Embedding Lattices with Top Preserved Below Non-GL_2 Degrees

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1 Introduction

One way to try to gain an understanding of the various degree-theoretic structures which recursion theorists study is to see what lattices can be embedded into them. Lattice embeddings have been used to show that such structures have an undecidable theory (via embeddings as initial segments) and to show that the theory of such structures is decidable up to a certain quantifier level. Many results in recursion theory can be stated as results about lattice embeddings even if they weren't originally phrased that way.

If $\mathbf{a}$ is a (Turing) degree, let $\mathcal{D}_{\leq}(\mathbf{a})$ stand for the degrees $\leq \mathbf{a}$. Lerman [16] has shown that every recursively presented (in fact $\mathbf{0}''$ presented) lattice with least element can be embedded into $\mathcal{D}_{\leq}(\mathbf{0}')$ as an initial segment. In Shore [20] a much easier argument is given which shows that every recursively presented lattice can be embedded into $\mathcal{D}_{\leq}(\mathbf{0}')$ (not as an initial segment). Shore's construction can easily be modified so that the embedding preserves least element. It is a natural question to ask whether the embeddings can also be made to preserve greatest element. In this paper, we answer this question by showing that every recursively presented lattice can be embedded into $\mathcal{D}_{\leq}(\mathbf{0}')$ preserving both least and greatest elements (Theorem 3). (This extends Cooper's result [5] that the diamond lattice can be embedded into $\mathcal{D}_{\leq}(\mathbf{0}')$ preserving both least and greatest elements.) In Theorem 6, we generalize Theorem 3 by showing that if $\mathbf{a}$ is not in GL_2 (a degree $\mathbf{a}$ is in GL_n iff $\mathbf{a}^{(n)} = (\mathbf{a} \cup \mathbf{0}')^{(n-1)}$ and is in GH_n iff $\mathbf{a}^{(n)} = (\mathbf{a} \cup \mathbf{0}')^{(n)}$), then every recursively presented lattice can be embedded into $\mathcal{D}_{\leq}(\mathbf{a})$ preserving both least and greatest elements. (Actually, Theorems 3 and 6 are stated in terms of recursive partial orders for slightly more generality.) As a corollary (Corollary 10), we get that if $\mathbf{a}$ is not in GL_2, then the $\exists$-theory of $\mathcal{D}_{\leq}(\mathbf{a})$ in the language $(\leq, \vee, \wedge, 0, 1)$ is decidable.

Since there are minimal degrees in both GL_1 (Yates [25]) and $GL_2 - GL_1$ (Sasso [18]), our result cannot be extended to any other part of the GL_n/GH_n hierarchy. Since the minimal degrees of the previous sentence can be taken to be $\leq \mathbf{0}'$, our result cannot be extended to any other part of the L_n/G_n hierarchy of degrees $\leq \mathbf{0}'$. ($\mathbf{a} \in L_n$ iff $\mathbf{a} \leq \mathbf{0}'$ and $\mathbf{a} \in GL_n; \mathbf{a} \in H_n$ iff $\mathbf{a} \leq \mathbf{0}'$ and $\mathbf{a} \in GH_n$.) One might think that if $\mathbf{a}$ is recursively enumerable (r.e.), then a permitting argument would allow the construction of Theorem 3 to be done below $\mathbf{a}$, thereby embedding any recursively presented lattice into $\mathcal{D}_{\leq}(\mathbf{a})$ preserving top and bottom element. (After all, as pointed out by Shore in [20], the argument given there which embeds recursively presented lattices into $\mathcal{D}_{\leq}(\mathbf{0}')$ can be modified in exactly this way to embed recursively presented lattices into $\mathcal{D}_{\leq}(\mathbf{a})$ for $\mathbf{a}$ a nonzero r.e. degree.) However, permitting does not combine well with the coding needed to preserve the top element, and in fact Slaman [21] has shown, using a $\mathbf{0}'''$-injury argument, that there is an r.e. degree $\mathbf{a}$ such that the diamond lattice cannot be embedded into $\mathcal{D}_{\leq}(\mathbf{a})$ preserving both top and bottom elements. By our result, Slaman's degree $\mathbf{a}$ must be in L_2. Slaman can make his degree $\mathbf{a}$ be either low (i.e., in L_1) or nonlow, so our result again cannot be extended to any other classes in the L_n/H_n hierarchy of r.e. degrees.

Before we give our results, we survey the related results known to us. Because of the existence of minimal degrees, any general result on embedding lattices into $\mathcal{D}_{\leq}(\mathbf{a})$ must make some sort of hypothesis on $\mathbf{a}$. Lerman [16] has shown that if $\mathbf{a}$ is high (i.e., in H_1), then any $\mathbf{0}'$ presented lattice with least element can be embedded into $\mathcal{D}_{\leq}(\mathbf{a})$ as an initial segment. Shore has observed that if $\mathbf{a}$ is one-generic, (see Jockusch [8] for the definition) then every recursively presented lattice can be embedded into $\mathcal{D}_{\leq}(\mathbf{a})$ preserving both least and greatest elements. (See Haught [7].) By a result of Jockusch and Posner [9], every degree $\mathbf{a}$ not in GL_2 bounds a one-generic degree, so it follows from these two facts that if $\mathbf{a}$ is not in GL_2, then every recursively presented lattice can be embedded into $\mathcal{D}_{\leq}(\mathbf{a})$ preserving least element. Since every one-generic degree is in GL_1, our result does not overlap with Shore's.

Lerman [16] has shown that if $\mathbf{a}$ is a nonzero r.e. degree, then every $\mathbf{0}'$-presentable lattice with least element can be embedded into $\mathcal{D}_{\leq}(\mathbf{a})$ as an initial segment. Thus when $\mathbf{a}$ is a nonzero r.e. degree, we have a good idea as to which recursively presented lattices can be embedded into $\mathcal{D}_{\leq}(\mathbf{a})$ preserving least element (all of them) and preserving least and greatest element (all of them if $\mathbf{a}$ is not L_2, but maybe not even the diamond if $\mathbf{a}$ is in L_2). However, if $\mathbf{a}$ is in L_2, we don't have a very good idea of which recursively presented lattices can be embedded into $\mathcal{D}_{\leq}(\mathbf{a})$ preserving top element. By Lachlan [12], we do know that the diamond lattice can be embedded this way.

