Embedding Distributive Lattices Preserving 1 Below a Nonzero Recursively Enumerable Turing Degree

Klaus Ambos-Spies
Universität Heidelberg
Mathematisches Institute
Im Neuenheimer Feld 294
D-6900 Heidelberg
Germany

Ding Decheng
Department of Mathematics
Nanjing University
Nanjing 210008
Peoples' Republic of China

Peter A. Fejer
Department of Mathematics and Computer Science
University of Massachusetts at Boston
Boston, MA 02125
U.S.A.

1 Introduction

One way to try to gain an understanding of the various degree-theoretic structures which recursion theorists study is to see what lattices can be embedded into them. Lattice embeddings have been used to show that such structures have an undecidable theory (via embeddings as initial segments) and to show that the theory of such structures is decidable up to a certain quantifier level. Many results in recursion theory can be stated as results about lattice embeddings even if they were not originally phrased that way.

We write $\mathcal{R}$ for the set of all recursively enumerable (r.e.) Turing degrees. If $\mathbf{a}$ is an r.e. Turing degree, then we write $\mathcal{R}(\leq \mathbf{a})$ for the set of all r.e. Turing degrees which are less than or equal to $\mathbf{a}$. Lattice embeddings into such initial segments of the r.e. degrees have been widely studied.

The existence of a so-called "minimal pair" (shown by Lachlan [12] and Yates [23]) is equivalent to the fact that the diamond lattice (the four element Boolean algebra) can be embedded into $\mathcal{R}$ preserving 0. The "nondiamond theorem" (shown in the same paper by Lachlan) says that the diamond lattice cannot be embedded into $\mathcal{R}$ preserving both 0 and 1. The obvious next question, whether the diamond lattice could be embedded into $\mathcal{R}$ preserving 1, was answered affirmatively by Shoenfield and Soare [19]. The Lachlan Splitting Theorem [16], proven about the same time, generalized this result by showing that the diamond lattice can be embedded preserving 1 into $\mathcal{R}(\leq \mathbf{a})$ for every nonzero r.e. degree $\mathbf{a}$. Ambos-Spies [1] generalized the result of Shoenfield and Soare in another direction by showing that every countable distributive lattice can be embedded into $\mathcal{R}$ preserving 1.

It is natural to ask whether the two generalizations of the Shoenfield-Soare result given by Lachlan and Ambos-Spies can be combined. In this paper, we show that they can be, by proving that for every nonzero $\mathbf{a} \in \mathcal{R}$, every countable distributive lattice can be embedded into $\mathcal{R}(\leq \mathbf{a})$ preserving 1 (Theorem 18).

The long gap between the appearance of [16] and our paper is perhaps explained by the fact that the techniques of Lachlan's paper have not been well-understood. Our first step in proving our result was to reprove the Lachlan Splitting Theorem using techniques which have been developed since Lachlan's paper appeared. In Section 2 of this paper, we give this reproof of the Splitting Theorem. In Section 3, we extend the techniques of Section 2 to show our result, and in Section 4, we discuss related results and open questions. In particular, in this section, we show how our results, together with some other recent and older results, provide a complete answer to all questions of the form

> Which finite distributive lattices can be embedded into A preserving B?

where A can be:

- $\mathcal{R}$;
- every nontrivial initial segment of $\mathcal{R}$;
- every nontrivial final segment of $\mathcal{R}$; or
- every interval of $\mathcal{R}$,

and B can be:

- meets and joins;
- meets, joins and 0;
- meets, joins and 1; or
- meets, joins, 0 and 1.

In fact, we get answers to the question for all but one combination of A and B even if we remove the word "finite".

The reader will find unexplained recursion-theoretic notations and terminology in Soare [21]. When we refer to an *initial segment* of $\mathcal{R}$, we mean a subset of the form $\mathcal{R}(\leq \mathbf{a})$ with $\mathbf{a} \in \mathcal{R}$. We call the initial segment *nontrivial* if $\mathbf{a} \neq \mathbf{0}$.

