0,s} \oplus C_s}(x)$ and $\{e_1\}_s^{B_{1,s} \oplus \widehat{B}_{1,s} \oplus C_s}(x)$ both predict the same value for $D(x)$. This value can be false only if *both* computations will give a different value later. Hence at a stage at which the latter happens, we lift the marker $\gamma(x)$. Formally, the marker $\gamma(x)$ is lifted at stage $s+1$ if $\gamma(x,s)\downarrow$ and, for $t+1 \leq s$ maximal such that $\gamma(x)$ was put down at stage $t+1$,

$$\forall i \leq 1 \ (\{e_i\}_s^{B_{i,s} \oplus \widehat{B}_{i,s} \oplus C_s}(x) \neq \{e_0\}_t^{B_{0,t} \oplus \widehat{B}_{0,t} \oplus C_t}(x)).$$

If this happens, $\gamma(x,s)$ is put into C at stage $s+1$ thereby ensuring that C controls γ via $\{C_s\}_{s \in \omega}$. Now, if $D = \{e_0\}^{B_0 \oplus \widehat{B}_0 + C} = \{e_1\}^{B_1 \oplus \widehat{B}_1 + C}$ then $\gamma(x)$ will reach a final position, whence γ is a marker system. Moreover, γ computes D via $\{D_s\}_{s \in \omega}$, where $D_s(x) = \{e_0\}_t^{B_{0,t} \oplus \widehat{B}_{0,t} \oplus C_t}(x)$ for the least e-expansionary stage $t \geq s$ with $l_e(t) > x$.

Thus, the strategy to meet the requirements N_e is an infinitary positive one. This creates problems for the P-requirements (and the $\widehat{P}$-requirements discussed below). When they try to diagonalize, there may be marker positions which might later go into C and destroy the computation against which the diagonalization is made. If these markers are associated with lower priority meet requirements, they can be cancelled, but if a meet requirement has higher priority, its markers must be left. If $\gamma(x,s)$ is such a higher priority marker and both of the computations $\{e_i\}_s^{Bi,s \oplus \widehat{B}_{i,s} \oplus C}(x)$ converge, then the P-requirement can complete its attack and possibly destroy one of these computations, while putting on restraint to hold the other computation and thereby prevent the meet marker from going in and destroying the attack. However, if only one of these two computations exists, and it is the one which the attack could destroy, then the attack may immediately be injured by the marker going into C. To avoid this problem, the construction is put on a tree, with nodes guessing about the lengths of agreements in the N-requirements in the usual way. Then, when the strategy for a P-requirement which is on the true path wishes to begin an attack, every marker for a higher priority meet strategy which might act infinitely often is associated with computations on both sides, and so can be held from going into C by the P-requirement after it makes its attack.

Condition (3.5) is satisfied as follows. The only numbers which enter C are the positions of markers defined by the strategies that meet the N-requirements. The moves of these markers cannot be subject to permitting or the meet strategy will not work. Instead extra restraint is put on the sets $B_0, \widehat{B}_0, B_1, \widehat{B}_1$ throughout the construction by requirements which attempt to freeze meet markers. This extra restraint will create "permanent configurations", which allow an A oracle to find infinitely many numbers x and stages s such that $C_s \restriction x = C \restriction x$ and hence $C \leq_T A$.

We now turn to conditions (3.6) and (3.7). To satisfy the former we have to meet the requirements

$$\widehat{P}_e: \; \widehat{B}_0 \neq \{e_0\}^{B_0 \oplus C} \quad \text{or} \quad \widehat{B}_1 \neq \{e_1\}^{B_1 \oplus C}$$

for all numbers $e = \langle e_0, e_1 \rangle$. Just as the P-requirements, these requirements are met by a finitary diagonalization strategy using followers targeted for $\widehat{B}_0$ or $\widehat{B}_1$ as potential diagonalization witnesses. These followers are the only numbers which may enter the sets $\widehat{B}_0$ and $\widehat{B}_1$ whence (3.7) can be guaranteed by allowing a follower x to enter such a set only at a stage at which A changes below the current position of the marker $\varepsilon(x)$. Since A controls ε this obviously implies $\widehat{B}_i \leq_T A$ (for $i = 0, 1$).

The strategy for meeting the $\widehat{P}$-requirements is much more involved than the Friedberg-Muchnik strategy used for the P-requirements. In order to explain the difficulties we first describe the strategy for meeting the first disjunct

$$\widehat{P}_{e,0}: \; \widehat{B}_0 \neq \{e\}^{B_0 \oplus C}$$

of a $\widehat{P}_e$-requirement if A were under our control, i.e., if we construct an embedding of the lattice N_5 into $\mathbf{E}$ represented by the degrees $\{\mathbf{c}, \mathbf{b}_0, \mathbf{b}_1, \widehat{\mathbf{b}}_0, \mathbf{a}\}$ all of which have to be constructed by us.

Here we assume that $A = B_0 \oplus B_1 \oplus C$ and we ensure $\widehat{B}_0 \leq_T B_0 \oplus B_1$, by constructing a total marker ε which is controlled by B_0 and B_1 and computes $\widehat{B}_0$. The constructed sets have to satisfy the P- and N-requirements from above (by the above described strategies), where $\widehat{B}_1$ is empty here. Now to meet requirement $\widehat{P}_{e,0}$ we use a refinement of the Friedberg-Muchnik strategy. As there we appoint a follower x (targeted for $\widehat{B}_0$) at some stage s and wait for an expansionary stage (i.e. a stage at which the markers of the infinitary higher priority N-requirements can be controlled by both sides) $t+1 > s$ at which $\{e_0\}_t^{B_{0,t} \oplus C_t}(x) = 0$. We then want to preserve this computation by restraining

$B_0\restriction t$ and $C\restriction t$ and put x into $\widehat{B}_0$. Now, putting x into $\widehat{B}_0$ requires putting $\varepsilon(x)$ into either B_0 or B_1. On the other hand, in order to restrain $C\restriction t$ we have to restrain $B_0\oplus\widehat{B}_0\restriction t$ or $B_1\restriction t$. Since both x and $\varepsilon(x)$ might be less than t, however, we cannot complete the diagonalization (by putting x into $\widehat{B}_0$ and $\varepsilon(x)$ into B_0 or B_1) without injuring some of these restraints. This problem is overcome as follows. At stage $t+1$ we put $\varepsilon(x,t)$ into B_1, which will lift $\varepsilon(x)$ to a position above t, while we protect the computation by restraining $B_0\restriction t$ and $B_0\oplus\widehat{B}_0\restriction t$. Then at a later expansionary stage $v+1$, the protection of the computation $\{e_0\}_t^{B_{0,t}+C_t}(x)$ against higher priority markers entering C can be taken over by restraining $B_1\restriction v$ (while restraining $B_0\restriction t$ continues to protect the computation against changes in B_0). So now we are free to put x into $\widehat{B}_0$ and at the same time put $\varepsilon(x)$ into B_0.

So a completed attack on $\widehat{P}_{e,0}$ based on the follower x goes through the following 5 states:

State 1: The follower $x\geq s$ is appointed at stage $s+1$.

State 2: At some expansionary stage $t+1>s+1$, $\{e_0\}_t^{B_{0,t}\oplus C_t}(x)=0$. Restrain $B_0\restriction t$ and $B_0\oplus\widehat{B}_0\restriction t$ to protect this computation (against changes in B_0 and C).

State 3: At some stage $u+1>t+1$ put $\varepsilon(x)$ into B_1 and lift the marker to a position $\geq u$.

State 4: At some expansionary stage $v+1>u+1$ replace the restraint $B_0\oplus\widehat{B}_0\restriction t$ for preserving the C-part of the computation by the restraint $B_1\restriction v$.

State 5: At some stage $w+1>v+1$ complete the attack by putting x into $\widehat{B}_0$ and $\varepsilon(x)$ into B_0.

Now we return to the strategy for meeting the $\widehat{P}$-requirements in the actual construction. The main problems that arise in adapting the basic N_5-strategy given above to this situation are resolved using techniques from Downey and Lempp [8]. The main difference is that now the marker ε is given and the given set A has to permit the follower x to enter $\widehat{B}_0$. So, to attain state 3 we have to wait for A to lift the marker $\varepsilon(x)$ at a stage $u+1>t+1$, and, to attain state 5 we have to wait for A to change below $\varepsilon(x)$ at a stage $w+1>v+1$. As in a standard permitting argument, we can overcome these problems by starting an additional attack with some new bigger follower whenever a follower gets stuck in one of these situations.

But there is another more serious problem. When a follower x is in state 4 (or 5) then in order to preserve the computation $\{e\}^{B_0\oplus C}(x)$ we impose restraints on both B_0 and B_1, namely we restrain $B_0\restriction t$ to hold B_0 and we restrain $B_1\restriction v$ in order to hold C. (In the terminology of Downey and Lempp [8], numbers $<t$ restrained from B_0 for the n-th attack are called n-small and numbers in the interval $[t,v)$ restrained from B_1 are called n-medium.) Now we can restrain

odd numbers (i.e. P-followers) from entering B_0 and B_1 in the usual way, but by our strategy for satisfying (3.1) we cannot prevent even numbers (i.e. numbers $2\varepsilon(y, s')$ for markers $\varepsilon(y)$ lifted at stage $s' + 1$) from entering B_0 or B_1, though we may choose the set which they enter. So, for such a number z with $t \leq z < v$ we can put z into B_0 but if $z < t$ we cannot maintain both restraints.

In order to solve this problem we modify the strategy for the n-th attack as follows. Instead of just one we appoint $n + 1$ followers $x_{n,0}, \ldots, x_{n,n}$, where the next follower is appointed after the previous one is realized. So in state 1 of the attack we go through stages $s_{n,0} < t_{n,0} < \ldots < s_{n,n} < t_{n,n}$ where $x_{n,i}$ is appointed at stage $s_{n,i} + 1$, $t_{n,i} + 1$ is an expansionary stage at which the computation $\{e_0\}_{t_{n,i}}^{B_{0,t_{n,i}} \oplus C_{t_{n,i}}}(x_{n,i}) = 0$ is protected by restraining $B_0 \oplus \widehat{B}_0 \restriction t_{n,i}$ and $x_{n,i+1}$ is chosen bigger than $t_{n,i}$ so that enumerating a follower into $\widehat{B}_0$ will not injure the restraints imposed by the previous followers. The attack enters state 2 at stage $t + 1 = t_{n,n} + 1$. Then we wait for $\varepsilon(n)$ to be lifted at some stage $u + 1$ which will simultaneously clear all computations of markers $\varepsilon(m)$ with $m \geq n$ so that after stage u at most n numbers less than u have to be put into B_0 or B_1. (The n-th attack enters state 3 at this point.) Now we continue the attack using the greatest follower, calling it the active follower.

If the active follower $x_{n,i}$ is in state 4 or 5, i.e., when it imposes restraints on $B_0 \restriction t_{n,i}$ and $B_1 \restriction v_{n,i}$ where $v_{n,i} + 1 > t_{n,i}$ is the stage at which the follower attained state 4 and if coding some $\varepsilon(m)$ into B_0 or B_1 will force a number $y < t_{n,i}$ into one of those sets we put y into B_1, cancel the follower and repeat this process for the next smaller follower $x_{n,i-1}$. As pointed out above, y entering B_1 may destroy the computation $\{e_0\}_{t_{n,i}}^{B_{0,t_{n,i}} \oplus C_{t_{n,i}}}(x_{n,i})$ for the active follower since here C is held by restraining B_1 but it does not hurt the computations related to the smaller followers since here the computation is preserved by restraining B_0 and $\widehat{B}_0$ only. So in the end there will be at least one follower x left such that the restraints imposed by this follower will never be injured. The attack will succeed if this follower gets A-permission to enter state 5.

Now, however, a standard permitting argument does not suffice to show that if we make infinitely many attacks then one of these attacks can be completed. (For an arbitrary nonrecursive set A in general we cannot construct a marker system ε which computes A and which is controlled by A such that for any recursive increasing sequence $\{t_n\}_{n \geq 0}$ there will be infinitely many n such that a number $\leq \varepsilon(n)$ will enter A after A has moved all marker positions $\varepsilon(m)$ below t_n which will ever move.) Here we use that the degree **a** is not contiguous. Recall that we have chosen A and ε to satisfy condition (ii) of Theorem 2.9. If there we let $g(n) = n$ (in the actual construction, we have to choose $g(n)$ larger to take account of the interplay between attacks which we will discuss further below) and s_n be the stage $t_{n,n}$ when all followers $x_{n,i}$ are realized then, for infinitely many n, $\varepsilon(n)$ will be lifted after stage s_n (by (2.7)) and there will be a later stage $s_{f(n)}$ after which no marker $\varepsilon(m)$ below s_n will move (by (2.8)) and n will enter A (by (2.9)).