Finally, if $\mathbf{a}$ is an r.e. degree, let $\mathcal{D}^{r.e.}_{\leq}(\mathbf{a})$ be the r.e. degrees which are $\leq$ $\mathbf{a}$. We quickly look at results on lattice embeddings into $\mathcal{D}^{r.e.}_{\leq}(\mathbf{a})$ when $\mathbf{a}$ is a nonzero r.e. degree. All countable distributive lattices (Thomason [24] and Lerman and Lachlan unpublished) and some finite nondistributive (Lachlan [13]) can be embedded into $\mathcal{D}^{r.e.}_{\leq}(\mathbf{0}')$ with least element preserved. However, there are some finite lattices which cannot be embedded into $\mathcal{D}^{r.e.}_{\leq}(\mathbf{0}')$ (Lachlan and Soare [15]). (See Ambos-Spies and Lerman [4] for the latest on which lattices cannot be embedded into $\mathcal{D}^{r.e.}_{\leq}(\mathbf{0}')$.) For $\mathbf{a}$ any nonzero r.e. degree, every countable distributive lattice can be embedded into $\mathcal{D}^{r.e.}_{\leq}(\mathbf{a})$ (Ambos-Spies [1]) and so can the five element nonmodular lattice (Ambos-Spies and Fejer [3]). Since there are nonzero r.e. degrees which do not bound a minimal pair (Lachlan [11]), there are obviously restrictions on which lattices can be embedded into $\mathcal{D}^{r.e.}_{\leq}(\mathbf{a})$ preserving least element. A description of which countable distributive lattices can be embedded into $\mathcal{D}^{r.e.}_{\leq}(\mathbf{a})$ preserving least element for $\mathbf{a}$ a nonzero r.e. degree is given in Ambos-Spies [1]. In Shoenfield and Soare [19] it is shown that the diamond lattice can be embedded into $\mathcal{D}^{r.e.}_{\leq}(\mathbf{0}')$ with top preserved, and in Lachlan [12] it is shown that the diamond lattice can be embedded with top preserved into $\mathcal{D}^{r.e.}_{\leq}(\mathbf{a})$ for any nonzero r.e. $\mathbf{a}$. Ambos-Spies [2] extends [19] by showing that any countable distributive lattice can be embedded into $\mathcal{D}^{r.e.}_{\leq}(\mathbf{0}')$ preserving greatest element. The nondiamond theorem of Lachlan [14] shows that the diamond lattice cannot be embedded into $\mathcal{D}^{r.e.}_{\leq}(\mathbf{0}')$ preserving both top and bottom elements.

Our notation is for the most part standard as in Soare [22, 23]. We use $\subseteq$ for inclusion, $\subset$ for strict inclusion, and when we write $\{e\}^\sigma(x)\downarrow$ where σ is a string, this automatically means that the computation takes $\leq |\sigma|$ steps and $x < |\sigma|$ where $|\sigma|$ denotes the length of the string σ. Thus $\{(e,\sigma,x) : \{e\}^\sigma(x)\downarrow\}$ is recursive, and for any fixed e and σ, there are only finitely many x with $\{e\}^\sigma(x)\downarrow$. We denote concatenation of strings by $*$. When σ is a string and n is a number, $\sigma * n$ denotes $\sigma * \langle n\rangle$ where $\langle n\rangle$ is the string of length one whose only entry is n. When we speak of the first string σ such that ... or the first triple of integers i, j, k such that ... we assume some effective (but unspecified) enumeration of whatever type of object is being discussed.

2 Results

Results on embedding partially ordered sets and lattices into degree theoretic structures depend on representations. The following definition describes the type of representation we use. (By the **trivial poset** we mean the poset with one element.)

Definition 1 *Let $\mathcal{P} = (P, \leq)$ be a nontrivial bounded poset with $P = \{p_i : i \in I\}$ and either $I = n$ for some $n \geq 2$ or $I = \omega$. Assume that p_0 is the least element of $\mathcal{P}$ and that p_1 is the greatest element. In addition, assume that $\mathcal{I}$ is a list of (true) statements of the form $p_i \wedge p_j = p_k$. Then by a* **representation** *of this system ($\mathcal{P}$ and $\mathcal{I}$), we mean a collection $\{\alpha_n\}_{n\in J}$ of maps α_n in ${}^I\omega$ where $J \in \omega$ or $J = \omega$, satisfying the following properties:*

For all $i, j \in I$,

$$p_i \leq p_j \leftrightarrow (\forall n, m \in J)(\alpha_n(j) = \alpha_m(j) \to \alpha_n(i) = \alpha_m(i)). \tag{1}$$

For all $i, j, k \in I$,

$$p_i \vee p_j = p_k \to (\forall n, m \in J)((\alpha_n(i) = \alpha_m(i) \& \alpha_n(j) = \alpha_m(j)) \to (\alpha_n(k) = \alpha_m(k)). \tag{2}$$

For all i, j, k with $p_i \wedge p_j = p_k$ in $\mathcal{I}$,

$$(\forall n, m \in J)(\alpha_n(k) = \alpha_m(k) \to (\exists q_1, q_2, q_3 \in J)(\alpha_n(i) = \alpha_{q_1}(i), \alpha_{q_1}(j) = \alpha_{q_2}(j), \alpha_{q_2}(i) = \alpha_{q_3}(i), \alpha_{q_3}(j) = \alpha_m(j))). \tag{3}$$

In (3), the $\alpha_{q_i}, i = 1, 2, 3$ *are called* **interpolants** *between* α_n *and* α_m.

In Shore [20], it is shown that if $\mathbf{a}$ is a degree, $\leq$ is $\mathbf{a}$-recursive and $\mathcal{I}$ is enumerable effectively in $\mathbf{a}$, then there is a representation as in Definition 1 which is recursive in $\mathbf{a}$ (i.e., $\lambda n, i.\alpha_n(i)$ is $\mathbf{a}$-recursive). (As pointed out in Shore [20], this is a restatement of a result in Jonsson [10].)

In order to embed partial orders in a way that preserves least and greatest elements, we need representations which have additional properties, as specified in the following theorem.