Similarly, a *final segment* of $\mathcal{R}$ is a subset of the form $\mathcal{R}(\geq \mathbf{a}) = \{\mathbf{b} \in \mathcal{R} : \mathbf{b} \geq \mathbf{a}\}$ with $\mathbf{a} \in \mathcal{R}$. The final segment is *nontrivial* if $\mathbf{a} \neq \mathbf{0}'$. An *interval* of $\mathcal{R}$ is a subset of the form $\{\mathbf{c} : \mathbf{a} \leq \mathbf{c} \leq \mathbf{b}\}$ for $\mathbf{a}, \mathbf{b} \in \mathcal{R}$ with $\mathbf{a} \leq \mathbf{b}$. The interval is *nontrivial* if $\mathbf{a} < \mathbf{b}$.

We use the "hat trick", that is we define $\{\hat{e}\}_s^{A_s}(x)$ to be $\{e\}_s^{A_s}(x)$ if $s > 0$, $\{e\}_{s-1}^{A_{s-1}}(x) \downarrow$ and $A_s \restriction u(A_{s-1}; e, x, s-1) = A_{s-1} \restriction u(A_{s-1}; e, x, s-1)$; otherwise, $\{\hat{e}\}_s^{A_s}(x) \uparrow$.

When we refer to an *embedding* of a lattice into an interval of $\mathcal{R}$, we mean an injective map which preserves (finite) meets and joins. An embedding of a lattice $\mathcal{L}$ into an interval of $\mathcal{R}$ is said to *preserve least element* if either $\mathcal{L}$ does not have a least element or else the embedding takes the least element of $\mathcal{L}$ to the least element of the interval. We sometimes say "preserves 0" for "preserves least element". The phrases "preserves greatest element", "preserves 1" are defined dually.

An element of a partial order is called *meet-reducible* if it is the meet of two elements strictly above it. An element which is not meet-reducible is called *meet-irreducible*. *Join-reducible* and *join-irreducible* are defined dually.

If $\mathcal{L}$ is a lattice, then we denote by $\mathcal{L}_{+0}$ ($\mathcal{L}_{+1}$) the lattice obtained from $\mathcal{L}$ by adding a new least (greatest) element and by $\mathcal{L}_{-0}$ ($\mathcal{L}_{-1}$) the partial order obtained from $\mathcal{L}$ by removing the least (greatest) element, if any. (So if $\mathcal{L}$ has no least element, then $\mathcal{L}_{-0} = \mathcal{L}$ and if $\mathcal{L}$ has no greatest element, then $\mathcal{L}_{-1} = \mathcal{L}$.) If $\mathcal{L}$ has at least two elements and does not have a meet-reducible least element (join-reducible greatest element) then $\mathcal{L}_{-0}$ ($\mathcal{L}_{-1}$) is again a lattice.

2 The Lachlan Splitting Theorem

In this section, we give a proof of the following theorem (the Lachlan Splitting Theorem ([16])) using the techniques which we need for our main theorem.

Theorem 1 *If* $\mathbf{a}$ *is a nonzero r.e. degree, then the diamond lattice can be embedded into* $\mathcal{R}(\leq \mathbf{a})$ *preserving 1.*

We provide an intuitive discussion before giving the formal proof.

We are given an r.e. degree $\mathbf{a}$ and we wish to construct r.e. degrees $\mathbf{b}_0, \mathbf{b}_1$, and $\mathbf{c}$ such that $\mathbf{b}_0$ and $\mathbf{b}_1$ are incomparable, $\mathbf{b}_0 \cap \mathbf{b}_1 = \mathbf{c}$, and $\mathbf{b}_0 \cup \mathbf{b}_1 = \mathbf{a}$. Let A be an r.e. set in $\mathbf{a}$, say $(a_0, a_1, \ldots)$ is an effective enumeration of A without repetitions and let $A_s = \{a_0, \ldots, a_{s-1}\}$ for all $s \geq 0$. We must construct r.e. sets C, B_0, B_1 so that

(2.1) $B_0, B_1 \not\leq_T C$,

(2.2) for all sets E, if $E \leq_T B_0 \oplus C, B_1 \oplus C$, then $E \leq_T C$,

(2.3) $B_0, B_1 \leq_T A$,

(2.4) $C \leq_T A$, and

(2.5) $A \leq_T B_0 \oplus B_1$.