There is still one problem left. In the above analysis we ignored the possible interactions (or necessary cooperation) between different attacks. When the existing attack does not seem to be successful we have to start a new attack. This is the case if, for every existing attack, either the attack waits in state 2 for some $\varepsilon(n)$ to be lifted or in state 4 for A to permit the active follower. Now given $n < m$ such that the m-th attack is started while the n-th attack waits in state 2, no conflicts will arise unless the n-th attack reaches state 3 at some later stage $u_n + 1$. Since then $\varepsilon(n)$ is moved above m, we can cancel the m-th attack. Namely, if m had been one of the numbers for which condition (ii) of Theorem 2.9 had allowed completion of the attack, then the n-th attack can be completed.

Problems arise if the m-th attack is started while the n-th attack waits in state 4. Then the n-th attack imposes restraints of length t_n on B_0 and of length $v_n > t_n$ on B_1. Now if the m-th attack is progressing it will also impose restraints on B_0 of length $t_{m,i} > v_n$. So, if a number y with $t_n \leq y < v_n$ has to enter $B_0 \cup B_1$, then the n-th attack will target y for B_0, whereas the m-th attack will target y for B_1. Since we do not know which of n and m (if any) will satisfy condition (ii) of Theorem 2.9, we cannot resolve this conflict.

This is the reason why we replaced the requirement $\widehat{P}_{e,0}$ by the weaker symmetric version $\widehat{P}_e$. Then in the above situation the m-th attack attempts to meet the second disjunct

$$\widehat{P}_{e,1}: \ \widehat{B}_1 \neq \{e_1\}^{B_1 \oplus C}.$$

(This is done by following the same strategy as for $\widehat{P}_{e,0}$ with e_1, B_i and $\widehat{B}_i$ in place of e_0, B_{1-i} and $\widehat{B}_{1-i}$.) So both attacks now want to target a number y as above in B_0. The only conflict which can arise is when a number $y < t_n$ has to enter B_0 or B_1. But then the marker $\varepsilon(n)$ is lifted at this stage whence it is safe to cancel the m-th attack.

We now turn to the formal construction.

Let $T = 2^{<\omega}$ and let T_0 consist of those elements of T which end with a 0. For each $\alpha \in T_0$ there is a strategy N_α which tries to satisfy the meet requirement $N_{|\alpha|-1}$ using a marker system γ_α in the above described way. To *initialize* N_α *at stage* s means that all marker positions $\gamma_\alpha(x, s)$ are undefined at the end of stage s. The guesses at which of the N-strategies are infinitary are formalized as follows. For each $s \in \omega$ we inductively define a binary string δ_s of length s by letting

$$\delta_s(e) = 0 \ \Leftrightarrow \ l_e(s) > l_e(t) \text{ for all } t < s \text{ such that } \delta_t \restriction e = \delta_s \restriction e$$

for $e < s$, where l_e is the length function of N_e defined above. We call $s + 1$ an *α-stage* or α *accessible* at stage $s + 1$ if $\alpha \subseteq \delta_s$.

The *true path* τ of the construction is defined by letting, for each $e \in \omega$, $\tau \restriction e = \tau_e$, where τ_e is the least string of length e such that $\tau_e \subseteq \delta_s$ for infinitely many $s \in \omega$. Note that, for a requirement N_e with a correct hypothesis, $\tau(e) = 0$, whence there will be a strategy N_α with $\alpha \subseteq \tau$ working on N_e. We will show that this strategy meets N_e.

The following definition will be used for deciding when a meet marker has to be lifted. Let $\alpha \in T_0$, $x, s, u \in \omega$ and $i \leq 1$, and fix $e = \langle e_0, e_1 \rangle$ such that $|\alpha| - 1 = e$. Then $\gamma_\alpha(x)$ is *i-controllable at stage s by u* if for all $y \leq x$

(3.8) $\gamma_\alpha(y, s)\!\downarrow$.

(3.9) $\{e_i\}_s^{B_{i,s} \oplus \widehat{B}_{i,s} \oplus C_s}(y) = \{e_i\}^{B_{i,t} \oplus \widehat{B}_{i,t} \oplus C_t}(y)$, where t is the greatest stage less than s such that $\gamma_\alpha(y)$ was put down at stage $t+1$.

(3.10) $u(B_{i,s} \oplus \widehat{B}_{i,s} \oplus C_s; e_i, y, s) < u$.

$\gamma_\alpha(x)$ is *i-controllable at stage s* if it is i-controllable at stage s by some u, and $\gamma_\alpha(x)$ is *controllable at stage s (by u)* if $\gamma_\alpha(x)$ is i-controllable at stage s (*by u*) for some i.

A number x is an *α-i-configuration at stage s* if $x \leq s$ and

(3.11) $\forall y\ (\gamma_\alpha(y, s)\!\downarrow < x \Rightarrow \gamma_\alpha(y)$ is i-controllable at stage s by x).

x is a *configuration at stage s* if $x \leq s$ and

(3.12) $\forall \alpha \in T_0\ \forall y\ (\gamma_\alpha(y, s)\!\downarrow < x \Rightarrow \gamma_\alpha(y)$ is controllable at stage s by x).

An (α-i-)configuration x is *permanent* if x is an (α-i-)configuration at almost all stages. In order to make sure that $C \leq_T A$ we will meet the requirements

$$K_e: \text{ There is a permanent configuration } x \geq e.$$

for all $e \in \omega$.

We will assume that all reductions follow the "hat trick" (2.4). We let $\{R_e\}_{e \in \omega}$ be an effective listing of the P-, $\widehat{P}$-, and K-requirements by letting $R_{3(2e+i)} = P_{e,i}$, $R_{3e+1} = \widehat{P}_e$ and $R_{3e+2} = K_e$. For each $\alpha \in T$ we will have a strategy R_α which tries to meet $R_{|\alpha|}$, and we will show that the strings α on the true path, $\alpha \subseteq \tau$, will succeed in doing so. We call R_α a P-($\widehat{P}$-,K-)*strategy* if $R_{|\alpha|}$ is a P-($\widehat{P}$-,K-)requirement.

Each strategy R_α may impose certain *restraints* on $B_0 \oplus \widehat{B}_0$ and/or $B_1 \oplus \widehat{B}_1$ during the construction. We let $r_\alpha^i(s)$ denote R_α's restraint on $B_i \oplus \widehat{B}_i$ at the end of stage s where $r_\alpha^i(s) = 0$ indicates that no restraint is imposed, and we let $r_\alpha(s) = \max(r_\alpha^0(s), r_\alpha^1(s))$. To *reset a restraint* r at stage $s+1$ means to let $r(s+1) = 0$. In particular, to *reset* R_α's *restraint* at stage $s+1$ means to let $r_\alpha^0(s+1) = r_\alpha^1(s+1) = 0$.

A strategy R_α with $|\alpha| = 3(2e+i)$ which attempts to meet the requirement $R_{|\alpha|} = P_{e,i}$ ($e \in \omega, i \leq 1$) may have certain followers, called *α-followers*, targeted for B_i and it may restrain certain numbers from $B_{1-i} \oplus \widehat{B}_{1-i}$ (whereas $r_\alpha^i(s) = 0$

at all stages s). To *initialize R_α at stage s* means to cancel all α-followers and to reset the restraint of R_α.

A strategy R_α with $|\alpha| = 3e+1$, which attemps to meet the requirement $\widehat{P}_e$, say $e = \langle e_0, e_1 \rangle$, will have *$n$-modules* for all $n \in \omega$. The n-module represents the n-th attack in the informal discussion of the strategy for meeting $\widehat{P}_e$. At the end of each stage s, a *state* $state_{\alpha,n}(s) \in \{0,1,2,3,4,5,6\}$ and an *index* $i_{\alpha,n}(s) \in \{0,1\}$ are assigned to the n-module. The state describes the progress currently made by the attack. State 0 indicates that the attack has not yet been started, while state 6 indicates that the attack has been cancelled. States 1-5 are called *active*, states 4 and 5 *critical.* If an n-module is in an active state, we refer to the module as being active. While the n-module is in some active state, the index will not change and it will indicate the part of the requirement $\widehat{P}_e$ which is attacked, i.e., for $i = i_{\alpha,n}(s)$ the n-module tries to achieve $\widehat{B}_i \neq \{e_i\}^{B_i \oplus C}$. For this sake, while in an active state, the n-module may have up to $n+1$ followers, called *α-n-followers* which are targeted for $\widehat{B}_i$ for $i = i_{\alpha,n}(s)$. We let $x_{\alpha,n,0}(s), \ldots, x_{\alpha,n,k}(s)$ denote the α-n-followers at the end of stage s in order of magnitude. Each follower $x_{\alpha,n,j}(s)$ may impose restraints on $B_0 \oplus \widehat{B}_0$ and/or $B_1 \oplus \widehat{B}_1$ which we denote by $r^0_{\alpha,n,j}(s)$ and $r^1_{\alpha,n,j}(s)$, respectively. In a noncritical state, $r^{1-i}_{\alpha,n,j}(s) = 0$ for $i = i_{\alpha,n}(s)$, while in a critical state there may be proper restraints on both sides. We call $r^{i_{\alpha,n}(s)}_{\alpha,n,j}(s)$ the *prime restraint* of $x_{\alpha,n,j}(s)$ at the end of stage s. We let $r^i_{\alpha,n}(s) = \max\{r^i_{\alpha,n,j}(s) : j \leq k\}$ be the maximum restraint imposed on $B_i \oplus \widehat{B}_i$ by some α-n-follower at the end of stage s, where $r^i_{\alpha,n}(s) = 0$ if no α-n-follower exists, and we let $r_{\alpha,n}(s) = \max\{r^0_{\alpha,n}(s), r^1_{\alpha,n}(s)\}$. Moreover, we let $r^i_\alpha(s) = \sup\{r^i_{\alpha,n}(s) : n \in \omega\}$ $(i = 0,1)$ and $r_\alpha(s) = \sup\{r_{\alpha,n}(s) : n \in \omega\} = \max\{r^0_\alpha(s), r^1_\alpha(s)\}$.

To *cancel the n-module* of R_α at stage s means to let $state_{\alpha,n}(s) = 6$. To *initialize the n-module of R_α* at stage s means to let $state_{\alpha,n}(s) = 0$, $ind_{\alpha,n}(s) = 0$ and to cancel all α-n-followers. To *initialize the strategy R_α* means to initialize all n-modules of R_α.

A strategy R_α with $|\alpha| = 3e+2$ which attempts to meet the requirement K_e will only impose restraints on $B_i \oplus \widehat{B}_i$ in order to maintain i-control over certain markers (for $i \leq 1$). So these strategies will be purely negative. To *initialize* such a strategy R_α amounts to resetting its restraints.

For the definition of when a strategy R_α requires attention we distinguish three cases according to the type of the requirement attached to R_α.

The strategy R_α, $|\alpha| = 3(2e+i)$, $i \leq 1$, which attempts to meet the requirement $P_{e,i}$ *requires attention at stage $s+1$* if

(3.13) $\alpha \leq \delta_s$ and there is no realized α-follower in B_i at the end of stage s,

and one of the following hold.

(3.14) There is a realized α-follower x at the end of stage s such that $A_{s+1} \restriction x \neq A_s \restriction x$.

(3.15) $\alpha \subseteq \delta_s$ and there is an α-follower x which is not yet realized and $\{e\}_s^{C_s}(x) = 0$.

(3.16) $\alpha \subseteq \delta_s$ and all α-followers – if there are any – are realized at the end of stage s.

The strategy R_α, $|\alpha| = 3e + 1$, for the requirement $\widehat{P}_e$, say $e = \langle e_0, e_1 \rangle$, *requires attention at stage* $s + 1$ *via* n if $\alpha \leq \delta_s$, $n \leq s$, and at the end of stage s one of the following holds.

(3.17) $\alpha \subseteq \delta_s, state_{\alpha,n}(s) = 0,$ and $state_{\alpha,m}(s) \notin \{0, 1, 5\}$ for all $m < n$.

(3.18) $\alpha \subseteq \delta_s, state_{\alpha,n}(s) = 1$, and, for all α-n-followers x, $\{e_i\}_s^{B_{i,s} \oplus C_s}(x) = 0$ for $i = i_{\alpha,n}(s)$.

(3.19) $state_{\alpha,n}(s) = 2$ and $\varepsilon(n, s) < \varepsilon(n, s + 1)$.

(3.20) $\alpha \subseteq \delta_s, state_{\alpha,n}(s) = 3$, and
(∗) $\quad \exists m(\varepsilon(m, s) < \varepsilon(m, s + 1)\ \&\ \varepsilon(m, s) < r^i_{\alpha,n}(s))$, where $i = i_{\alpha,n}(s)$
does not hold.

(3.21) $state_{\alpha,n}(s) \in \{3, 4, 5\}$ and (∗) holds.

(3.22) $state_{\alpha,n}(s) = 4, A_s \restriction \varepsilon(n, s) \neq A_{s+1} \restriction \varepsilon(n, s),$ and (∗) does not hold.

The strategy R_α, $|\alpha| = 3e + 1$, *requires attention at stage* $s + 1$ if it requires attention at stage $s + 1$ via some number n.