Theorem 2 *Let* $\mathcal{P} = (P, \leq)$ *be a nontrivial bounded poset with* $\leq$ *recursive, and let* $\mathcal{I}$ *be a recursively enumerable list of inf relationships in* $\mathcal{P}$. *Then there is a recursive representation for* $\mathcal{P}$ *and* $\mathcal{I}$ *as in Definition 1 with these additional properties:*

$$(\forall n \in J)(\alpha_n(0) = 0). \tag{4}$$

$$(\forall n, m \in J)(\alpha_n = \alpha_m \rightarrow n = m). \tag{5}$$

$$\begin{aligned}&(\forall i, j \in I)(p_i \not\leq p_j \rightarrow \\ &\quad (\exists n, m \in J)(n \neq 0, m \neq 0, \alpha_n(j) = \alpha_m(j), \alpha_n(i) \neq \alpha_m(i))\end{aligned} \tag{6}$$

In (3), if p_i *and* p_j *are incomparable then* $q_1, q_2,$ *and* q_3 *can be taken to be nonzero.* (7)

$$(\forall j \in I, j \neq 1)(\exists r \in J, r \neq 0)(\alpha_r(j) = \alpha_0(j)) \tag{8}$$

Proof: Let $\mathcal{P}$ and $\mathcal{I}$ be as in the statement of the Theorem. For $s \in I, s \neq 1$, define $\beta_{2s}, \beta_{2s+1} \in {}^I\omega$ by

$$\begin{aligned}\beta_{2s}(i) &= \left\{\begin{array}{ll} 0 & \text{if } i = 0 \\ 2^{2s} & \text{if } i > 0 \end{array}\right\} \\ \beta_{2s+1}(i) &= \left\{\begin{array}{ll} 0 & \text{if } i = 0 \\ 2^{2s} & \text{if } i > 0, p_i \leq p_s \\ 2^{2s+1} & \text{if } i > 0, p_i \not\leq p_s \end{array}\right\}\end{aligned}$$

Then it is easy to check that $\{\beta_i\}_{i\neq 2,3}$ satisfies (1),(2),(4),(5). Now define δ and $\gamma_j, j \in I, j \neq 1$ by

$$\begin{aligned}
\delta(i) &= \left\{ \begin{array}{ll} 0 & \text{if } i = 0 \\ 4 & \text{if } i > 0 \end{array} \right\} \\
\gamma_j(i) &= \left\{ \begin{array}{ll} \delta(i) & \text{if } p_i \leq p_j \\ 3^j & \text{if } p_i \not\leq p_j \end{array} \right\}
\end{aligned}$$

Then it is again easy to check that $\{\beta_i\}_{i\neq 2,3} \cup \{\delta\} \cup \{\gamma_j\}_{j\neq 1}$ satisfies (1),(2), (4),(5) and there are infinitely many numbers not in the range of any of these functions. We can now add interpolants using the method in Shore [20], i.e., suppose that we have a family of functions satisfying (1),(2), (4),(5) and whose combined range is coinfinite. Suppose that $p_i \wedge p_j = p_k$ is in $\mathcal{I}$ and that ρ and η are in the family of functions and $\rho(k) = \eta(k)$. If p_i and p_j are comparable, then the desired interpolants already exist (e.g., if $p_i \leq p_j$, take all the interpolants to be equal to η), so suppose that p_i and p_j are incomparable. Let w, x, y, z be numbers not in the range of any of the functions and define

$$\begin{aligned}
\rho_1(n) &= \left\{ \begin{array}{ll} \rho(n) & \text{if } p_n \leq p_i \\ w & \text{if } p_n \not\leq p_i \end{array} \right\} \\
\rho_2(n) &= \left\{ \begin{array}{ll} \rho_1(n) & \text{if } p_n \leq p_j \\ x & \text{if } p_n \leq p_i \text{ and } p_n \not\leq p_j \\ y & \text{otherwise} \end{array} \right\} \\
\rho_3(n) &= \left\{ \begin{array}{ll} \eta(n) & \text{if } p_n \leq p_j \\ x & \text{if } p_n \leq p_i \text{ and } p_n \not\leq p_j \\ z & \text{otherwise} \end{array} \right\}
\end{aligned}$$

Then ρ_1, ρ_2, ρ_3 interpolate between ρ and η, and it is easy to check that (1),(2),(4),(5) are still met by the set of functions obtained by adding ρ_1, ρ_2, ρ_3 to the old set, (and, of course, the new set of functions still has a combined range which is coinfinite).

Starting with $\{\beta_i\}_{i\neq 2,3} \cup \{\delta\} \cup \{\gamma_j\}_{j\neq 1}$ and repeating the above process systematically, we get a uniformly recursive collection of functions such that for each $p_i \wedge p_j = p_k$ in $\mathcal{I}$ and each ρ, η in the collection with $\rho(k) = \eta(k)$, there are interpolants between ρ and η in the collection.

Let $\{\alpha_i\}_{i\in J}$ be some effective listing of this family of functions with $\alpha_0 = \delta$. Then we have seen that $\{\alpha_i\}_{i\in J}$ is a representation for $\mathcal{P}$ and $\mathcal{I}$ which satisfies

(4) and (5). Also, (6) holds since the α_n and α_m can be chosen from among the β_i's, so won't have index 0; (7) holds since if p_i and p_j are incomparable, then we have added interpolants between them which are $\neq \delta$; and (8) holds because the γ_j's are included in the representation. ■

Theorem 3 *Let $\mathcal{P} = (P, \leq)$ be a nontrivial bounded poset and let $\mathcal{I}$ be a list of infs in $\mathcal{P}$. Suppose that $\leq$ is recursive and $\mathcal{I}$ is r.e. Then $\mathcal{P}$ can be embedded into $\mathcal{D}_{\leq}(\mathbf{0}')$ preserving $0, 1$, all joins which exist in $\mathcal{P}$ and all infs in $\mathcal{I}$.*