If we do this, and we define $\mathbf{c} = \deg_T(C)$, $\mathbf{b}_i = \deg_T(B_i \oplus A)$, for $i < 2$, then $\mathbf{c}, \mathbf{b}_0, \mathbf{b}_1$ are clearly as desired.

Requirements and Techniques

In order to meet (2.1), we meet, for each $e \in \omega$ and $i < 2$, the requirement

$$P_{e,i} : \{e\}^C \neq B_i$$

using the standard Friedberg-Muchnik technique.

To meet (2.2), we meet, for each $e = \langle e_0, e_1 \rangle$ in ω, the requirement

$$N_e : \{e_0\}^{B_0 \oplus C} = \{e_1\}^{B_1 \oplus C} = f, \text{total} \rightarrow f \leq_T C.$$

We meet these requirements with a strategy as in([10]). This strategy will be explained shortly.

We meet (2.3) by permitting. This means that when we meet the requirements $P_{e,i}$ by the Friedberg-Muchnik technique, we need to appoint followers and wait for a realized follower to be permitted before we begin an attack on the requirement.

In order to meet (2.4), we use Slaman's idea (introduced in [20]) of creating configurations; this will be discussed in more detail later in this section.

We meet (2.5) by direct coding; that is, for all x, $x \in A$ if and only if $2x + 1 \in B_0 \cup B_1$.

We now discuss the positive meet strategy of [10] in more detail. First we show an easy lemma

Lemma 2 *Let A be an r.e. set with recursive enumeration $\{A_s\}_{s \in \omega}$ and let f be a total function. Then $f \leq_T A$ if and only if there are partial recursive functions $\lambda x, s[\Gamma(x,s)]$, $\lambda x, s[\gamma(x,s)]$ with recursive domains such that*

R1. for all x, s, $\Gamma(x,s)\downarrow$ if and only if $\gamma(x,s)\downarrow$,

R2. for all x, $\lim_s \gamma(x,s)$ exists (by this we mean that $\gamma(x,s)$ converges to a common value for all large enough s), and

R3. for all x and s, if $\Gamma(x,s)\downarrow$ and $\Gamma(x,s) \neq f(x)$, then $A_s \restriction \gamma(x,s) + 1 \neq A \restriction \gamma(x,s) + 1$.

Proof. First, suppose that $f \leq_T A$, say $f = \{e\}^A$. Define $\Gamma(x,s) = \{e\}_s^{A_s}(x)$ and $\gamma(x,s) = u(A_s; e, x, s)$. It is easy to see that R1,R2, and R3 are met.

Conversely, suppose that $\Gamma(x,s), \gamma(x,s)$ satisfy the hypotheses. To compute $f(x)$, given an A oracle, find an s such that $A_s \restriction \gamma(x,s) + 1 = A \restriction \gamma(x,s) + 1$.

By R2, such an s must exist, and since $\gamma(x,s)$ is partial recursive, the s can be found recursively in A. By R1, $\Gamma(x,s)\downarrow$, so by R3, $\Gamma(x,s) = f(x)$. ∎

The method we use to meet the N_e requirements is described abstractly in the following lemma.