The strategy R_α, $|\alpha| = 3e + 2$, for the requirement K_e *requires attention at stage* $s + 1$ if

(3.23) $\alpha \subseteq \delta_s$ and either $r^0_\alpha(s) = r^1_\alpha(s) = 0$ or there is a $\beta \in T_0$ with $\beta <_L \alpha$ and a number x such that $\gamma_\beta(x)$ is controllable at stage s but $\gamma_\beta(x)$ is neither 0-controllable by $r^0_\alpha(s)$ nor 1-controllable by $r^1_\alpha(s)$ at stage s

We are now ready to describe the construction.

Stage 0: Initialize all strategies.

Stage s+1: The stage consists of four steps. During these steps, when we refer to one of the parameters of the construction, such as a restraint r_α, without specifying a stage number, then we are referring to the value of the parameter at this particular point in the construction, taking into account action earlier in the stage.

> **Step 1** (Action for the R-strategies): Fix the least α such that R_α requires attention at stage $s + 1$. (If no such α exists, then initialize all R_β and N_β with $\delta_s < \beta$ and proceed to step 2.) Then R_α *receives attention* or *acts at stage* $s + 1$. Initialize all R_β and N_β with $\alpha < \beta$. Distinguish the following three cases according to the length of α.

Case 1: $|\alpha| = 3(2e+i)$, where $i \leq 1$, i.e., $R_{|\alpha|} = P_{e,i}$. If (3.14) holds then for the least follower x satisfying (3.14) put x into B_i. If (3.14) fails and (3.15) holds then for the least x satisfying (3.15) declare x to be *realized* and set $r_\alpha^{1-i} = s+1$. Otherwise, appoint $2s+1$ as follower of R_α.

Case 2: $|\alpha| = 3e+1$, say $e = \langle e_0, e_1 \rangle$. Then R_α is a strategy for $\widehat{P}_e$. Fix n minimal such that R_α requires attention via n and let $x_{\alpha,n,0}(s), \ldots, x_{\alpha,n,k}(s)$ be the α-n-followers at the end of stage s in order ($k \geq -1$). Say that R_α *receives attention via* n, and, for each $m > n$ such that the m-module of R_α is in an active state at the end of stage s, initialize the m-module if $\mathit{state}_{\alpha,m}(s) = 1$ and cancel the m-module otherwise. Distinguish the following cases according to the state of the n-module.

Case 2.1: (3.17) holds. Appoint $s+1$ as α-n-follower, let $\mathit{state}_{\alpha,n} = 1$ and define $i_{\alpha,n}$ as follows. Fix $m < n$ maximal such that the m-module of R_α is in an active state at the end of stage s and let

$$i_{\alpha,n} = \begin{cases} i_{\alpha,m}(s) & \text{if } \mathit{state}_{\alpha,m}(s) = 2 \\ 1 - i_{\alpha,m}(s) & \text{if } \mathit{state}_{\alpha,m}(s) \in \{3,4\} \end{cases}$$

If no such m exists, then let $i_{\alpha,n} = 0$.

Case 2.2: (3.18) holds. For $i = i_{\alpha,n}(s)$ let $r^i_{\alpha,n,k} = s+1$ and declare $x_{\alpha,n,k}(s)$ to be realized. Moreover, if $k < n$ then appoint $s+1$ as the next α-n-follower, and if $k = n$ then let $\mathit{state}_{\alpha,n} = 2$.

Case 2.3: (3.19) holds. Let $\mathit{state}_{\alpha,n} = 3$.

Case 2.4: (3.20) holds. Let $\mathit{state}_{\alpha,n} = 4$. Moreover, for $i = i_{\alpha,n}(s)$ let $r^{1-i}_{\alpha,n,k} = s+1$ and call $x_{\alpha,n,k}(s)$ the *active* α-n-follower.

Case 2.5: (3.21) holds. Let $\mathit{state}_{\alpha,n} = 3$. Moreover, if $\mathit{state}_{\alpha,n}(s) > 3$ then cancel $x_{\alpha,n,k}(s)$.

Case 2.6: (3.22) holds. Let $\mathit{state}_{\alpha,n} = 5$ and put $x_{\alpha,n,k}(s)$ into $\widehat{B}_{i_{\alpha,n}(s)}$.

Case 3: $|\alpha| = 3e+2$: Take the least $\beta <_L \alpha$ and then the least x as in (3.23). (If no such β and x exist then let $r^0_\alpha = s+1$.) Take $i \leq 1$ minimal such that $\gamma_\beta(x)$ is i-controllable at stage s and let $r^i_\alpha = s+1$.

Step 2 (Control ε by B_0 and B_1): First take the least x, if any, such that $\varepsilon(x,s) < \varepsilon(x,s+1)$ and then take the least α, if any, such that $2\varepsilon(x,s) < r_\alpha$. If no such x exists then step 2 is vacuous, and if no such α exists then put $2\varepsilon(x,s)$ into B_0. Otherwise, initialize all strategies R_β with $\alpha < \beta$ and distinguish the following cases.

Case 1: $|\alpha| = 3(2e+i)$ for some $e \in \omega$ and $i \leq 1$. Then put $2\varepsilon(x, s)$ into B_i.

Case 2: $|\alpha| = 3e + 1$ for some e. Then fix n minimal such that $2\varepsilon(x, s)$ is less than the current restraint $r_{\alpha,n}$ of the n-module of R_α and let $i = i_{\alpha,n}$. If $2\varepsilon(x, s) < r^i_{\alpha,n}$ then put $2\varepsilon(x, s)$ into B_{1-i}, otherwise into B_i.

Case 3: $|\alpha| = 3e + 2$ for some e. If $2\varepsilon(x, s) > r^i_\alpha$ for some $i \leq 1$ then put $2\varepsilon(x, s)$ into B_i for this i. Otherwise put $2\varepsilon(x, s)$ into B_1 and reset $r^1_\alpha = 0$.

Step 3 (Lift meet markers): For each $\alpha \in T_0$ such that N_α was not initialized in step 1 and for each x such that $\gamma_\alpha(x, s)\downarrow$ and $\gamma_\alpha(x)$ is not controllable at stage s put $\gamma_\alpha(x, s)$ into C and let $\gamma_\alpha(x, s+1)\uparrow$. Moreover, initialize all strategies R_β with $\alpha <_L \beta$.

Step 4 (Put down meet markers): For each $\alpha \in T_0$ such that $\alpha \subseteq \delta_s$ and N_α was not initialized in step 1 and for each x, if any, such that $x < l_{|\alpha|-1}(s)$ and $\gamma_\alpha(x, s)\uparrow$, define $\gamma_\alpha(x, s+1) = \langle |\alpha| - 1, x, s+1 \rangle$.

This completes the construction. We show the correctness of the construction by a sequence of lemmas. By effectivity of the construction the constructed sets are c.e. whence it suffices to show that they satisfy the conditions (3.1)–(3.7). We first prove that the required reductions (3.1), (3.3) and (3.7) are realized.

Lemma 3.1 $A \leq_T B_0 \oplus B_1$.

Proof. By step 2 in stage $s+1$, the even part $B_E = \{x : 2x \in B_0 \cup B_1\}$ of the sets B_0 and B_1 controls the marker ε via the enumeration $\{B_{E,s}\}_{s \in \omega}$ given by

$$B_{E,s} = \{x : 2x \in (B_{0,s} \cup B_{1,s})\}.$$

Since ε computes A, it follows that $A \leq_T B_E$ by Lemma 2.6. Since, obviously, $B_E \leq_T B_0 \oplus B_1$ this implies the claim. □

Lemma 3.2 *For $i \leq 1$, $B_i \leq_T A$.*

Proof. Let x enter B_i at stage $s+1$. If x is odd then x is a follower of a P-strategy and $A_{s+1}\restriction x+1 \neq A_s\restriction x+1$. If x is even then $x = 2y$ for some position $y = \varepsilon(n, s)$ of the marker ε which is lifted at stage $s+1$, i.e., $\varepsilon(n, s) < \varepsilon(n, s+1)$. Since A controls ε via $\{A_s\}_{s \in \omega}$, the latter implies that $A_s\restriction\varepsilon(n, s) + 1 \neq A\restriction\varepsilon(n, s) + 1$. So, in either case, $A_s\restriction x+1 \neq A\restriction x+1$ whence $B_i \leq_T A$ by permitting. □

Lemma 3.3 *For $i \leq 1$, $\widehat{B}_i \leq_T A$.*

Proof. If x enters $\widehat{B}_i$ at stage $s+1$ then x is an α-n-follower of some $\widehat{P}$-strategy requiring attention via (3.22), whence $A_s \restriction \varepsilon(n,s) \neq A_{s+1} \restriction \varepsilon(n,s)$. Since any α-n-follower x is greater than n, it follows that $A_s \restriction \varepsilon^*(x) \neq A \restriction \varepsilon^*(x)$. Since A controls ε, whence $\varepsilon^* \leq_T A$ by Lemma 2.3, this implies the claim. □

In order to show that the remaining conditions (3.2), (3.4), (3.5) and (3.6) are satisfied we will show that the corresponding P-, N-, K- and $\widehat{P}$-requirements are satisfied by the strategies for these requirements on the true path. For this sake we first analyze the possible actions of the $\widehat{P}$-strategies.

Note that the state of the n-module of a $\widehat{P}$-strategy R_α only changes at stage $s+1$ if either R_α is initialized (in which case $state_{\alpha,n}(s+1) = 0$), or R_α receives attention via some number $m < n$ (in which case $state_{\alpha,n}(s+1) = 0$ if $state_{\alpha,n}(s) < 2$ and $state_{\alpha,n}(s+1) = 6$ otherwise), or R_α receives attention via n. In the last case the following transitions are possible.

Lemma 3.4 *Let R_α be a $\widehat{P}$-strategy which receives attention via n at stage $s+1$ and suppose that R_α is not initialized at stage $s+1$. Then one of the following holds.*

(3.24) $state_{\alpha,n}(s) \in \{0,1,2,3,4\}$ *and* $state_{\alpha,n}(s+1) = state_{\alpha,n}(s) + 1$.

(3.25) $state_{\alpha,n}(s) = state_{\alpha,n}(s+1) = 1$.

(3.26) $state_{\alpha,n}(s) \in \{3,4,5\}$ *and* $state_{\alpha,n}(s+1) = 3$.

Proof. Straightforward by step 1 of stage $s+1$ in the construction. □

Moreover, as shown in the following lemma, cases (3.25) and (3.26) can apply only n times each, unless the n-module is initialized between.

Lemma 3.5 *Assume that $t < t'$ and the n-module of the $\widehat{P}$-strategy R_α is not initialized at any stage u with $t < u \leq t'$. Then there are at most n stages s, $t \leq s < t'$, such that R_α receives attention via n at stage $s+1$ and (3.25) holds, and there are at most n stages s, $t \leq s < t'$, such that R_α receives attention via n at stage $s+1$ and (3.26) holds.*

Proof. For the first claim observe that the first α-n-follower is appointed when the n-module reaches state 1 first, and any activity of the n-module in state 1 adds one more follower. With the accumulation of $n+1$ followers, however, the n-module reaches state 2.

For the second claim observe that when the n-module reaches state 3 first, say at stage $v+1$, then $v+1 \leq \varepsilon(n, v+1)$, $r^i_{\alpha,n} \leq v$ for $i = i_{\alpha,n}(v)$, and $i_{\alpha,n}$ will not change and $r^i_{\alpha,n}$ will not increase later (unless the n-module is initialized). So, if (3.26) applies to a stage $s \geq v+1$, then, by $(*)$ in (3.21), for some number m, $\varepsilon(m,s) < r^i_{\alpha,n}$ and $\varepsilon(m)$ is lifted at stage $s+1$ whence $r^i_{\alpha,n} < \varepsilon(m, s+1)$. But, such an m has to be $< n$ and for each m this can happen at most once, whence the number of occurrences of (3.26) is bounded by n. □

Lemma 3.6 *Assume that the n-module of R_α is initialized only finitely often. Then R_α receives attention via n only finitely often and the n-module reaches a final state.*

Proof. By Lemmas 3.4 and 3.5. □

Another consequence of Lemmas 3.4 and 3.5 is that the n-module of R_α has some follower whenever it is in an active state.

Lemma 3.7 *If the n-module of the $\widehat{P}$-strategy R_α is in an active state at the end of stage s then there is at least one α-n-follower at the end of stage s.*

Proof. When the module reaches state 1 first the first follower is appointed. When state 2 is reached first, the number of followers has grown to $n+1$. This number is unchanged when state 3 is reached first. Once in a state ≥ 3 the number of followers is stable, unless (3.26) occurs with $state_{\alpha,n}(s) \in \{4,5\}$ in which case the greatest follower is cancelled. By Lemma 3.5, however, the latter can happen at most n times, whence at least one follower is permanently left. □

Lemma 3.7 shows that in cases 2.4, 2.5 and 2.6 of step 1 in stage $s+1$ of the construction the follower $x_{\alpha,n,k}(s)$ actually exists.