Proof: Let $\{\alpha_n\}_{n\in J}$ be a recursive representation for $\mathcal{P}$ as given by Theorem 2. If $g \in {}^{\omega}J$, then for each $i \in I$, define $(g)_i \in {}^{\omega}\omega$ by $(g)_i(n) = \alpha_{g(n)}(i)$. If $\sigma \in {}^{<\omega}J$ and $i \in I$, define $(\sigma)_i$ similarly. By standard arguments (as in Shore [20]), we have, for all i, j, k in I,

$$p_i \leq p_j \to (g)_i \leq_T (g)_j \tag{9}$$

$$p_i \vee p_j = p_k \to (g)_i \oplus (g)_j \equiv_T (g)_k \tag{10}$$

In addition, by (4), we have

$$(g)_0 \text{ is recursive} \tag{11}$$

and by (5)

$$(g)_1 \equiv_T g \tag{12}$$

In order to obtain an embedding of $\mathcal{P}$ into $\mathcal{D}_{\leq}(\mathbf{0}')$, we will construct $g \in {}^{\omega}J$ and map p_i to $\deg_T((g)_i)$. By (9)-(12), in order for this to be the desired embedding, we need only assure that for each i, j, k in I,

$$p_i \not\leq p_j \to (g)_i \not\leq_T (g)_j \tag{13}$$

$$p_i \wedge p_j = p_k \text{ in } \mathcal{I} \to \deg_T((g)_i), \deg_T((g)_j) \text{ have inf } \deg_T((g)_k) \tag{14}$$

$$g \equiv_T \emptyset' \tag{15}$$

For each $i, j \in I$ with $p_i \not\leq p_j$, we break up (13) into the requirements

$$(g)_i \neq \{e\}^{(g)_j} \tag{16}$$

for $e \in \omega$.

For each $i, j, k \in I$ with $p_i \wedge p_j = p_k$ in $\mathcal{I}$, if p_i and p_j are comparable, then (14) is met automatically. Otherwise, we break (14) into requirements

$$\{e\}^{(g)_i} = \{e\}^{(g)_j}, \text{ total } \to \{e\}^{(g)_i} \leq_T (g)_k \tag{17}$$

for $e \in \omega$.

The construction of g will be a finite extension construction with a $\emptyset'$ oracle. This will ensure that $g \leq_T \emptyset'$. To get $\emptyset' \leq_T g$, we let K be an r.e. set in $\mathbf{0}'$ and let $\{K_s\}_{s\in\omega}$ be a recursive approximation to K. We will ensure that g has infinitely many 0's and if $t_0, t_1, \dots$ are the 0's of g, then $K_{t_n}\lceil n = K\lceil n$. This will give $\emptyset' \leq_T g$.

Our strategy to meet (16) and (17) is essentially that of Shore [20]; however, we must ensure that our action in meeting these requirements does not add any new zeros to g so that the coding strategy will work. This is where the additional properties of the representation $\{\alpha_n\}$ given in Theorem 2 are needed.

Suppose that we are given g_s and a requirement (16) where $p_i \not\leq p_j$. We describe how, using a $\emptyset'$ oracle, to obtain a string $g_{s+1} \supseteq g_s$ such that if $g_{s+1} \subset g$, then g will meet the requirement (16) and such that if $|g_s| \leq t < |g_{s+1}|$, then $g_{s+1}(t) \neq 0$.

Let $|g_s| = t_0$. Since $p_i \not\leq p_j$, by (6), we can find $n, m \in J$, both nonzero, such that $\alpha_n(j) = \alpha_m(j)$ but $\alpha_n(i) \neq \alpha_m(i)$. Using $\emptyset'$, see if there is a string $\sigma \supseteq g_s * n$ such that $\{e\}^{(\sigma)_j}(t_0)\downarrow$. If not, set $g_{s+1} = g_s * n$. We then have met (16) by divergence, and $n \neq 0$. If so, take the least such σ. If, for some t with $|g_s| < t < |\sigma|, \sigma(t) = 0$, then by (8), we may replace $\sigma(t)$ by a number $r \neq 0$ in such a way that $(\sigma)_j$ is not changed. (Note that since $p_i \not\leq p_j, j \neq 1$.) Thus, we may assume that $\sigma(t) \neq 0$ for $|g_s| < t < |\sigma|$. If $\{e\}^{(\sigma)_j}(t_0) \neq (\sigma)_i(t_0) = \alpha_n(i)$, then set $g_{s+1} = \sigma$. Otherwise, let g_{s+1} be the same as σ, but with $g_{s+1}(t_0) = m$. Either way, g_{s+1} is as desired.

Now suppose that we are given g_s and a requirement (17) where p_i and p_j are incomparable. We describe how to obtain g_{s+1} as before, but now meeting (17). Ask $\emptyset'$ if there are extensions σ_1, σ_2 of g_s of the same length such that $(\sigma_1)_k = (\sigma_2)_k$ and for some x, $\{e\}^{(\sigma_1)_i}(x)\downarrow \neq \{e\}^{(\sigma_2)_j}(x)\downarrow$. If there are no such σ_1, σ_2, set $g_{s+1} = g_s$. Then if $g \supseteq g_{s+1}$ and $\{e\}^{(g)_i} = \{e\}^{(g)_j}$ is total, then we have $\{e\}^{(g)_i} \leq_T (g)_k$ by the following procedure: given y, find a $\sigma \supseteq g_s$ such that $(\sigma)_k \subseteq (g)_k$ and $\{e\}^{(\sigma)_j}(y)\downarrow$. (A large enough initial segment of g is one such σ.) Then $\{e\}^{(g)_i}(y) = \{e\}^{(\sigma)_j}(y)$.

If such σ_1, σ_2 exist, take the first such pair and first corresponding x and let $\{e\}^{(\sigma_1)_i}(x) = A, \{e\}^{(\sigma_2)_j}(x) = B$. Let ρ_1, ρ_2, ρ_3 be interpolant strings for σ_1, σ_2, (i.e., for $r = 1, 2, 3$, $g_s \subseteq \rho_r, |\rho_r| = |\sigma_1|$, and for $|g_s| \leq y < |\sigma_1|, \alpha_{\rho_1(y)}, \alpha_{\rho_2(y)}, \alpha_{\rho_3(y)}$ are interpolants between $\alpha_{\sigma_1(y)}$ and $\alpha_{\sigma_2(y)}$). Then by (7), if $|g_s| \leq y < |\rho_r|$, $\rho_r(y) \neq 0$. Also, $(\sigma_1)_i = (\rho_1)_i, (\rho_1)_j = (\rho_2)_j, (\rho_2)_i = (\rho_3)_i, (\rho_3)_j = (\sigma_2)_j$, so $\{e\}^{(\rho_1)_i}(x) = A, \{e\}^{(\rho_3)_j}(x) = B$.