Lemma 3 *Let A, B_0, and B_1 be r.e. sets with recursive enumerations $\{A_s\}_{s\in\omega}$, $\{B_{0,s}\}_{s\in\omega}$, $\{B_{1,s}\}_{s\in\omega}$, respectively, and let f be a total function with $\{e_0\}^{B_0} = \{e_1\}^{B_1} = f$. Suppose that $\Gamma(x,s)$, $\gamma(x,s)$ are partial recursive functions with recursive domains such that for all x and s,*

M1. $\Gamma(x,s)\downarrow$ *if and only if* $\gamma(x,s)\downarrow$,

M2. *if* $\Gamma(x,s)\downarrow$ *and* $\Gamma(x,s) = \Gamma(x,s+1)$, *then* $\gamma(x,s) = \gamma(x,s+1)$,

M3. *if* $\Gamma(x,s)\downarrow$ *and* $\neg(\Gamma(x,s) = \Gamma(x,s+1))$, *then* $\gamma(x,s) \in A_{s+1}$,

M4. *if* $\gamma(x,s)\downarrow$, *then* $\gamma(x,s) \notin A_s$,

M5. *if* $\Gamma(x,s)\downarrow$ *and there is an* $i < 2$ *such that* $\Gamma(x,s) = \{e_i\}_s^{B_{i,s}}(x)$, *then* $\Gamma(x,s) = \Gamma(x,s+1)$,

M6. *if* $\Gamma(x,s+1)\downarrow$, *then there is an* $i < 2$ *such that* $\Gamma(x,s+1) = \{e_i\}_s^{B_{i,s}}(x)$, *and*

M7. *there are infinitely many* s *such that if* x *is the least number with* $\Gamma(x,s)\uparrow$, *then* $\Gamma(x,s+1)\downarrow$.

Then $f \leq_T A$.

Proof. We must show that $\Gamma(x,s), \gamma(x,s)$ satisfy R1,R2,R3 from Lemma 2. We have R1 since it is the same as M1. To see that R2 is true, suppose that $\lim_s \gamma(x',s)$ exists for all $x' < x$. Take s_0 such that

$$(\forall x' < x)(\forall s \geq s_0)(\gamma(x',s)\downarrow)$$

and

$$(\forall s \geq s_0)(\forall i < 2)(\{e_i\}_s^{B_{i,s}}(x) = f(x))$$

and take $s > s_0$ as in M7. Then by M1 and choice of s_0, $\Gamma(x,s'+1)\downarrow$ for some $s' \geq s_0$ (either $s' = s-1$ or $s' = s$). By M6 and choice of s_0, $\Gamma(x,s'+1) = f(x)$. By M5 and choice of s_0,

$$(\forall s \geq s'+1)(\Gamma(x,s) = \Gamma(x,s'+1)),$$

so by M2, R2 is met.

To see that R3 is met, suppose that $\Gamma(x,s)\downarrow$ and $\Gamma(x,s) \neq f(x)$. We can't have $\Gamma(x,t) = \Gamma(x,s)$ for all $t \geq s$ since this would contradict M6. If $t+1 > s$ is the least number with $\neg(\Gamma(x,s) = \Gamma(x,t+1))$, then by M3, $\gamma(x,t) \in A_{t+1}$, by

M2, $\gamma(x,t)=\gamma(x,s)$, and by M4, $\gamma(x,t)=\gamma(x,s)\notin A_s$. Thus $A_s\restriction\gamma(x,s)+1\neq A\restriction\gamma(x,s)+1$, so R3 is met. ∎

Our strategy to meet requirement N_e is to build a system of marker positions $\gamma(x,s)$ and values $\Gamma(x,s)$ as in Lemma 3 (with $B_0\oplus C, B_1\oplus C$ in place of B_0, B_1 and C in place of A) if the hypothesis of the requirement is met. When the length of agreement gets bigger than x, a marker $\gamma(x,s)$ is defined and the value $\Gamma(x,s)$ set equal to the common value of $\{e_i\}_s^{B_{i,s}\oplus C_s}(x)$. If at some later point, neither side is giving the same value, then the marker position is put into C and the marker and value are cancelled, to be redefined when agreement comes back.