The above observations on the activity of $\widehat{P}$-strategies suffice to show that strategies on the true path are initialized by actions of strategies to the left of them only finitely often.

Lemma 3.8 *For $\alpha \in T$ with $\alpha \subseteq \tau$ there is a stage s_α such that no strategy R_β with $\beta <_L \alpha$ receives attention after stage s_α and such that no marker $\gamma_\beta(x)$ where $\beta <_L \alpha$ and $x \in \omega$ is lifted or put down after stage s_α.*

Proof. Fix s_0 such that $\alpha \leq \delta_s$ for all $s \geq s_0$. There are only finitely many strategies R_β with $\beta <_L \alpha$ which have received attention at some stage $\leq s_0$. Moreover, no other strategy R_β with $\beta <_L \alpha$ will require attention after stage $s+1$, since by (3.16), (3.17) or (3.23) the first action of a strategy R_β is limited to stages at which β is accessible. So for the first part of the claim it suffices to show that every strategy R_β, $\beta <_L \alpha$, will act only finitely often.

First assume that R_β is a P-strategy. Since β-followers are appointed at β-stages only, R_β has only finitely many followers, and each β-follower x causes R_β to act after its appointment at most twice: first when it becomes realized and second if it is enumerated into its target set. So R_β is finitary. Next assume that R_β is a $\widehat{P}$-strategy. Since, by (3.17), an n-module of R_β can reach an active state only at a β-stage, there will be only finitely many numbers n such that R_β will receive attention via n. So it suffices to show that for any given n, R_β receives attention via n only finitely often. If the n-module is initialized at some stage $s > s_0$ then, by (3.17), R_β will not require attention via n after stage s. Otherwise, the claim follows from Lemma 3.6. Finally, if R_β is a K-strategy then, by (3.23), R_β will not require attention after stage s_0.

For the second part of the claim note that a marker $\gamma_\beta(x)$ is put down only at a β-stage, and at each stage only finitely many markers are put down. So only finitely many markers $\gamma_\beta(x)$ with $\beta <_L \alpha$ are ever defined and no such marker is put down after stage s_0. Moreover, we may fix a stage $s_1 > s_0$ such that all of these markers which are lifted after stage s_0 are lifted before stage s_1. $\square$

Next we make two simple observations on the meet markers which then will be used to show that the i-restraints of P- and K-strategies – if not reset – actually restrain numbers from $B_i \oplus \widehat{B}_i \oplus C$. (The corresponding more technical claim for the $\widehat{P}$-strategies is given later.)

Lemma 3.9 *For all x and s, if $\gamma_\alpha(x,s)\downarrow$, then $\gamma_\alpha(y,s)\downarrow$ for all $y < x$ and if further $\alpha \subseteq \delta_s$, then $\gamma_\alpha(x)$ is both 0-controllable and 1-controllable at stage s.*

Proof. The proof is by induction on s and is trivial for $s = 0$. Suppose the result holds for $s-1$ and $\gamma_\alpha(x,s)\downarrow$. Fix $y < x$. If $\gamma_\alpha(x,s) = \gamma_\alpha(x,s-1)$, then, by inductive hypothesis, $\gamma_\alpha(y,s-1)\downarrow$ and since $\gamma_\alpha(x)$ was controllable at stage $s-1$, so was $\gamma_\alpha(y)$, so $\gamma_\alpha(y,s) = \gamma_\alpha(y,s-1)\downarrow$. If, on the other hand, $\gamma_\alpha(x,s)$ is put down at step 4 of stage s, then, by construction, $\gamma_\alpha(y,s)$ is also put down at step 4 or $\gamma_\alpha(y,s-1)\downarrow$. In the latter case, $\gamma_\alpha(y)$ is controllable at stage $s-1$ (by inductive hypothesis, since $\alpha \subseteq \delta_{s-1}$) and, since α is not initialized at stage s, $\gamma_\alpha(y,s) = \gamma_\alpha(y,s-1)\downarrow$.

Now suppose in addition that $\alpha \subseteq \delta_s$ and fix $y \le x$. We have shown that $\gamma_\alpha(y,s)\downarrow$. Let $e = \langle e_0, e_1\rangle = |\alpha| - 1$ and let t be the greatest stage $< s$ such that $\gamma_\alpha(y)$ was put down at stage $t+1$. Then, by construction, $\alpha \subseteq \delta_t$. Since $\alpha(e) = 0$, by definition of δ_s, $y < l_e(t) < l_e(s)$, whence $\{e_0\}_t^{B_{0,t}\oplus\widehat{B}_{0,t}\oplus C_t}(y) = \{e_1\}_t^{B_{1,t}\oplus\widehat{B}_{1,t}\oplus C_t}(y)\downarrow$ and $\{e_0\}_s^{B_{0,s}\oplus\widehat{B}_{0,s}\oplus C_s}(y) = \{e_1\}_s^{B_{1,s}\oplus\widehat{B}_{1,s}\oplus C_s}(y)\downarrow$. So it suffices to show that $\{e_i\}_t^{B_{i,t}\oplus\widehat{B}_{i,t}\oplus C_t}(y) = \{e_i\}_s^{B_{i,s}\oplus\widehat{B}_{i,s}\oplus C_s}(y)\downarrow$ for some $i \le 1$. In fact, by the "hat trick" it suffices to show that $\{e_i\}_t^{B_{i,t}\oplus\widehat{B}_{i,t}\oplus C_t}(y) = \{e_i\}_{s-1}^{B_{i,s-1}\oplus\widehat{B}_{i,s-1}\oplus C_{s-1}}(y)$ for some $i \le 1$, since $\{e_i\}_{s-1}^{B_{i,s-1}\oplus\widehat{B}_{i,s-1}\oplus C_{s-1}}(y) \neq \{e_i\}_s^{B_{i,s}\oplus\widehat{B}_{i,s}\oplus C_s}(y)$ implies that one of the computations diverges. Now, for $t+1 = s$, this is trivial since $t = s-1$; and for $t+1 < s$ this must hold, since otherwise $\gamma_\alpha(y)$ would have been lifted at stage s. Thus, $\gamma_\alpha(x)$ is 0-controllable and 1-controllable at stage s. $\square$

Lemma 3.10 *Assume that $\gamma_\alpha(x)$ is i-controllable at stage s by u, $s \le t$, N_α is not initialized at any stage v with $s < v \le t$, and*

$$B_{i,s} \oplus \widehat{B}_{i,s} \oplus C_s \restriction u = B_{i,t} \oplus \widehat{B}_{i,t} \oplus C_t \restriction u.$$

Then $\gamma_\alpha(x)$ is i-controllable at stage t by u.

Proof. By a straightforward induction on t' with $s \leq t' \leq t$, $\gamma_\alpha(y, s) = \gamma_\alpha(y, t')$ for all $y \leq x$ and $\gamma_\alpha(x)$ is i-controllable at stage t' by u. □

Lemma 3.11 *Let R_α be a P- or K-strategy, let $i \leq 1$, and assume that $r^i_\alpha(s+1) = s+1$ and r^i_α is not reset after stage $s+1$. Then $B_{i,s}\lceil s+1 = B_i\lceil s+1$, $\widehat{B}_{i,s}\lceil s+1 = \widehat{B}_i\lceil s+1$ and $C_s\lceil s+1 = C\lceil s+1$.*

Proof. For a contradiction assume that a number $y < s+1$ enters B_i, $\widehat{B}_i$ or C after stage s. Let y be the first such number and let $t+1$ be the stage at which y enters B_i, $\widehat{B}_i$ or C. We rule out all the possibilities.

Case 1: y is a follower of some strategy R_β. If $\alpha < \beta$ then in step 1 of stage $s+1$ all followers of R_β are cancelled and any followers assigned later are $\geq s+1$. So this is impossible. The case of $\beta < \alpha$ is impossible too, since then r^i_α will be reset at stage $t+1$. This leaves the case that $\alpha = \beta$. But if α is a P-strategy then α-followers are targeted for B_{1-i} and if α is a K-strategy then there are no α-followers. So this case is impossible too.

Case 2: $y = 2\varepsilon(x, t)$ enters B_i at stage $t+1$. Since r^i_α is not reset after stage $s+1$, $r_\alpha \geq r^i_\alpha \geq s+1$ at the beginning of step 2 of stage $t+1$. Moreover, since r^i_α is not reset, α is the least string α' with $2\varepsilon(x,t) < r_{\alpha'}$. So, if R_α is a P-strategy, y is put into B_{1-i}, a contradiction. If R_α is a K-strategy then, since y enters B_i, $i = 1$ and r^1_α is reset. So this is impossible too.

Case 3: $y = \gamma_\beta(x, t)$ enters C. If $\alpha < \beta$ then N_β is initialized in step 1 of stage $s+1$ whence $\gamma_\beta(x)$ was put down after stage s, i.e., $\gamma_\beta(x,t) = \langle|\beta| - 1, x, u+1\rangle \geq s+1$ for some $u \geq s$. So this is impossible. If $\beta <_L \alpha$ then R_α is initialized, hence r^i_α reset at stage $t+1$ which is impossible too. This leaves the case that $\beta \subseteq \alpha$. Since $y = \gamma_\beta(x,t) \leq s$, $\gamma_\beta(x)$ was put down on $\gamma_\beta(x,t)$ before stage $s+1$, i.e., $\gamma_\beta(x,s) = \gamma_\beta(x,t)$. Since $\beta \subseteq \alpha \subseteq \delta_s$ it follows by Lemma 3.9 that $\gamma_\beta(x)$ is i-controllable at stage s by s. Moreover, by choice of y, $B_{i,s} \oplus \widehat{B}_{i,s} \oplus C_s\lceil s+1 = B_{i,t} \oplus \widehat{B}_{i,t} \oplus C_t\lceil s+1$ whence, by Lemma 3.10, $\gamma_\beta(x)$ is i-controllable at stage t. So $y = \gamma_\beta(x,t)$ is not put into C at stage $t+1$ contrary to assumption. □

Next we will show that the strategies for the R-requirements on the true path are finitary. This is shown by induction by distinguishing the different types of strategies. We consider the P- and K-strategies first before we look at the more involved situation for the $\widehat{P}$-strategies.

Lemma 3.12 *Let R_α be the strategy for the requirement $P_{e,i}$ on the true path, i.e., $|\alpha| = 3(2e+i)$ and $\alpha \subseteq \tau$, and assume that R_α is initialized only finitely often. Then R_α requires attention only finitely often.*

Proof. Fix s_0 such that R_α is not initialized after stage s_0, and, for a contradiction, assume that R_α requires attention infinitely often. Since, by choice of s_0, R_α receives attention whenever it requires attention after stage s_0, we can compute an increasing sequence $\{t_n\}_{n\in\omega}$ such that $t_n > s_0$ and $x_n = 2t_n + 1$ is appointed α-follower at stage $t_n + 1$. By choice of s_0 none of these followers is ever cancelled, whence, by (3.15) and (3.16), all followers x_n are eventually realized, at stage u_n say, but none of them ever enters B_i. By the latter, (3.14) will fail for all $x = x_n$ and $s \geq u_n$, whence $x_n > n$ implies that $A_{u_n}\restriction n = A\restriction n$. Since $f(n) = u_n$ is computable, it follows that A is computable. This contradicts the assumption that $\mathbf{a} = \deg_T(A)$ is noncontiguous. □

Lemma 3.13 *Let R_α be the strategy for the requirement K_e on the true path, i.e., $|\alpha| = 3e + 2$ and $\alpha \subseteq \tau$, and assume that R_α is initialized only finitely often. Then R_α requires attention only finitely often.*

Proof. Let s_0 be a stage such that R_α is not initialized after stage s_0, and, for a contradiction, assume that R_α requires attention infinitely often. Then R_α will receive attention whenever it requires attention after stage s_0 and r^0_α will not be reset after stage s_0. (Note that in step 2 of stage $s+1$, if making B_0 and B_1 control ε forces a number into B_0 or B_1 which is less than both r^0_α and r^1_α then this number is put into B_1 and r^1_α is reset.) Moreover, when R_α receives attention at stage $s+1$ then r^0_α or r^1_α is set equal to $s+1$ in step 1 of this stage. So, for some $i \leq 1$, r^i_α is set to $s+1$ at infinitely many stages s.