Ask $\emptyset'$ if there is a ν_1 containing ρ_1 such that $\{e\}^{(\nu_1)_j}(x) \downarrow$. If not, set $g_{s+1} = \rho_1$. If so, then by (8), we may chose ν_1 so that the part which extends ρ_1 contains no zeros. (Note that since p_i and p_j are incomparable, neither i nor j is 1.) If $\{e\}^{(\nu_1)_j}(x) \neq A$, then set $g_{s+1} = \nu_1$. Otherwise, we have $\{e\}^{(\nu_1)_j}(x) = A = \{e\}^{(\nu_1)_i}(x)$; let ν_2' be obtained from ρ_2 by adding the same string to ρ_2 as was added to ρ_1 to get ν_1. Then $(\nu_2')_j = (\nu_1)_j$, so $\{e\}^{(\nu_2')_j}(x) = A$. Ask $\emptyset'$ if there is an extension ν_2 of ν_2' such that $\{e\}^{(\nu_2)_i}(x) \downarrow$. If not, set $g_{s+1} = \nu_2'$. If so, we may take ν_2 such that it introduces no new zeros. If $\{e\}^{(\nu_2)_i}(x) \neq A$, set $g_{s+1} = \nu_2$. Otherwise, $\{e\}^{(\nu_2)_i}(x) = A = \{e\}^{(\nu_2)_j}(x)$. Let ν_3' be obtained from ρ_3 by adding the same string that was added to ρ_2 to get ν_2. Then $(\nu_3')_i = (\nu_2)_i$, so $\{e\}^{(\nu_3')_i}(x) = A$, but $\{e\}^{(\nu_3')_j}(x) = \{e\}^{(\rho_3)_j}(x) = B$. Set $g_{s+1} = \nu_3'$.

In any of the latter ways of defining g_{s+1}, (17) is met by making the antecedent false, and in all cases, no new zeros are added to g.

Now, to construct g, list all the requirements (16), (17) effectively, and at even stages use the appropriate strategy given above to meet the next requirement on the list. At odd stages $s+1$, using a $\emptyset'$ oracle, let $g_{s+1} = g_s * 1^k 0$ where k is chosen so that if $t = |g_{s+1}| - 1$ is the n^{th} zero of g_{s+1}, then $K_t \lceil n = K \lceil n$. Since no zeros are added to g at even stages, the coding strategy is met and g is as desired. ■

Corollary 4 *Any nontrivial finite partial order $\mathcal{P}$ can be embedded into $\mathcal{D}_{\leq}(\mathbf{0}')$ preserving least and greatest element if they exist in $\mathcal{P}$ and all joins and infs which exist in $\mathcal{P}$.* ■

Theorems 2 and 3 relativize to an arbitrary degree $\mathbf{a}$, giving embeddings of an $\mathbf{a}$-recursive partial order $\mathcal{P}$ with an $\mathbf{a}$-effective list of infs $\mathcal{I}$ into $[\mathbf{a}, \mathbf{a}']$.

We now show how to generalize Theorem 3 by replacing $\mathbf{0}'$ with an arbitrary degree $\mathbf{a}$ which is not in GL_2 (i.e., $\mathbf{a}'' \neq (\mathbf{a} \cup \mathbf{0}')'$). We use a technique introduced in Jockusch and Posner [9] for taking a construction which can be done below $\mathbf{0}'$ and pushing it below a non-GL_2 $\mathbf{a}$. This technique depends on the following result of Martin [17].

Lemma 5 *If* $\mathbf{a}$ *is not in* GL_2*, then no function recursive in* $\mathbf{a} \cup \mathbf{0}'$ *dominates every function recursive in* $\mathbf{a}$.

See Jockusch and Posner [9] for a proof. ■

The outline of our construction is as follows. We effectively list all requirements (16), (17) as $\{R_n\}_{n\in\omega}$ and we define $f(n, \sigma)$ to be the length of the resulting g_{s+1} if we take σ as g_s and use $\mathbf{0}'$ to meet R_n. We let $F(s) = \max\{f(n, \sigma) : n \leq s, |\sigma| \leq s\}$, so F is recursive in $\mathbf{0}'$, and hence by Lemma 5, there is an $\mathbf{a}$-recursive h which is not dominated by F, i.e., there are infinitely many s with $F(s) \leq h(s)$. At stage $s + 1$, we bound our search for strings to meet requirements to strings of length $\leq h(|g_s|)$. If $h(|g_s|) \geq F(|g_s|)$, then we have searched for long enough. There are two problems with this. First, if we allow the lengths of g_s to grow by leaps and bounds, then the set of lengths of g_s and the set of s such that $h(s) \geq F(s)$ may be disjoint, and we may never search long enough to meet a requirement. Thus we must make sure that $|g_s| = s$. This means that if we find a $\sigma \supset g_s$ such that σ meets some requirement, we cannot just set $g_{s+1} = \sigma$. We must instead set $g_{s+1} = g_s * \sigma(|g_s|)$ and in later stages, we keep heading towards σ one step at a time. Meanwhile, we keep looking for a higher priority requirement which we can attack. If we find one before we reach σ, we abandon our attempt to reach σ, and we begin heading for the new goal.

A second problem is that if the representation is infinite, then there are infinitely many strings of a given finite length, and hence the definition of F doesn't make sense. We must first analyze the construction closely to see that there is a recursive *a priori* bound $B(x)$ such that $g(x) \leq B(x)$ for all x.

We now give the details.

Theorem 6 *Let* $\mathbf{a}$ *be a degree which is not in* GL_2*, (i.e.,* $\mathbf{a}'' \neq (\mathbf{a} \cup \mathbf{0}')'$*). Let* $\mathcal{P} = (P, \leq)$ *be a recursive nontrivial bounded partial order with* $\mathcal{P} = \{p_i : i \in I\}$*,* $I = [0, n], n \geq 1$ *or* $I = \omega$ *and* p_0 *and* p_1 *the least and greatest elements of* $\mathcal{P}$*. Let* $\mathcal{I}$ *be an effective list of infs in* $\mathcal{P}$*. Then there is an embedding of* $\mathcal{P}$ *into* $\mathcal{D}_{\leq}(\mathbf{a})$ *which preserves least and greatest element, all joins, and all infs in* $\mathcal{I}$.