Thus, the strategy to meet the N_e requirements is an infinitary positive one. This creates problems for the $P_{e,i}$ requirements. When they try to diagonalize, there may be marker positions which might later go into C and destroy the computation against which the diagonalization is made. If these markers are associated with lower priority meet requirements, they can be cancelled, but if a meet requirement has higher priority, its markers must be left. If $\gamma(x,s)$ is such a higher priority marker and both of the computations $\{e'_j\}_s^{B_{j,s}\oplus C_s}(x)$ converge (where the marker is associated with $N_{e'}$ and $e'=\langle e'_0,e'_1\rangle$), then $P_{e,i}$ can make its attack and possibly destroy one of these computations, while putting on restraint to hold the other computation and thereby prevent the meet marker from going in and destroying the attack. However, if only one of these two computations exists, and it is the one which the attack on $P_{e,i}$ could destroy, then the attack may immediately be injured by the marker going into C. To avoid this problem, the construction is put on a tree, with nodes guessing about the lengths of agreements in the $N_{e'}$ requirements in the usual way. Then, when the strategy for $P_{e,i}$ which is on the true path wishes to begin an attack, every marker for a higher priority meet strategy which might act infinitely often is associated with computations on both sides, and so can be held from going into C by $P_{e,i}$ after it makes its attack.

Numbers put into the B_i's by the $P_{e,i}$'s are subject to permitting by A, but numbers (i.e., marker positions) put into C by the N_e's cannot be subject to permitting or the meet strategy will not work. Instead, extra restraint is put on throughout the construction by requirements which attempt to freeze meet markers. This extra restraint creates "permanent configurations", which allow an A oracle to find infinitely many x's with stages s such that $C_s\restriction x = C\restriction x$ and hence $C\leq_T A$.

The Tree and Strategies

Let $T=2^{<\omega}$ and let $\{R_{2n}\}_{n\in\omega}$ be some effective listing of the $P_{e,i}$'s. For each $\alpha\in T$ with $|\alpha|$ even, we will have a strategy R_α which tries to meet $R_{|\alpha|}$ by a Friedberg-Muchnik approach. Each such R_α can have followers. From time to time, an attack can be begun on R_α through one of its followers.

Let T_0 consist of those elements of T which end with a 0. For each $\alpha \in T_0$, there is a strategy N_α which tries to meet $N_{|\alpha|-1}$ with a strategy as in Lemma 3, using markers γ_x^α and values Γ_x^α.

We write $\gamma_x^\alpha(s)$ and $\Gamma_x^\alpha(s)$ for the values of γ_x^α and Γ_x^α, if any, at the end of stage s.

Definition 4 *Let $\alpha \in T_0$, $x, s, u \in \omega$, $i < 2$. Then*

1. *γ_x^α is i-*controllable at stage s by u *if*

   (a) $\gamma_x^\alpha(s)\downarrow$,

   (b) $\{e_i\}_s^{B_{i,s}\oplus C_s}(x) = \Gamma_x^\alpha(s)$, and

   (c) $u(B_{i,s} \oplus C_s; e_i, x, s) \leq u$,

   where $\langle e_0, e_1\rangle = |\alpha| - 1$.

2. *γ_x^α is* controllable at stage s by u *if it is i-controllable at stage s by u for some $i < 2$.*

3. *γ_x^α is i-*controllable at stage s *if it is i-controllable at stage s by u for some u.*

4. *γ_x^α is* controllable at stage s *if it is i-controllable at stage s for some $i < 2$.*

Definition 5 *Let $x, s \in \omega$. Then x* is a configuration at stage s *if*

1. *$x \leq s$, and*

2. *$(\forall \alpha \in T_0)(\forall y)(\gamma_y^\alpha(s)\downarrow < x \rightarrow \gamma_y^\alpha$ is controllable at stage s by $x)$.*

x is a permanent configuration *if x is a configuration at stage s for all but finitely many s.*

For each $\alpha \in T$ with $|\alpha|$ odd, R_α is a strategy which tries to build a permanent configuration.