First assume that this applies to $i = 0$. Then, since r^0_α is not reset after stage s_0, $r^0_\alpha(s)$ is nondecreasing in s after stage s_0 and $r^0_\alpha(s+1) = s+1$ for infinitely many $s \geq s_0$. Fix such an s with $r^0_\alpha(s) > 0$. Then, by (3.23), there is a marker $\gamma_\beta(x,s)$ such that $\beta <_L \alpha$ and $\gamma_\beta(x)$ is 0-controllable at stage s, but $\gamma_\beta(x)$ is not 0-controllable at stage s by $r^0_\alpha(s)$. So, by $r^0_\alpha(s+1) = s+1$, $\gamma_\beta(x)$ is 0-controllable at stage s by $r^0_\alpha(s+1)$, and, since r^0_α is not reset later, $B_{0,s} \oplus \widehat{B}_{0,s} \oplus C_s\restriction r^0_\alpha(s+1) = B_0 \oplus \widehat{B}_0 \oplus C\restriction r^0_\alpha(s+1)$ by Lemma 3.11. Since $r^0_\alpha(t) \geq r^0_\alpha(s+1)$ for $t \geq s+1$, it follows by Lemma 3.10 that $\gamma_\beta(x)$ is 0-controllable at all stages $t \geq s+1$ by $r^0_\alpha(t)$. It follows that there are infinitely many pairs (β, x) such that $\beta <_L \alpha$ and $\gamma_\beta(x)$ is 0-controllable at some stage t, whence $\gamma_\beta(x,t)\downarrow$. So there are infinitely many stages at which a marker $\gamma_\beta(x)$ with $\beta <_L \alpha$ is put down. But this contradicts Lemma 3.8.

So, there must be infinitely many stages s such that $r^1_\alpha = s+1$ in step 1 of stage $s+1$. Moreover, we may fix $s_1 \geq s_0$ such that $r^0_\alpha(s) = r^0_\alpha(s_1)$ for all $s \geq s_1$ and we may fix a stage $s_2 \geq s_1$ such that

$$\forall n\, \forall s \geq s_2\, (\varepsilon(n,s) < \varepsilon(n,s+1) \;\Rightarrow\; \varepsilon(n,s) > r^0_\alpha(s)).$$

Then r^1_α is not reset after stage s_2. Hence we can argue as above with 1 in place 0 to obtain a contradiction to Lemma 3.8. □

In order to show that the $\widehat{P}$-strategies on the true path act only finitely often too, we first state two more facts on the states of the modules of these strategies.

Lemma 3.14 *The n-module of a $\widehat{P}$-strategy R_α is initialized at stage $s+1$ if and only if R_α is initialized or $state_{\alpha,n}(s) = 1$ and R_α receives attention via some $m < n$ at stage $s+1$. The n-module of R_α is cancelled at stage $s+1$ if and only if $2 \leq state_{\alpha,n}(s) \leq 5$ and R_α receives attention via some $m < n$ at stage $s+1$.*

Proof. Immediate by construction. $\square$

Lemma 3.15 *Assume that a $\widehat{P}$-strategy R_α receives attention via n at stage $s+1$. Then, for all $m < n$, $state_{\alpha,m}(s) \in \{2,4,6\}$ and, unless R_α is initialized at stage $s+1$, $state_{\alpha,m}(s) = state_{\alpha,m}(s+1)$.*

Proof. Take $t \leq s$ minimal such that R_α receives attention via n at stage $t+1$ and the n-module of R_α is neither initialized nor cancelled at a stage u with $t+1 \leq u \leq s$. Then R_α acts in step 1 of stage $t+1$ via n according to case 2.1, whence $\alpha \subseteq \delta_t$ and $state_{\alpha,m}(t) \in \{2,3,4,6\}$ for all $m < n$. Moreover, by Lemma 3.14, $state_{\alpha,m}(t) = state_{\alpha,m}(s)$ for $m < n$. So, given $m < n$, to show that $state_{\alpha,m}(s) \in \{2,4,6\}$ it suffices to show that $state_{\alpha,m}(t) \neq 3$. But if $state_{\alpha,m}(t) = 3$ then, by $\alpha \subseteq \delta_t$, R_α requires attention via m at stage $t+1$. Since $m < n$ this contradicts R_α receiving attention via n at this stage.

Moreover, since R_α receives attention via n at stage $s+1$, it follows from Lemma 3.14 that $state_{\alpha,m}(s) = state_{\alpha,m}(s+1)$ for $m < n$ unless R_α is initialized at stage $s+1$. Finally, by Lemma 3.14, the m-module of R_α can be cancelled only if R_α receives attention via some smaller number m'. Hence, if $n > 0$, then $state_{\alpha,0}(s) \in \{2,4\}$. $\square$

Lemma 3.16 *Let R_α be the strategy for the requirement $\widehat{P}_e$, $e = \langle e_0, e_1 \rangle$, on the true path, i.e., $|\alpha| = 3e+1$ and $\alpha \subseteq \tau$, and assume that R_α is initialized only finitely often. Then R_α requires attention only finitely often.*

Proof. Let s_{-1} be the greatest stage at which R_α is initialized, and, for a contradiction, assume that R_α requires attention infinitely often. Then, whenever R_α requires attention after stage s_{-1}, R_α receives attention, whence R_α acts infinitely often. By Lemma 3.14, the n-module of R_α is initialized at a stage $s+1 > s_{-1}$ only if R_α receives attention via some number $m < n$ at stage $s+1$. By Lemma 3.6 this implies (by induction on n) that R_α requires attention via n only finitely often and that the n-module of R_α eventually reaches a final state, in the following denoted by $fs_{\alpha,n}$. Since R_α acts infinitely often, it follows by Lemma 3.15 that R_α receives attention via all numbers n after stage s_{-1} and that $fs_{\alpha,n} \in \{2,4,6\}$ for all $n \geq 0$.

Note that in order to reach state 4 or 6 the n-module has to reach state 2 first. Moreover, since s_{-1} is the last stage at which R_α is initialized, once the n-module reaches state 2 after stage s_{-1}, its state will be ≥ 2 forever (see

Lemmas 3.4 and 3.14). In addition, if $m < n$ then, by Lemma 3.15, the m-module reaches state 2 first. So, if we let s_n be the unique stage $s > s_{-1}$ such that $state_{\alpha,n}(s) = 1$ and $state_{\alpha,n}(s+1) = 2$ then the sequence $S = \{s_n\}_{n\in\omega}$ is strictly increasing and computable.

By choice of A and ε and by Theorem 2.9 this implies that there is a total function $f : \omega \to \omega$ with $f(n) > n$ such that for infinitely many $n \in \omega$,

(3.27) $\varepsilon(n, s_n + 1) < \varepsilon(n, s_{f(n)})$

(3.28) $\forall m\ (\varepsilon(m, s_{f(n)}) \leq \max(\varepsilon(n, s_n + 1), s_n) \ \Rightarrow\ \varepsilon(m, s_{f(n)}) = \varepsilon^*(m))$

and

(3.29) $n \in A - A_{s_{f(n)}+1}$.

(Since $s_n + 1 > s_n \geq n$ this is achieved by letting $g(n) = \max(\varepsilon(n, s_n + 1), n + 1)$ in Theorem 2.9.)

For the remainder of the proof fix $n > 0$ satisfying (3.27) – (3.29) and let $m \leq n$ be the least number such that $\varepsilon(m, s_n + 1) < \varepsilon^*(m)$ and the m-module is in an active state at the end of stage $s_n + 1$. (Note that n is among these numbers whence such an m actually exists.)

We will show that for some number $m' \leq m$ the final state of the m'-module will be 5 contrary to our observation above that $fs_{\alpha,m'} \in \{2, 4, 6\}$.

To show this we start with some observations on the states of the m'-modules of R_α for $m' \leq m$ after stage s_n. By choice of s_n and by Lemma 3.15, $state_{\alpha,m'}(s_n + 1) \in \{2, 4, 6\}$ and – as observed already above – $state_{\alpha,m'}(s) \geq 2$ for $s > s_n$. Moreover, for $m' < m$, the m'-module of R_α will not be in state 3 after stage s_n. This is shown as follows. For a contradiction let s be the least stage $s > s_n$ such that $state_{\alpha,m'}(s) = 3$. Then, by the above, $s > s_n + 1$ and R_α receives attention via m' according to case 2.3 or case 2.5 at stage s. So $\varepsilon(m', s-1) < \varepsilon(m', s)$ (in case 2.5, by the argument of Lemma 3.5, some $\varepsilon(m'')$ with $m'' < m'$ was lifted at stage s, so by marker rule (M2), $\varepsilon(m')$ was lifted at stage s) and the m'-module was in an active state at stage $s_n + 1$. But, since $m' < m$, this contradicts the minimality condition in the choice of m.

The above together with Lemma 3.4 shows that if R_α is active via $m' < m$ at a stage $s + 1 > s_n$ then the m'-module of R_α reaches state 5 at stage $s + 1$.

Now in order to show that $fs_{\alpha,m'} = 5$ for some $m' \leq m$, we distinguish two cases depending on whether or not R_α will act via some number less than m after stage s_n.

First assume that

(3.30) $\exists m' < m\ \exists s > s_n(state_{\alpha,m'}(s) = 5)$

holds. Pick $m' < m$ and $s > s_n$ minimal (in this order) such that $state_{\alpha,m'}(s) = 5$. Then, by minimality of m', R_α will not become active via any number $m'' < m'$ after stage s_n, whence the m'-module will never be cancelled. Since, moreover, $state_{\alpha,m'}(s') \neq 3$ for $s' > s_n$, it follows from Lemma 3.4 that $state_{\alpha,m'}(s') = 5$ for all $s' \geq s$, i.e., $fs_{\alpha,m'} = 5$.

Finally assume that (3.30) fails. Then, by the above, the m-module of R_α is never cancelled and R_α will receive attention via m at any stage $s > s_n$ at which R_α requires attention via m. We will show that in this case $fs_{\alpha,m} = 5$.

By choice of m and n we can fix stages t_n, u_n, v_n such that

$$s_n < t_n < u_n < v_n$$

and

(3.31) $\varepsilon(m, t_n) < \varepsilon(m, t_n + 1)$

(3.32) $u_n \in S \;\&\; \forall m'(\,\varepsilon(m', u_n) \leq s_n \Rightarrow \varepsilon(m', u_n) = \varepsilon^*(m')\,)$

(3.33) $n \in A_{v_n+1} - A_{v_n}$

hold. (This follows from (3.27) – (3.29) if we let $u_n = s_{f(n)}$. Then (3.32) and (3.33) are immediate by (3.28) and (3.29), respectively. Moreover, by choice of m, there is a stage $t \geq s_n + 1$ such that $\varepsilon(m, t) < \varepsilon(m, t + 1)$. Since $m \leq n$, whence $\varepsilon(m, s_n + 1) \leq \varepsilon(n, s_n + 1)$, it follows from (3.28) that the least such stage t is less than $s_{f(n)} = u_n$. So (3.31) holds.)

Now, as shown above, $state_{\alpha,m}(s) \in \{2, 3, 4, 5\}$ for all $s > s_n$. Moreover, if $state_{\alpha,m}(t_n) = 2$ then, by (3.31), R_α will receive attention via m according to case 2.3 of the construction at stage $t_n + 1$, whence $state_{\alpha,m}(t_n + 1) = 3$. Since a module which has reached state 3 will never be in a lesser state later unless the corresponding strategy is initialized, this shows that $state_{\alpha,m}(s) \in \{3, 4, 5\}$ for $s \geq t_n + 1$. Moreover, the index $i = i_{\alpha,m}(s)$ for the m-module at stages $s > s_m$ is determined by its value at stage s_m which is the last stage at which the m-module is in state 1. Since the primary restraint of a module is not increased after the module enters state 2 (unless the module is initialized later), this implies that

$$r^i_{\alpha,m}(s) \leq r^i_{\alpha,m}(s_m + 1) = s_m + 1 \leq s_n + 1$$

for all $s > s_n \geq s_m$. So, by the second part of (3.32), R_α will not require attention via m according to case 2.5 of the construction at any stage $s > u_n$.

Now if $state_{\alpha,m}(u_n) = 3$ then R_α will act via m at stage $u_n + 1$ according to case 2.4 of the construction since, by the first part of (3.32), $\alpha \subseteq \delta_{u_n}$. So $state_{\alpha,m}(u_n + 1) = 4$ in this case, whence either $state_{\alpha,m}(u_n) \in \{4, 5\}$ or $state_{\alpha,m}(u_n + 1) = 4$ (or both). By the above observation this implies that $state_{\alpha,m}(s) \in \{4, 5\}$ for $s \geq u_n + 1$.

Finally, if the m-module is not in state 5 at stage v_n then it will reach state 5 at stage $v_n + 1$. Namely, by (3.31),

$$\varepsilon(m, v_n) \geq \varepsilon(m, t_n + 1) \geq t_n + 1 > s_n \geq n.$$

So, by (3.33) R_α will become active via m according to case 2.6 of the construction at stage $v_n + 1$. Since a transition from state 5 to state 4 is impossible (see Lemma 3.4) this shows that $fs_{\alpha,m} = 5$. $\square$

Lemma 3.17 *For $\alpha \subseteq \tau$, R_α is initialized only finitely often and R_α requires attention only finitely often.*

Proof. The proof is by induction on $|\alpha|$. Fix $\alpha \subseteq \tau$. By inductive hypothesis and by Lemma 3.8 we can take s_0 such that for all $s \geq s_0$

(3.34) $\forall \beta < \alpha\, (R_\beta$ does not receive attention at stage $s)$,

(3.35) $\forall \beta <_L \alpha\, \forall x\, (\gamma_\alpha(x)$ is not lifted at stage $s)$,

(3.36) $\alpha \leq \delta_s$,

and

(3.37) $\forall n\, (\varepsilon(n,s) < \varepsilon(n,s+1) \Rightarrow \varepsilon(n,s) > s_0)$.