Proof: Let $\{\alpha_i\}_{i\in J}$ be a representation for $\mathcal{P}$ as in Theorem 2. We will construct $g \in {}^{\omega}J$ by a construction recursive in $\mathbf{a}$. We will get our embedding from g just as in Theorem 3. Thus we must meet requirements (13),(14) and also make $g \in \mathbf{a}$. We will automatically have $\deg_T(g) \leq \mathbf{a}$. To get $\mathbf{a} \leq \deg_T(g)$, let A be a set in $\mathbf{a}$, and let $B = \{\langle A(0), \ldots, A(n)\rangle : n \geq 0\}$. Then $B \in \mathbf{a}$ and B is recursive in each of its infinite subsets. We will make sure that g has infinitely many zeros and that they are all in B. It follows that $B \leq_T g$, so $\mathbf{a} \leq \deg_T(g)$. (We thank Carl Jockusch for suggesting this coding method to us.)

We break up each requirement (13) with $p_i \not\leq p_j$ as in (16) and we break up each requirement (14) with $p_i \wedge p_j = p_k$ in $\mathcal{I}$ and p_i and p_j incomparable as in (17). Let $\{R_n\}_{n\in\omega}$ be some effective listing of the requirements (16) and (17). If $R_{n'}$ is of type (16), then we define $K(n')$ to be the first pair $\langle n, m\rangle$ of nonzero elements of J (in some effective enumeration of pairs of elements of J) which satisfy (6) and we let $K'(n') = \{n, m\}$ for the same n, m. If $R_{n'}$ is of type (17) and $r_1, r_2 \in J$, then we define $L(n', r_1, r_2)$ to be $\langle\rangle$ if $\alpha_{r_1}(k) \neq \alpha_{r_2}(k)$ and otherwise to be the first triple $\langle q_1, q_2, q_3\rangle$ (in some effective enumeration of triples in J) of nonzero elements of J such that $\alpha_{q_1}, \alpha_{q_2}, \alpha_{q_3}$ interpolate between α_{r_1} and α_{r_2}. (Such a triple exists by (3) and (7).) Let $L'(n', r_1, r_2)$ be the set consisting of those elements of J which are entries in $L(n', r_1, r_2)$. For each $j \in J$ with $j \neq 1$, let $v_j \in J$ be such that $v_j \neq 0$ and $\alpha_{v_j}(j) = \alpha_0(j)$. (Such a v_j exists by (8) and can be found effectively given j.) Now we define $S(x, n)$ for each x, n as follows. (The idea is that $S(x, n)$ consists of those elements of J which $g(x)$ could equal if no requirement $R_{n'}$ with $n' < n$ affects $g(x)$.)

$$S(x, n) = \{0, 1\} \text{ if } n \geq x + 1.$$

If $S(x, n+1)$ has been defined (with $0 \leq n \leq x$), then define

$$S(x,n) \;=\; \left\{ \begin{array}{l} S(x,n+1) \cup K'(n) \cup \{v_j\} \text{ if } R_n \text{ is (16)} \\ S(x,n+1) \cup \{v_i, v_j\} \cup \bigcup\{L'(n, r_1, r_2) : r_1, r_2 \in S(x, n+1)\} \\ \quad \text{if } R_n \text{ is (17)} \end{array} \right\}$$

Note that S is recursive, $S(x, n+1) \subseteq S(x, n)$ and each $S(x, n)$ is finite. In the construction, we will only consider strings $\sigma \in {}^{<\omega}J$ such that $(\forall t < |\sigma|)(\sigma(t) \in S(t, 0))$. Let T be the set of such strings. There are only finitely many strings in T of a given length.

Next, we define a function $f : T \times \omega \to \omega$. The idea is that $f(\sigma, n)$ will be the length of g_{s+1} if $g_s = \sigma$ and we define g_{s+1} using a $\mathbf{0}'$ oracle so as to meet R_n. More precisely, suppose first that R_n is (16). Let $K(n) = \langle r, q\rangle$. If there is a τ such that i) $\tau \supseteq \sigma * r$, ii) $\{e\}^{(\tau)_j}(|\sigma|)\downarrow$, iii) $(\forall x)(|\sigma| < x < |\tau| \to \tau(x) \in S(x, n+1))$, then let $f(\sigma, n)$ be the length of the first such τ. If no such τ exists, let $f(\sigma, n) = |\sigma|$.

Now suppose that R_n is (17). If there are no strings σ_1, σ_2 such that i) $\sigma \subseteq \sigma_1, \sigma_2$, ii) $|\sigma_1| = |\sigma_2|$, iii) $(\sigma_1)_k = (\sigma_2)_k$, iv) $|\sigma| \leq x < |\sigma_1| \to \sigma_1(x), \sigma_2(x) \in S(x, n{+}1)$, v) for some y, $\{e\}^{(\sigma_1)_i}(y)\downarrow \neq \{e\}^{(\sigma_2)_j}(y)\downarrow$, then let $f(\sigma, n) = |\sigma|$. If such σ_1, σ_2 exist, take the first such, let y be the least as in (v) and let ρ_1, ρ_2, ρ_3 be interpolant strings between σ_1, σ_2 obtained as in the proof of Theorem 3. If there is no extension $\hat{\nu_1}$ of ρ_1 such that i) $|\rho_1| \leq x < |\hat{\nu_1}| \to \hat{\nu_1}(x) \in S(x, n{+}1)$, and ii) $\{e\}^{(\hat{\nu_1})_j}(y)\downarrow$, then let $f(\sigma, n) = |\rho_1|$. If such $\hat{\nu_1}$ exists, then take the first such and let ν_1 be obtained from $\hat{\nu_1}$ by replacing 0's which occur in $\hat{\nu_1}$ beyond ρ_1 with v_j. If $\{e\}^{(\nu_1)_i}(y) \neq \{e\}^{(\nu_1)_j}(y)$, then set $f(\sigma, n) = |\nu_1|$. Otherwise, let ν_2' be obtained as in the proof of Theorem 3. If there is no string ν_2 with i) $\nu_2 \supseteq \nu_2'$, ii)$|\nu_2'| \leq x < |\nu_2| \to \nu_2(x) \in S(x, n+1)$, and iii) $\{e\}^{(\nu_2)_i}(y)\downarrow$, then let $f(\sigma, n) = |\nu_2'|$. Otherwise, let $f(\sigma, n)$ be the length of the first such ν_2.