For each $\alpha \in T$, there are numbers k_0^α, k_1^α which can be defined, cancelled, and redefined throughout the construction. We write $k_i^\alpha(s)$ for the value, if any, of k_i^α at the end of stage s. The values of k_i^α, $i < 2$, are restraints protecting B_i. If $|\alpha|$ is even and $R_{|\alpha|}$ is $P_{e,i}$, then $k_i^\alpha(s)\uparrow$ for all s and k_{1-i}^α will be set to help preserve computations diagonalized against. If $|\alpha|$ is odd, R_α will try to make $\max(k_0^\alpha, k_1^\alpha)$ a permanent configuration.

Definitions

For $e, s \in \omega$, we define

$$\ell(e,s) = \max\{x : (\forall y < x)(\{\widehat{e_0}\}_s^{B_{0,s}\oplus C_s}(y) = \{\widehat{e_1}\}_s^{B_{1,s}\oplus C_s}(y))\}.$$

For each $s \in \omega$, we define a string δ_s in 2^s in the standard way using ℓ; namely, if $e < s$ and $\delta_s \restriction e$ is defined, we define:

$$\delta_s(e) = 0 \leftrightarrow \ell(e,s) > \ell(e,t) \text{ for all } t < s \text{ such that } \delta_t \restriction e = \delta_s \restriction e.$$

To *initialize* R_α means to cancel k_0^α, k_1^α and (if $|\alpha|$ is even) all followers of R_α.
To *initialize* N_α means to cancel all $\gamma_x^\alpha, \Gamma_x^\alpha$ (if $\alpha \in T_0$).
R_α is *satisfied at stage* s if for some $t \leq s$, an attack is begun on R_α at stage t and for no v with $t \leq v \leq s$ is R_α initialized at stage v.
R_α *requires attention at stage* $s+1$ if

A) $|\alpha|$ is even, R_α is not satisfied at stage s and either

  A1) $\alpha \leq \delta_s$ and there is a follower x of R_α at the end of stage s such that

    A1.1) x is not the largest follower of R_α at the end of stage s, and

    A1.2) $a_s < x$,

    or

  A2) $\alpha \subseteq \delta_s$ and for every follower x of R_α at the end of stage s, $\{e\}_s^{C_s}(x) = 0$ where $R_{|\alpha|}$ is $P_{e,i}$,

  or

B) $|\alpha|$ is odd, $\alpha \subseteq \delta_s$ and either

  B1) $(\exists \alpha' \in T_0)(\exists x)(\alpha' <_L \alpha \wedge \gamma_x^{\alpha'}$ is controllable at stage $s\ \wedge$ $\neg(\exists i < 2)(k_i^\alpha(s)\downarrow \wedge\ \gamma_x^{\alpha'}$ is i-controllable at stage s by $k_i^\alpha(s)))$, or

  B2) $\neg(\exists i)(k_i^\alpha(s)\downarrow)$.

Construction

Stage 0: Do nothing.
Stage $s+1$: The stage consists of 4 steps.

Step 1 (Action for R_α's): Find the least α such that R_α requires attention at stage $s+1$. Then R_α receives attention at stage $s+1$. Initialize all $R_{\alpha'}$, $N_{\alpha'}$ with $\alpha < \alpha'$. Distinguish the following two cases according to the length of α.

Case 1: $|\alpha|$ is even, say $R_{|\alpha|} = P_{e,i}$. Then if (A1) holds, put the least x satisfying (A1.1),(A1.2) into B_i. If (A1) does not hold (so (A2) holds), then appoint $2(s+1)$ to be a follower of R_α and set k_{1-i}^α to $s+1$.