By Lemmas 3.12, 3.13 and 3.16 it suffices to show that R_α is not initialized after stage s_0. Let $s > s_0$. Then, by (3.34) and (3.36), R_α is not initialized in step 1 of stage s and, by (3.35), R_α is not initialized in step 3 of stage s. Moreover, by (3.34), $r_\beta(s) \leq s_0$ for all $\beta < \alpha$, whence, by (3.37), R_α is also not initialized in step 2 of stage s. $\square$

We are now ready to show that the strategies on the true path satisfy the requirements attached to them. We show this first for the N-requirements, then for the P-, K- and $\widehat{P}$-requirements. Again the case of the $\widehat{P}$-requirements will be the most involved one and will require some auxiliary lemmas.

Lemma 3.18 *For all $e \in \omega$, N_e is met.*

Proof. Fix $e = \langle e_0, e_1 \rangle$ and w.l.o.g. assume that

(3.38) $D = \{e_0\}^{B_0 \oplus \widehat{B}_0 \oplus C} = \{e_1\}^{B_1 \oplus \widehat{B}_1 \oplus C}$.

We have to show that $D \leq_T C$. By (3.38), $\lim_s l_e(s) = \omega$ whence $\tau(e) = 0$. So, for α the string of length $e+1$ on τ, the strategy N_α tries to meet N_e. We will show that N_α produces a marker system γ which computes D and which is controlled by C. By Lemma 2.6 this will imply the claim.

By Lemmas 3.8 and 3.17 let s_0 be the greatest stage at which N_α is initialized. Define the function γ as follows. For $s \leq s_0$ let $\gamma(x,s) = \gamma_\alpha(x, s_0)$ and for $s > s_0$, let $\gamma(x,s) = \gamma_\alpha(x,s)$. We will show first that γ is a marker system in the sense of Definition 2.1. Obviously the function $\lambda x, s.\gamma(x,s)$ and its domain are computable. The marker properties (M1) - (M4) are straightforward to establish. (Note that Lemma 3.9 shows one part of (M3).) For a proof of (M5) fix

x and let $s+1$ be an α-stage such that $s > s_0$, $l_e(s) > x$ and the computations $\{e_i\}_s^{B_{i,s}\oplus\widehat{B}_{i,s}\oplus C_s}\restriction x+1$, $i = 0, 1$, are true, i.e., $B_{i,s}\oplus\widehat{B}_{i,s}\oplus C_s\restriction u_i = B_i\oplus\widehat{B}_i\oplus C\restriction u_i$, where $u_i = \max\{u(B_{i,s}\oplus\widehat{B}_{i,s}\oplus C_s;\, e_i, y, s) : y \le x\}$. Now, if $\gamma_\alpha(x,s)\downarrow$ then, by Lemma 3.9, $\gamma_\alpha(x)$ is controllable at stage s whence $\gamma_\alpha(x, s+1) = \gamma_\alpha(x, s)$ by construction. Otherwise, $\gamma_\alpha(x)$ is put down at stage $s+1$. In either case, since the computations $\{e_i\}_s^{B_{i,s}\oplus\widehat{B}_{i,s}\oplus C_s}(y)$, $y \le x$, are true computations, by Lemma 3.10, $\gamma_\alpha(x)$ is controllable at all stages $\ge s+1$ whence $\gamma^*(x) = \gamma(x, s+1) = \gamma_\alpha(x, s+1)$.

Obviously γ is controlled by C via $\{C_s\}_{s\in\omega}$ since whenever x is such that $\gamma_\alpha(x)$ is lifted at stage $s+1 > s_0$, then $\gamma_\alpha(x,s)$ is put into C in step 3 of this stage.

Finally, to show that γ computes D we define a computable approximation $\{D_s\}_{s\in\omega}$ of D by letting

$$D_s(x) = \{e_0\}_{f(s,x)}^{B_{0,f(s,x)}\oplus\widehat{B}_{0,f(s,x)}\oplus C_{f(s,x)}}(x)$$

where $f(s,x)$ is the least stage $t > \max(s, s_0)$ such that $\alpha \subseteq \delta_t$ and $l_e(t) > x$. Then, to prove that γ computes D via $\{D_s\}_{s\in\omega}$ it suffices to show that, for x, s with $D_s(x) \ne D_{s+1}(x)$, $\gamma^*(x) \ge s+1$. So fix such x, s. By definition of γ it suffices to show that $\gamma_\alpha(x)$ is lifted or put down at some stage $t \ge s+1$. For a contradiction assume that this will not happen. Then $\gamma^*(x) = \gamma_\alpha(x, f(s,x)) = \gamma_\alpha(x,t)\downarrow$ for all $t \ge f(s,x)$. In particular, $\gamma_\alpha(x, f(s,x)) = \gamma_\alpha(x, f(s+1,x))\downarrow$. By choice of x and s, however $D_s(x) \ne D_{s+1}(x)$ whence

$$\{e_0\}_{f(s,x)}^{B_{0,f(s,x)}\oplus\widehat{B}_{0,f(s,x)}\oplus C_{f(s,x)}}(x) \ne \{e_0\}_{f(s+1,x)}^{B_{0,f(s+1,x)}\oplus\widehat{B}_{0,f(s+1,x)}\oplus C_{f(s+1,x)}}(x).$$

It follows that $\gamma_\alpha(x)$ is not 0-controllable at stage $f(s,x)$ or $f(s+1,x)$. Since $\alpha \subseteq \delta_{f(s,x)}$ and $\alpha \subseteq \delta_{f(s+1,x)}$ this contradicts Lemma 3.9. $\square$

Lemma 3.18 immediately implies that (3.4) is satisfied.

Lemma 3.19 *For all $e \in \omega$ and $i \le 1$, $P_{e,i}$ is met.*

Proof. Let R_α, $\alpha \subseteq \tau$ and $|\alpha| = 3(2e+i)$, be the strategy for $P_{e,i}$ on the true path and, by Lemma 3.17, let s_0+1 be the least α-stage such that R_α neither is initialized nor requires attention after stage s_0. Then one of the following must hold.

(3.39) There is a realized α-follower x at the end of stage s_0 such that $x \in B_{i,s_0}$.

(3.40) There is an α-follower x at the end of stage s_0 which is not yet realized.

Moreover, by choice of s_0 the α-followers existing at the end of stage s_0 are permanent, they do not change their status later and no new α-followers are appointed later.

Now first assume that (3.39) holds and let $s+1$ be the stage at which the follower x was realized. Then $\alpha \subseteq \delta_s$, $\{e\}_s^{C_s}(x) = 0$, and $r_{1-i}^{\alpha}(s+1) = s+1$. Moreover, since x is permanent, the restraint r_{1-i}^{α} is not reset after stage s, whence $C_s \upharpoonright s+1 = C \upharpoonright s+1$ by Lemma 3.11. It follows that $B_i(x) = 1 \neq 0 = \{e\}^C(x)$, whence x witnesses that $P_{e,i}$ is met.

Finally assume that (3.39) fails and (3.40) holds. By the former, (3.13) will hold for all α-stages $s+1 > s_0$. By the latter the follower x in (3.40) will never be realized whence $x \notin B_i$. On the other hand, however, $\{e\}^C(x) \neq 0$. Namely, otherwise, (3.15) will hold for all sufficiently large α-stages, whence R_α will require attention after stage s_0 contrary to choice of s_0. So, again x witnesses that $P_{e,i}$ is met. $\square$

Lemma 3.19 immediately implies that condition (3.2) is satisfied.

To show that the configuration requirements are met too and that this implies that (3.5) is satisfied we first show that an oracle for A can recognize whether or not a configuration is permanent.

Lemma 3.20 *Let x be a configuration at stage s and assume that $A_s \upharpoonright \varepsilon^*(x) = A \upharpoonright \varepsilon^*(x)$. Then*

(i) $B_{i,s} \upharpoonright x = B_i \upharpoonright x$ *for* $i \leq 1$,

(ii) $\widehat{B}_{i,s} \upharpoonright x = \widehat{B}_i \upharpoonright x$ *for* $i \leq 1$,

(iii) $C_s \upharpoonright x = C \upharpoonright x$, *and*

(iv) x *is a configuration at stage* t *for all* $t \geq s$.

Proof. Note that $\varepsilon^*(x) \geq x$, whence (i) and (ii) follow from $A_s \upharpoonright \varepsilon^*(x) = A \upharpoonright \varepsilon^*(x)$ as shown in the proofs of Lemmas 3.2 and 3.3. For a proof of (iii) and (iv) we show by induction on t that for all $t \geq s$, $C_t \upharpoonright x = C_s \upharpoonright x$ and x is a configuration at stage t.

For $t = s$ the claims are immediate. So fix $t+1 > s$ and by inductive hypothesis assume that $C_s \upharpoonright x = C_t \upharpoonright x$ and x is a configuration at stage t.

Note that if a number y enters C at stage $t+1$ then $y = \gamma_\alpha(z,t)$ for some α and z. Moreover, by construction, $\gamma_\alpha(z)$ is not controllable at stage t. Since x is a configuration at stage t, it follows that $y \geq x$. Hence $C_{t+1} \upharpoonright x = C_t \upharpoonright x = C_s \upharpoonright x$.

It remains to show that x is a configuration at stage $t+1$. Since x is a configuration at stage s, we have that $x \leq s < t+1$, whence it suffices to show that (3.12) holds (for $t+1$ in place of s). So fix α and y such that $\gamma_\alpha(y, t+1)\downarrow < x$. We have to show that $\gamma_\alpha(y)$ is controllable at stage $t+1$ by x. Since $x \leq s$, $\gamma_\alpha(y)$ was put down the last time at a stage $\leq s$, i.e., $\gamma_\alpha(y,s) = \gamma_\alpha(y,t+1)$. Since x is a configuration at stage s, it follows that $\gamma_\alpha(y,s)$ is i-controllable at stage s by x for some $i \leq 1$. Since $B_{i,s} \upharpoonright x = B_{i,t+1} \upharpoonright x$ and $\widehat{B}_{i,s} \upharpoonright x = \widehat{B}_{i,t+1} \upharpoonright x$ by (i) and (ii), and $C_s \upharpoonright x = C_{t+1} \upharpoonright x$ as shown above, it follows from Lemma 3.10 that $\gamma_\alpha(y)$ is i-controllable at stage $t+1$ by x. $\square$

Lemma 3.21 *For all $e \in \omega$, K_e is met.*

Proof. Let R_α with $\alpha \subseteq \tau$ and $|\alpha| = 3e+2$ be the strategy for K_e on the true path. By Lemma 3.17 let s_0 be the last stage at which R_α is initialized and let s_1 be the least stage after which R_α does not require attention.

Note that, for $i \leq 1$, the restraint r^i_α of R_α changes at stage $s+1$ only if R_α receives attention – in which case, for some i, $r^i_\alpha(s+1) = s+1$ – or if r^i_α is reset to 0. So, $r^i = \lim_s r^i_\alpha(s)$ exists for $i \leq 1$. Moreover, $r^0 \neq 0$ or $r^1 \neq 0$, since otherwise (3.23) will hold for infinitely many s whence R_α will require attention infinitely often. For the remainder of the proof, fix $j \leq 1$ such that $r^j \geq r^{1-j}$ and let $r = r^j$. Then R_α receives attention at stage r whence $\alpha \subseteq \delta_{r-1}$, and $r^j_\alpha(s) = r^j_\alpha(r) = r$ for all $s \geq r$. In particular, R_α is not initialized after stage $r-1$, whence $s_0 < r \leq s_1$.

We will show that r is a permanent configuration. Since, by the first clause of (3.23), $r > e$, this will imply that K_e is met.

By Lemma 3.20 it suffices to give a stage s such that r is a configuration at stage s and

(3.41) $A_s \restriction \varepsilon^*(r) = A \restriction \varepsilon^*(r)$.

Such a stage s can be chosen as follows. Let s be the least stage such that $s > s_1 \geq r > s_0$, $\alpha \subseteq \delta_s$, and (3.41) and

(3.42) $\forall i \leq 1 \, \forall t \geq s \, (r^i_\alpha(t) = r^i)$

hold. To show that r is a configuration at stage s it suffices to verify (3.12) for $x = r$. So fix β and y such that $\gamma_\beta(y,s)\downarrow < r$. We have to show that $\gamma_\beta(y)$ is controllable at stage s by r. We distinguish the following cases.

If $\alpha < \beta$ then N_β is initialized in step 1 of stage r whence $\gamma_\beta(y,r)\uparrow$ and any values assigned to $\gamma_\beta(y)$ later are greater than r. So $\alpha < \beta$ is impossible.