Now, f is recursive in $\mathbf{0}'$. Define F by $F(s) = \max\{f(\sigma, n) : \sigma \in T, |\sigma| \leq s, n \leq s\}$, so F is also recursive in $\mathbf{0}'$. Since $\mathbf{a}$ is not in GL_2, by Lemma 5 there is a function h recursive in $\mathbf{a}$ such that h is not dominated by F, i.e., there are infinitely many t with $F(t) \leq h(t)$.

At each stage s in the construction, we may choose a requirement R_n which receives attention at stage s. At most one requirement receives attention at a given stage. One way for a requirement to receive attention is to begin an attack on the requirement. This attack remains in effect until it is canceled or else throughout the construction if the attack is not canceled. When an attack is begun, a **target** string for the attack is given and in addition, a condition may be given such that if the condition becomes true at a later stage, then the attack is **discredited**. If, at the end of stage s, an uncanceled attack has $g_s \subset \sigma$ where σ is the target of the attack, then we call the attack **ongoing** at the end of stage s. It will follow easily from the construction that there is at most one ongoing attack at the end of any stage.

We now give the construction. Set $g_0 = \langle\rangle$.

Stage $s{+}1$**.** Cancel all attacks which are discredited at stage $s{+}1$. Let n be the least number such that R_n is not under attack. We proceed according to which of the following cases holds.

Case 1. n is $\leq$ the number of zeros in g_s and, if there is an ongoing attack at the end of stage s (say on R_m) which is not discredited at stage $s+1$, then $n < m$.

Then an attack is begun on R_n at stage $s+1$ (so R_n receives attention at stage $s+1$). How we proceed depends on what form R_n has.

Case 1.1 R_n is (16).

Then let $K(n) = \langle r, q \rangle$. See if there is a string $\tau \in T$ such that i) $|\tau| \leq h(s)$, ii) $\tau \supseteq g_s * r$, iii) $(\forall x)(s < x < |\tau| \rightarrow \tau(x) \in S(x, n+1))$, and iv) $\{e\}^{(\tau)_j}(s) \downarrow$. If not, then the target string for the attack is $g_s * r$ and if for some $t > s$ there is a τ with $|\tau| \leq h(t)$ and (ii)-(iv) satisfied, then the attack is discredited at stage $t+1$.

If such a τ exists at stage $s+1$, then take the first such τ and obtain σ from τ by replacing all 0's in τ which occur beyond g_s with v_j (so $(\sigma)_j = (\tau)_j$). If $\{e\}^{(\sigma)_j}(s) \neq (\sigma)_i(s)$ $(= \alpha_r(i))$, then σ is the target string of the attack. Otherwise, the target string is $\hat{\sigma}$, which is the same as σ except that $\hat{\sigma}(s) = q$. Either way, the attack can never later be discredited.

Case 1.2 R_n is (17).

See if there are strings σ_1, σ_2 such that i) $|\sigma_1|, |\sigma_2| \leq h(s)$, ii) $g_s \subseteq \sigma_1, \sigma_2$, iii) $|\sigma_1| = |\sigma_2|$, iv) $(\sigma_1)_k = (\sigma_2)_k$, v) $s \leq x < |\sigma_1| \rightarrow \sigma_1(x), \sigma_2(x) \in S(x, n+1)$, and vi) for some y, $\{e\}^{(\sigma_1)_i}(y) \downarrow \neq \{e\}^{(\sigma_2)_j}(y) \downarrow$. If not, then the target string is g_s and if for some $t > s$ there are σ_1, σ_2 with $|\sigma_1|, |\sigma_2| \leq h(t)$ and (ii)-(vi) satisfied, then the attack is discredited at stage $t+1$.

If such σ_1, σ_2 exist at stage $s+1$, take the least such and let ρ_1, ρ_2, ρ_3 be interpolant strings between σ_1 and σ_2, (i.e., for $s \leq x < |\sigma_1|$, $L(n, \sigma_1(x), \sigma_2(x)) = \langle \rho_1(x), \rho_2(x), \rho_3(x) \rangle$) and let y be the first number satisfying (vi). See if there is an extension $\hat{\nu_1}$ of ρ_1 satisfying i) $|\hat{\nu_1}| \leq h(s)$, ii) $\rho_1 \subseteq \hat{\nu_1}$, iii) $|\rho_1| \leq x < |\hat{\nu_1}| \rightarrow \hat{\nu_1}(x) \in S(x, n+1)$, and iv) $\{e\}^{(\hat{\nu_1})_j}(y) \downarrow$. If not, then ρ_1 is the target string for the attack and if $t > s$ is such that there is a $\hat{\nu_1}$ with $|\hat{\nu_1}| \leq h(t)$ satisfying (ii)-(iv), then the attack is discredited at stage $t+1$.

If such a $\hat{\nu_1}$ exists, then take the first such and obtain ν_1 from $\hat{\nu_1}$ by replacing each 0 which occurs beyond ρ_1 in $\hat{\nu_1}$ with v_j. If $\{e\}^{(\nu_1)_i}(y) \neq \{e\}^{(\nu_1)_j}(y)$, then the target string for the attack is ν_1 and the attack cannot be discredited.

If the equality does hold, then let ν_2' be obtained by adding to ρ_2 the string that was added to ρ_1 to get ν_1. See if there is a $\hat{\nu_2}$ such that i) $|\hat{\nu_2}| \leq h(s)$, ii) $\nu_2' \subseteq \hat{\nu_2}$, iii) $|\nu_2'| \leq x < |\hat{\nu_2}| \rightarrow \hat{\nu_2}(x) \in S(x, n+1)$, and iv) $\{e\}^{(\hat{\nu_2})_i}(y) \downarrow$. If not, then ν_2' is the target string and if for some $t > s$ there is a string $\hat{\nu_2}$

satisfying (ii)-(iv) with $|\hat{\nu_2}| \leq h(t)$, then the attack is discredited at stage $t+1$.