Case 2: $|\alpha|$ is odd. Then, if (B1) holds, proceed as follows: Take the least α' and then the least x satisfying (B1). Take $i < 2$ minimal so that $\gamma_x^{\alpha'}$ is i-controllable at stage s and set k_i^α to be $s+1$. If $i = 0$, cancel k_1^α.

If (B1) does not hold (so (B2) holds), then set k_0^α to be $s+1$.

Step 2 (Splitting): Find the least α, if any, such that for some $i < 2$, k_i^α is defined and $2a_s + 1$ is less than the current value of k_i^α. If no such α exists, put $2a_s + 1$ into B_0. Otherwise, take the least such α and initialize all $R_{\alpha'}$ with $\alpha < \alpha'$. If k_0^α is defined and $2a_s + 1$ is less than the current value of k_0^α, then put $2a_s + 1$ into B_1 and cancel k_1^α. Otherwise, put $2a_s + 1$ into B_0.

Step 3 (Cancel Meet Markers): For each $\alpha \in T_0$ and x such that $\gamma_x^\alpha(s)\downarrow$, γ_x^α was not cancelled at Step 1 and γ_x^α is not controllable at stage s, put $\gamma_x^\alpha(s)$ into C, cancel γ_x^α and Γ_x^α and initialize all $R_{\alpha'}$ with $\alpha <_L \alpha'$.

Step 4 (Assign Meet Markers): For each $e < s$ such that $\delta_s(e) = 0$, find the least x, if any, such that $x < \ell(e,s)$ and $\gamma_x^{\delta_s\upharpoonright e+1}(s)\uparrow$, and define $\gamma_x^{\delta_s\upharpoonright e+1}(s+1) = \langle e, s+1\rangle$ and $\Gamma_x^{\delta_s\upharpoonright e+1}(s+1) = \{e_0\}_s^{B_{0,s}\oplus C_s}(x)$. ∎

Note that at the end of stage s, there are only finitely many α which have followers, hence only finitely many α require attention at stage $s+1$ via (A1), so Step 1 can be carried out effectively.

Correctness Proof

For each n, let μ_n be the least $\alpha \in 2^n$ such that $\alpha \subseteq \delta_s$ for infinitely many s. By a standard (and purely combinatorial) proof, $\mu_n \subseteq \mu_{n+1}$ for every n.

We show the correctness of the construction by a sequence of lemmas. The first lemma states that for each n, there is only finitely much activity to the left of μ_n throughout the construction.

Lemma 6 $(\forall n)(\exists t)(\forall s \geq t)(\forall \alpha <_L \mu_n)(R_\alpha$ *does not receive attention at stage* $s \wedge (\forall x)(\gamma_x^\alpha$ *is not cancelled at stage* $s))$.

Proof. Take s_0 such that $(\forall s \geq s_0)(\mu_n \leq \delta_s)$. Then no α with $\alpha <_L \mu_n$ can be assigned a follower after stage s_0, so only finitely many such α are ever assigned followers. If an α with $\alpha <_L \mu_n$ receives attention at a stage $s \geq s_0$, then an attack is begun on R_α at stage s. If R_α is not initialized at any stage greater than or equal to s, then R_α is satisfied at all stages greater than or equal to s. If R_α is initialized at some stage greater than or equal to s, then R_α never requires attention after it is initialized, since R_α is not assigned any followers after stage s_0. Thus each R_α with $\alpha <_L \mu_n$ receives attention only finitely often and all but finitely many such α never receive attention.

If γ_x^α is assigned a value at stage $s+1$, then $\alpha \subseteq \delta_s$, and only finitely many markers are assigned values at any given stage, so there are only finitely many α, x with $\alpha <_L \mu_n$ such that $(\exists s)(\gamma_x^\alpha(s)\downarrow)$. Thus, there are only finitely many stages s such that for some x and $\alpha <_L \mu_n$, γ_x^α is cancelled at stage s. ∎

Lemma 7 $A \leq_T B_0 \oplus B_1$.

Proof. We have $x \in A \leftrightarrow 2x+1 \in B_0 \cup B_1$.