Next assume that $\beta <_L \alpha$ and for a contradiction assume that $\gamma_\beta(y,s)$ is not controllable at stage s by r. Then, by definition of r and by (3.42), $r^{1-j}_\alpha(s) \leq r^j_\alpha(s) = r$, whence, for $i = 0, 1$, $\gamma_\beta(y,s)$ is not i-controllable at stage s by $r^i_\alpha(s)$. So if $\gamma_\beta(y,s)$ is controllable at stage s then (3.23) is satisfied whence R_α requires attention at stage $s+1$ contrary to $s > s_1$. On the other hand, if $\gamma_\beta(y)$ is not controllable at stage s then, by step 3 in stage $s+1$ (or by initialization), $\gamma_\beta(y)$ is lifted at stage $s+1$ and R_α is initialized. But this contradicts $s > s_0$.

There remains the case that $\beta \subseteq \alpha$. Since $\gamma_\beta(y,s) < r$, $\gamma_\beta(y)$ was put down on $\gamma_\beta(y,s)$ at a stage $< r$, whence $\gamma_\beta(y, r-1) = \gamma_\beta(y,s)\downarrow$. Since $\beta \subseteq \alpha \subseteq \delta_{r-1}$, by Lemma 3.9, $\gamma_\beta(y, r-1)$ is j-controllable at stage $r-1$ by r. Moreover, since $r^j_\alpha(r) = r$ is not reset after stage r, $B_{j,r-1} \oplus \widehat{B}_{j,r-1} \oplus C_{r-1} \restriction r = B_j \oplus \widehat{B}_j \oplus C \restriction r$ by Lemma 3.11. So, by Lemma 3.10, $\gamma_\beta(y,s) = \gamma_\beta(y,r-1)$ is j-controllable via $r = r^j_\alpha(s)$ at stage s. $\square$

Using Lemmas 3.20 and 3.21 we can now show that (3.5) is satisfied.

Lemma 3.22 $C \leq_T A$.

Proof. Let

$$T = \{(x, s) : x \text{ is a configuration at stage } s \text{ and } A_s \upharpoonright \varepsilon^*(x) = A \upharpoonright \varepsilon^*(x)\}.$$

Then $T \leq_T A$ since A controls ε. Moreover, by Lemma 3.21, there are infinitely many x such that $(x, s) \in T$ for some s, and, by Lemma 3.20, if $(x, s) \in T$ then $C_s \upharpoonright x = C \upharpoonright x$. Thus, $C \leq_T A$. □

It remains to show that the $\widehat{P}$-requirements are met. We first show that the different modules of such a requirement choose the disjunct of $\widehat{P}$ to work on in such a way that they cooperate correctly.

Lemma 3.23 *Let R_α be a strategy for a $\widehat{P}$-requirement and let $m < n$ and s be given such that at the end of stage s the α-n-module is in an active state and either*

(3.43) *the α-m-module is in a critical state and $i_{\alpha,m}(s) = i_{\alpha,n}(s)$*

or

(3.44) *the α-m-module is in an active but not critical state and $i_{\alpha,m}(s) \neq i_{\alpha,n}(s)$.*

Then, there is a number m' such that $m < m' < n$, the α-m'-module is in a critical state at the end of stage s, and $i_{\alpha,m'}(s) \neq i_{\alpha,n}(s)$.

Proof. We prove the claim by induction on s assuming (3.43). The proof for (3.44) is similar. For $s = 0$ the claim is trivial. So suppose that the claim is true for s and assume that at the end of stage $s+1$ the α-n-module is in an active state, the α-m-module is in a critical state and $i_{\alpha,m}(s+1) = i_{\alpha,n}(s+1)$. Then the α-m-module already is in a critical state at the end of stage s because if it changed state at stage $s+1$, the α-n-module would be cancelled or initialized.

Now if the α-n-module is in an active state at the end of stage s then, by inductive hypothesis, for some m' with $m < m' < n$, the α-m'-module is in a critical state at the end of stage s and $i_{\alpha,m'}(s) \neq i_{\alpha,n}(s)$. Moreover, since the α-n-module is not initialized or cancelled at stage $s+1$, the state of the α-m'-module does not change at stage $s+1$. Since, moreover, the indices of the m'- and the n-module do not change at stage $s+1$, m' has the required properties in this case.

So, for the remainder of the proof, we may assume that the α-n-module is in an inactive state at the end of stage s. Then the α-n-module moves from state 0 to state 1 at stage $s+1$ according to case 2.1 in step 1 of the construction. So, in particular, the index of the α-n-module is defined at stage $s+1$. Namely, for $\widehat{m} < n$ maximal such that the α-$\widehat{m}$-module is in an active state at the end of stage s, $i_{\alpha,n}(s+1) = i_{\alpha,\widehat{m}}(s)$ if the α-$\widehat{m}$-module is not in a critical state and $i_{\alpha,n}(s+1) = 1 - i_{\alpha,\widehat{m}}(s)$ otherwise. Moreover, for any $m'' < n$, the index and state of the α-m''-module do not change at stage $s+1$. Since the α-m-module is in a critical state at the end of stage s and $i_{\alpha,m}(s+1) = i_{\alpha,n}(s)$, the above implies that $m < \widehat{m} < n$.

Now, if the α-$\widehat{m}$-module is in a critical state at the end of stage s (hence $s+1$), then we can let $m' = \widehat{m}$. So assume that $state_{\alpha,\widehat{m}}(s) \notin \{4,5\}$. Then, by Lemma 3.15, $state_{\alpha,\widehat{m}}(s) = 2$. Hence, by inductive hypothesis applied to m and $\widehat{m}$ (in place of m and n), there is a number m' with $m < m' < \widehat{m} < n$ such that the α-m'-module is in a critical state at the end of stage s and $i_{\alpha,m'}(s) = 1 - i_{\alpha,\widehat{m}}(s) = 1 - i_{\alpha,n}(s+1)$. Moreover, as observed above, state and index of the α-m'-module do not change at stage $s+1$, whence m' has the required properties. $\square$

The next four lemmas show that the restraints imposed by the $\widehat{P}$-modules have the desired effects.

Lemma 3.24 *Let R_α be a strategy for a $\widehat{P}$-requirement and let $m < n$ and s be given such that the α-m-module and the α-n-module are active at the end of stage s. Then $r_{\alpha,m}(s) < r_{\alpha,n}^{i_{\alpha,n}(s)}(s)$ or $r_{\alpha,n}(s) = 0$.*

Proof. Note that the prime restraint $r_{\alpha,n}^{i_{\alpha,n}}$ and the total restraint $r_{\alpha,n}$ of an α-n-module may change at a stage $s+1$ only if the α-n-module is initialized or cancelled – in which case the restraints are reset to 0 – or R_α acts via n. In the latter case prime and total restraint may be raised to $s+1$ if a new follower is realized (case 2.2), the prime restraint might be unchanged but the total restraint is raised to $s+1$ if one of the followers becomes active (case 2.4), or due to cancellation of a follower the corresponding part of the restraints is erased whence prime and total restraint may decrease (case 2.5). Since followers are cancelled in reverse order of appointment, however, it follows from Lemma 3.7 that the α-n-follower which is appointed first (after some initialization of the α-n-module) cannot be cancelled later unless the α-n-module is initialized or cancelled whence once the first restraint is imposed (according to case 2.2) at some stage $t+1$, neither prime nor total restraint can drop below $t+1$ (by actions according to case 2.5).

The claim follows from these observations as follows. W.l.o.g. assume that $r_{\alpha,n}(s) > 0$, and let t be the greatest stage less than s such that $state_{\alpha,n}(t) = 0$. Then there is a stage u with $t < u < s$ such that at stage $u+1$ R_α acts via n according to case 2.2 for the first time after stage t. By choice of u,

$$u + 1 \leq r_{\alpha,n}^{i_{\alpha,n}(v)}(v) \leq r_{\alpha,n}(v)$$

for all v with $u+1 \leq v \leq s$, whence in particular

$$0 \leq t < r_{\alpha,n}^{i_{\alpha,n}(s)}(s).$$

So if $r_{\alpha,m}(s) = 0$ the claim is immediate. Finally if $r_{\alpha,m}(s) > 0$ then, as observed above, $r_{\alpha,m}(s) = w$ for a stage $w \leq s$ at which R_α receives attention via m. Since $m < n$, however, the α-n-module is initialized or cancelled at such a stage whence

$$r_{\alpha,m}(s) \leq t < r_{\alpha,n}^{i_{\alpha,n}(s)}(s)$$

which completes the proof of the lemma. □

Lemma 3.25 *Let R_α be a strategy for a $\widehat{P}$-requirement and let n and s be given such that the α-n-module is active at the end of stage $s+1$. Then there is no marker position $\varepsilon(x,s)$ such that, for $i = i_{\alpha,n}(s+1)$, $2\varepsilon(x,s) < r_{\alpha,n}^i(s+1)$ and $2\varepsilon(x,s)$ is put into B_i in step 2 of stage $s+1$.*

Proof. For a contradiction assume that $y = 2\varepsilon(x,s) < r_{\alpha,n}^i(s+1)$ enters B_i at step 2 of stage $s+1$. Let t be the greatest stage $\leq s$ at which the α-n-module is in state 0. Since a module which moves from state 0 to state 1 does not impose any restraint, $t < s$ whence, the α-n-module is in an active state at the end of stage s. Moreover, since the α-n-module is active at the end of stage $s+1$, $r_{\alpha,n}^i(s+1)$ coincides with the current value of $r_{\alpha,n}^i$ after step 1 of stage $s+1$ and R_α is not initialized in step 2 of stage $s+1$. So, in step 2 of stage $s+1$, the decision whether y is put into B_0 or B_1 is determined by the α-m-module for the least m such that y is less than the current restraint $r_{\alpha,m}$.

So fix this m. By the above, $m \leq n$. In fact, $m < n$ since otherwise y is put into B_{1-i}. Moreover, the α-m-module is in an active state at the end of stage $s+1$ but does not receive attention at stage $s+1$, since otherwise the α-n-module would be cancelled or initialized at stage $s+1$. So index, state and restraints of the α-m-module are not changed during stage $s+1$.

For the following we distinguish two cases depending on the index of the α-m-module.

First suppose that $i_{\alpha,m}(s+1) = i$, i.e., $i_{\alpha,m}(s+1) = i_{\alpha,n}(s+1)$. Then, since the α-m-module targets y into B_i, $r_{\alpha,m}^i(s+1) \leq y < r_{\alpha,m}(s+1)$, whence the α-m-module is in a critical state at stage $s+1$, hence at the end of stage s. So, by Lemma 3.23, there is a number m' with $m < m' < n$ such that the α-m'-module is in a critical state at the end of stage s and $i_{\alpha,m'}(s) = 1-i$. By $m < m'$ and Lemma 3.24,

$$\varepsilon(x,s) \leq y < r_{\alpha,m}(s+1) = r_{\alpha,m}(s) < r_{\alpha,m'}^{i_{\alpha,m'}(s)}(s).$$

It follows that R_α requires attention via m' according to (3.21) at stage $s+1$. Consequently the α-n-module is cancelled or initialized in step 1 of stage $s+1$ contrary to assumption.

Finally suppose that $i_{\alpha,m}(s+1) = 1-i$, i.e., $i_{\alpha,m}(s+1) \neq i_{\alpha,n}(s+1)$. Then, since the α-m-module targets y into B_i, $y < r_{\alpha,m}^{i_{\alpha,m}(s+1)}(s+1) = r_{\alpha,m}^{i_{\alpha,m}(s)}(s)$. So, if $state_{\alpha,m}(s) \in \{3,4,5\}$ then R_α requires attention via m according to (3.21) causing the α-n-module to be initialized or cancelled at stage $s+1$ contradicting our assumption. Otherwise, the α-m-module is in an active but not critical state at the end of stage s, whence Lemma 3.23 yields a number m' with $m < m' < n$ such that the α-m'-module is in a critical state at the end of stage s and $i_{\alpha,m'}(s) = 1 - i$. But, as shown in the first case already, this is impossible. $\square$

Lemma 3.26 *Let R_α be a strategy for a $\widehat{P}$-requirement and let n and s be given such that the α-n-module is in a critical state at the end of stage $s+1$. Then there is no marker position $\varepsilon(x,s)$ such that, for $i = i_{\alpha,n}(s+1)$, $2\varepsilon(x,s) < r^{1-i}_{\alpha,n}(s+1)$ and $2\varepsilon(x,s)$ is put into B_{1-i} in step 2 of stage $s+1$.*

Proof. For a contradiction assume that $y = 2\varepsilon(x,s) < r^{1-i}_{\alpha,n}(s+1)$ enters B_{1-i} in step 2 of stage $s+1$. Let $r^j_{\alpha,n}$ $(j = 0,1)$ denote the current restraints of the α-n-module at the end of step 1 of stage $s+1$. Since the α-n-module is not initialized or cancelled at stage $s+1$, $r^j_{\alpha,n} = r^j_{\alpha,n}(s+1)$. Moreover, since the prime restraint of a module is not changed by an activity resulting in a critical state, $r^i_{\alpha,n}(s) = r^i_{\alpha,n} = r^i_{\alpha,n}(s+1)$, and, by the state transition rules (Lemma 3.4), $state_{\alpha,n}(s) \in \{3,4,5\}$.