If such $\hat{\nu_2}$ exists, take the first such and let ν_2 be obtained from $\hat{\nu_2}$ by replacing any 0's in $\hat{\nu_2}$ beyond ν_2' with v_i. If $\{e\}^{(\nu_2)_i}(y) \neq \{e\}^{(\nu_2)_j}(y)$, then ν_2 is the target of the attack and the attack cannot be discredited.

If the equality holds, let ν_3' be obtained by adding to ρ_3 the same string which was added to ρ_2 to get ν_2. The target string is ν_3' and the attack cannot be discredited.

In either Case 1.1 or 1.2, let σ be the target string of the attack, so $g_s \subseteq \sigma$. If $g_s = \sigma$, set $g_{s+1} = g_s * 1$. Otherwise, define $g_{s+1} = \sigma \lceil s+1$ $(= g_s * \sigma(s))$.

Case 2. Case 1 fails to hold and there is an ongoing attack, say on R_n, at the end of stage s which is not discredited at stage $s+1$.

Then R_n receives attention at stage $s+1$. Let $g_{s+1} = \sigma\lceil s+1$ where σ is the target string of the ongoing attack.

Case 3. Neither Case 1 nor Case 2 holds.

Define $g_{s+1} = g_s * (1 - B(s))$.

In all cases, if R_n receives attention at stage $s+1$, cancel all attacks on requirements $R_{n'}$ with $n' < n$. $\square$

Now, if an attack is begun on R_n at stage $s+1$, then n is $\leq$ the number of zeros in g_s. Since $|g_s| = s$, this means that

$$\text{If an attack is begun on } R_n \text{ at stage } s+1, \text{ then } n \leq s. \tag{18}$$

From the construction, the definition of S and (18), it follows that

$$\begin{array}{l} \text{If } R_n \text{ receives attention at stage } s+1, \text{ then } g(s) \in S(s,n). \\ \text{If no requirement receives attention at stage } s+1, \\ \text{then } g(s) \in S(s,n) \text{ for every } n. \end{array} \tag{19}$$

Lemma 7 *For each n, g has at least n zeros and R_n is met and receives attention only finitely often.*

Proof: Suppose that the result is true for all $n' < n$. Let s_0 be such that no $R_{n'}$ with $n' < n$ receives attention at a stage $\geq s_0$.

If $t \geq s_0$, is such that g_t does not have at least n zeros, then at no stage $\leq t+1$ can an attack be begun on a requirement $R_{n'}$ with $n \leq n'$, so at stage $t+1$ no $R_{n'}$ with $n \leq n'$ can receive attention. Thus, by choice of s_0, no $R_{n'}$

receives attention at stage $s+1$, i.e., Case 3 holds. Since B is infinite, there must be some $t \geq t_0$ such that g_t has at least n zeros, and hence g has at least n zeros.

Now take $s_1 \geq s_0$ such that g_{s_1} has at least n zeros and let $s \geq s_1$ be such that $h(s) \geq F(s)$. (This is possible since F does not dominate h.) Then by the construction, either there is an attack made on R_n before stage $s+1$ which is not canceled by the end of stage s and is not discredited at stage $s+1$ or else an attack is made on R_n at stage $s+1$. Let $t+1 \leq s+1$ be the stage at which the attack on R_n which exists at the end of stage $s+1$ was made.

If, during the attack at stage $t+1$ on R_n, a search was made for a string or strings of length $\leq h(t)$ and no such string(s) were found, then we claim that no such strings of any length exist. For suppose that such strings do exist. Then by definition of f, the lengths of these strings are $\leq f(g_t, n)$. Now, by (18), $n \leq t$, so for any $t' \geq t$, we have that the string(s) have length $\leq F(t')$. In particular, the length is $\leq F(s) \leq h(s)$. Since the string(s) have length $> h(t)$, we must have $t < s$. But then the attack on R_n would be discredited at stage $s+1$, contradicting hypothesis. Thus, no such string(s) exist.

It follows that the attack made at stage $t+1$ can never be discredited. Since $s \geq s_0$, this means that the attack will never be canceled. Let σ be the target string of the attack. Since the attack is never canceled, it follows from the construction that $\sigma \subset g$. It now follows that if $t' \geq |\sigma|, t+1$, then R_n does not receive attention at stage $t'+1$. Thus, R_n receives attention only finitely often.

To see that R_n is met, note that by (19) and the construction, if $x \geq |\sigma|$, then $g(x) \in S(x, n+1)$. Also, we have already seen that if the attack at stage $t+1$ involved an unsuccessful search for a string or strings, then these string(s) don't exist, and that g extends the target string for the attack. By an argument similar to that given in Theorem 3, it follows from these facts that R_n is met. $\square$

Thus, all the R_n's are met. To finish the proof of the theorem, note that if some R_n receives attention at stage $t+1$, then $g(t) \neq 0$. Thus, the only way that $g(t)$ can be 0 is if Case 3 holds at stage $t+1$. It follows that all of g's 0's are in B. Since g has infinitely many zeros, we are done. $\blacksquare$

Corollary 8 *If* $\mathbf{a}$ *is not in* GL_2*, then any nontrivial recursively presented lattice can be embedded (as a lattice) into* $\mathcal{D}_{\leq}(\mathbf{a})$ *preserving least and greatest element, if they exist.* ■

Corollary 9 *If* $\mathbf{a}$ *is not in* GL_2*, then any finite nontrivial partial order can be embedded into* $\mathcal{D}_{\leq}(\mathbf{a})$*, preserving least and greatest elements, if they exist, and all joins and infs that exist in the partial order.* ■

Corollary 10 *If* $\mathbf{a}$ *is not in* GL_2*, then the* $\exists$ *theory of the structure* $\mathcal{D}_{\leq}(\mathbf{a})$ *in the language* $\mathcal{L} = (\leq, \vee, \wedge, 0, 1)$ *is decidable.*

Proof: This proof is essentially as in Fejer-Shore [6], Corollary to Theorem 3. That is, given an $\exists$ sentence $\phi = \exists x_1 \ldots \exists x_n \psi$ in $\mathcal{L}$ with ψ quantifier free (where $\vee$ is taken to be a three place relation symbol), we claim that ϕ is valid in $\mathcal{D}_{\leq}(\mathbf{a})$ if and only if ϕ is valid in some nontrivial bounded partial order with at most $2(n+2)^3 + (n+2)$ elements. We refer the reader to [6] for details. ■

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