So, if $y < r^i_{\alpha,n}$, then $\varepsilon(x,s) < r^i_{\alpha,n}(s)$ and $\varepsilon(x,s) < \varepsilon(x,s+1)$ (since $\varepsilon(x,s)$ was put into B_{1-i} in step 2 of stage $s+1$), whence R_α will require attention via n at stage $s+1$ according to (3.21). Since a module which requires attention but does not receive attention is cancelled or initialized, it follows that R_α actually receives attention. But then the α-n-module is put into the noncritical state 3 contrary to assumption.

It follows that $r^i_{\alpha,n} \leq y < r^{1-i}_{\alpha,n}$, whence the α-n-module wants to direct y into B_i in step 2 of stage $s+1$. Since y enters B_{1-i} but the α-n-module is not initialized, it follows that, for some $m < n$

$$y < r_{\alpha,m} = r_{\alpha,m}(s+1), \text{ whence } r^i_{\alpha,n}(s+1) < r_{\alpha,m}(s+1).$$

But, by $r^{1-i}_{\alpha,n}(s+1) > 0$, this contradicts Lemma 3.24. $\square$

We call an α-n-follower *permanent* if the follower is not cancelled after its appointment. Note that if an α-n-module reaches its final state $fs_{\alpha,n}$ for the last time at stage $s+1$ and $fs_{\alpha,n}$ is an active state greater than 1, then the permanent α-n-followers are just the α-n-followers existing at the end of stage $s+1$. Moreover, by Lemma 3.7, there will be at least one permanent α-n-follower in this case. In particular, if the final state is critical, then the α-n-module has a permanent active follower x. Note that the permanent active α-n-follower x (if there is any) is unique, it is appointed at stage x and there are stages $x < s+1 < t+1$ such that x is realized at stage $s+1$ for the first time and x becomes active at stage $t+1$. We will show next that the restraints imposed by such a permanent active follower will be respected.

Lemma 3.27 *Let R_α be a strategy for a $\widehat{P}$-requirement, let $n \geq 0$, assume that the α-n-module has final state $fs_{\alpha,n} \in \{4,5\}$, let x be the permanent active α-n-follower and let $s+1 < t+1$ be the stages at which x is realized first and* *at which x becomes active, respectively. Moreover, let i be the final index of the α-n-module. Then*

(3.45) $B_{i,s}\restriction s+1 = B_i\restriction s+1$ *and* $B_{1-i,t}\restriction t+1 = B_{1-i}\restriction t+1$

(3.46) $\widehat{B}_{i,s}\restriction s+1 = \widehat{B}_{i,t}\restriction s+1$ *and* $\widehat{B}_{1-i,t}\restriction t+1 = \widehat{B}_{1-i}\restriction t+1$

(3.47) $C_s\restriction s+1 = C\restriction s+1$ *and* $C_t\restriction t+1 = C\restriction t+1$

Proof. Note that no R_β with $\beta < \alpha$ receives attention at any stage $\geq s+1$, because otherwise R_α would be initialized. Also, all R_β with $\alpha < \beta$ are initialized at stages $s+1$ and $t+1$ by the activity of R_α. It follows that at no stage $\geq s+1$ does any strategy R_β with $\beta \neq \alpha$ put a number $< s+1$ into any set $B_j, \widehat{B}_j, j = 0, 1$, and similarly with $t+1$ in place of $s+1$. Also, since x is permanent, $r^i_{\alpha,n}(u) \geq s+1$ at all stages $u \geq s+1$, $r^{1-i}_{\alpha,n}(u) = t+1 > r^i_{\alpha,n}(u)$ at all stages $u \geq t+1$, the state of the α-n-module is critical at all stages $\geq t+1$, and no α-m-module with $m < n$ is active at a stage $\geq s+1$.

Now, for a proof of (3.45), by the above it suffices to show that there is no marker $\varepsilon(x,u)$ such that either $u \geq s$, $2\varepsilon(x,u) < r^i_{\alpha,n}(u+1)$ and $2\varepsilon(x,u)$ enters B_i in step 2 of stage $u+1$ or $u \geq t$, $2\varepsilon(x,u) < r^{1-i}_{\alpha,n}(u+1)$ and $2\varepsilon(x,u)$ enters B_{1-i} in step 2 of stage $u+1$. But this is guaranteed by Lemma 3.25 and Lemma 3.26, respectively.

For a proof of (3.46), by the observations in the first paragraph of the proof, it suffices to show that there is no follower y of an α-m-module with $m \geq n$ such that either $y \leq s$ and y enters $\widehat{B}_i$ at stage u with $s < u \leq t$ or $y \leq t$ and y enters $\widehat{B}_{1-i}$ at a stage $v \geq t+1$. Since, by the activity of the α-n-module, all α-m-modules with $m > n$ are cancelled or initialized at stages $s+1$ and $t+1$, the claim is trivial for $m > n$. By permanence of x, however, any α-n-follower $y \neq x$ which may enter $\widehat{B}_0$ or $\widehat{B}_1$ at stage $s+1$ or later has been appointed after x was realized first, i.e., $y \geq s+1$ and y will be targeted for $\widehat{B}_i$ and will enter $\widehat{B}_i$ at a stage u before x becomes active, i.e., $u \leq t$. So such a follower y will not interfere with (3.46). Finally, if x itself enters $\widehat{B}_0$ or $\widehat{B}_1$ then x will enter $\widehat{B}_i$ at a stage v after it has become active, i.e. $v > t+1$. So (3.46) holds.

Finally, for a proof of (3.47), recall that only marker positions $\gamma_\beta(z,u)$ can enter C, and for a contradiction assume that $u+1$ is the first stage at which a number $y = \gamma_\beta(z,u)$ violates one of the conditions in (3.47). Then $\beta \subseteq \alpha$ must hold. Namely, since R_α is not initialized after stage s, $\beta <_L \alpha$ is impossible, and, for $\alpha < \beta$, N_β is initialized at stages $s+1$ and $t+1$ so that $\gamma_\beta(z,u)$ will be too big to violate (3.47). We distinguish two cases depending on the side of (3.47) which is injured.

First assume that $s < u+1 \leq t$. Then $\gamma_\beta(z,u) \leq s$ whence $\gamma_\beta(z)$ was put down on this value at a stage $\leq s$ whence $\gamma_\beta(z,s) = \gamma_\beta(z,u)$. Since R_α requires attention via n based on (3.18) at stage $s+1$, it follows that $\beta \subseteq \alpha \subseteq \delta_s$, whence, by Lemma 3.9 $\gamma_\beta(z)$ is i-controllable at stage s by s. Since, by choice of u and by (3.45) and (3.46), $B_{i,s}\restriction s+1 = B_{i,u}\restriction s+1$, $\widehat{B}_{i,s}\restriction s+1 = \widehat{B}_{i,u}\restriction s+1$ and

$C_s \restriction s+1 = C_u \restriction s+1$, it follows by Lemma 3.10 that $\gamma_\beta(z)$ is i-controllable at stage u too. So $\gamma_\beta(z)$ is not put into C at stage $u+1$, contrary to assumption.

The case that $t < u+1$ is ruled out in a similar way by showing that $\gamma_\beta(z)$ is $(1-i)$-controllable at stage u. $\square$

Lemma 3.28 *For all e, $\widehat{P}_e$ is met.*

Proof. Fix $e = \langle e_0, e_1 \rangle$ and let R_α with $|\alpha| = 3e+1$ and $\alpha \subseteq \tau$ be the strategy for $\widehat{P}_e$ on the true path. By Lemma 3.17 R_α requires attention only finitely often, whence every α-n-module reaches a final state $fs_{\alpha,n}$. Moreover, by (3.17), there must be at least one number n such that $fs_{\alpha,n} \in \{1,5\}$. For the following fix such a number n and distinguish the following cases according to the final state of the α-n-module.

First assume that $fs_{\alpha,n} = 1$ and let v be the last stage at which R_α requires attention. Then (3.18) will fail for all $s \geq v$. Since there are infinitely many α-stages this implies that there is a permanent α-n-follower x such that $\{e_i\}^{B_i \oplus C}(x) \neq 0$, where $i = i_{\alpha,n}(v)$ is the final index of the α-n-module. Moreover, by $fs_{\alpha,n} = 1$, $x \notin \widehat{B}_i$, whence $\widehat{P}_e$ is met.

Finally assume that $fs_{\alpha,n} = 5$. By Lemma 3.7 pick the permanent active α-n-follower x and let $s+1$ and $t+1$ be the stages at which x is realized first and becomes active, respectively. Then $x \in \widehat{B}_i$ where $i = i_{\alpha,n}(s)$ is the final index of the α-n-module, and, by choice of $s+1$, (3.18) holds, whence $\{e_i\}_s^{B_{i,s} \oplus C_s}(x) = 0$. Moreover, by Lemma 3.27, $B_{i,s} \restriction s = B_i \restriction s$ and $C_s \restriction s = C \restriction s$, whence $\{e_i\}^{B_i \oplus C}(x) = 0$. So $\widehat{P}_e$ is met in this case too. $\square$

4 Open Problems

One obvious open question given our construction is the following:

Question 4.1 *Is there an effective construction that, given a presentation of a noncontiguous degree* $\mathbf{a}$, *produces uniformly an embedding of N_5 into the c.e. degrees with top* $\mathbf{a}$*?*

In Question 4.1, by a presentation of a noncontiguous degree $\mathbf{a}$, we mean that we are given c.e. indices of two c.e. sets $A_0, A_1 \in \mathbf{a}$ such that $A_0 \not\leq_{wtt} A_1$ and are also given indices e_0, e_1, such that $A_0 = \{e_0\}^{A_1}$, $A_1 = \{e_1\}^{A_0}$. (Note that the marker system γ of Theorem 2.9 can be produced effectively given a presentation of $\mathbf{a}$.) By producing the embedding of N_5 uniformly, we mean that indices for the sets whose degrees form the embedding can be computed effectively from the presentation of the noncontiguous degree. Our construction is not uniform, because we do not know how (3.6) is met.

Another obvious open question is

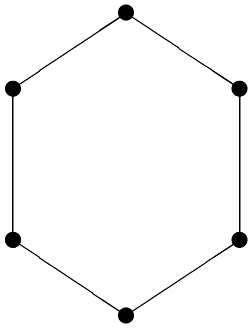

Figure 3

Question 4.2 *If* $\mathbf{a}$ *is a noncontiguous c.e. degree, is* $\mathbf{a}$ *the top of an embedding of the lattice* N_6 *(see Figure 3) into the c.e. degrees?*

One might suspect that a positive answer to Question 4.2 would give a positive answer to Question 4.1, as the natural ways of embedding N_6 are uniform, but there is no logical implication between the two questions.

Another interesting question concerns the class of the semicontiguous degrees. (A c.e. degree $\mathbf{a}$ is called *semicontiguous* if the upper semilattice of c.e. *wtt*-degrees in $\mathbf{a}$ has a greatest element.) By Lachlan's observation in [19] that the upper semilattice of the c.e. *wtt*-degrees is distributive (i.e., every such degree is locally distributive), every semicontiguous degree $\mathbf{a}$ possesses a splitting $\mathbf{a}_0 \vee \mathbf{a}_1 = \mathbf{a}$ that is distributive in the following sense.

$$\forall \mathbf{b} \leq \mathbf{a}\ \exists \mathbf{b}_0, \mathbf{b}_1 (\mathbf{b} = \mathbf{b}_0 \vee \mathbf{b}_1\ \&\ \mathbf{b}_0 \leq \mathbf{a}_0\ \&\ \mathbf{b}_1 \leq \mathbf{a}_1)$$

It is not known whether this property characterizes the semicontiguous degrees.

Question 4.3 *Are the semicontiguous degrees the degrees that possess a distributive splitting? More generally, is there a natural splitting property defining the semicontiguous degrees?*

It is known that every incomplete semicontigous degree is low$_2$ ([11] and [12]) and that above every low$_2$ c.e. degree there is a semicontiguous degree (Downey and Shore [9]). Since this gives a characterization of the class $\mathbf{L}_2$ of the low$_2$ c.e. degrees in terms of semicontiguity, an affirmative answer to Question 4.3 will provide a natural definition of $\mathbf{L}_2$ in the sense of Shore [17].

Alternatively, one might hope to use the definability of contiguity to give a natural definition of $\mathbf{L}_2$. Here, however, as Downey has observed, results on the array nonrecursive (anr) degrees show that $\mathbf{L}_2$ is not the downward closure of the contiguous degrees. Namely, anr degrees are noncontiguous, and the class of anr degrees is closed upwards and contains a low degree (see Downey, Jockusch, Stob [7] and Downey [6]).

A recent very interesting result obtained by Cholak, Downey and Walk, namely, that there is a maximal contiguous degree (see Theorem 2.15 of [17]), provides another proof that $\mathbf{L}_2$ is not the downward closure of the contiguous degrees.

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