Lemma 8 $B_0, B_1 \leq_T A$.

Proof. By permitting, since if a number x enters B_0 or B_1 at stage s, then a number less than x enters A at stage s. ∎

The next lemma states that at stage s, along δ_s, all meet markers are controllable on both sides.

Lemma 9 *$(\forall\alpha)(\forall s)((\alpha \subseteq \delta_s \wedge \gamma_x^\alpha(s)\downarrow) \rightarrow (\forall i < 2)(\gamma_x^\alpha$ is i-controllable at stage $s))$.*

Proof. Suppose that $\alpha \subseteq \delta_s$, $\gamma_x^\alpha(s)\downarrow$, and $e = |\alpha| - 1$. Let $t+1$ be the greatest stage $\leq s$ such that γ_x^α was appointed a value at stage $t+1$. Then, by construction, $\alpha \subseteq \delta_t$. Also, $\delta_s(e) = \alpha(e) = 0$, so, by definition of δ_s, $x < \ell(e,t) < \ell(e,s)$. Thus, $\{\widehat{e_0}\}_s^{B_{0,s}\oplus C_s}(x) = \{\widehat{e_1}\}_s^{B_{1,s}\oplus C_s}(x)$. By the "hat trick", $\{e_i\}_{s-1}^{B_{i,s-1}\oplus C_{s-1}}(x) = \{e_i\}_s^{B_{i,s}\oplus C_s}(x)$ for $i = 0,1$. If $t+1 = s$, then $\Gamma_x^\alpha(s) = \{e_0\}_{s-1}^{B_{0,s-1}\oplus C_{s-1}}(x) = \{e_0\}_s^{B_{0,s}\oplus C_s}(x) = \{e_1\}_s^{B_{1,s}\oplus C_s}(x)$. If $t+1 < s$, then, since Γ_x^α was not cancelled at stage s, for some $i < 2$, $\Gamma_x^\alpha(s) = \Gamma_x^\alpha(s-1) = \{e_i\}_{s-1}^{B_{i,s-1}\oplus C_{s-1}}(x) = \{e_i\}_s^{B_{i,s}\oplus C_s}(x) = \{e_{1-i}\}_s^{B_{1-i,s}\oplus C_s}(x)$. In either case, Γ_x^α is both 0-controllable and 1-controllable at stage s. ∎

Lemma 10 *If γ_x^α is i-controllable at stage s by u, $t \geq s$, and*

$$\begin{aligned} B_{i,t}\restriction u &= B_{i,s}\restriction u, \\ C_t\restriction u &= C_s\restriction u, \end{aligned}$$

then γ_x^α is i-controllable at stage t by u.

Proof. By induction on t' with $s \leq t' \leq t$, we show that γ_x^α is i-controllable at stage t' by u. The basis is immediate. Suppose that $s \leq t' < t$ and the result holds for t'. Since γ_x^α is i-controllable at stage t' by u, it is controllable at stage t', so γ_x^α, Γ_x^α are not cancelled at stage $t'+1$ and, by hypothesis, the computation existing at the end of stage t' which makes γ_x^α i-controllable at stage t' by u is not destroyed at stage $t'+1$, so γ_x^α is i-controllable at stage $t'+1$ by u. ∎

The next lemma says that the numbers k_i^α really do act as restraints.

Lemma 11 *Suppose that for some α and i, k_i^α is assigned a value at stage $s+1$ and is never later cancelled. Then*

$$\begin{aligned} B_{i,s}\restriction s+1 &= B_i\restriction s+1, \\ C_s\restriction s+1 &= C\restriction s+1. \end{aligned}$$

Proof. Suppose that k_i^α is assigned a value at stage $s+1$ and is never later cancelled, but that the conclusion does not hold, say that y is the first number $< s+1$ which enters B_i or C at a stage $> s$, say at stage $t+1$. We rule out all the possibilities.

- y is a follower of some $R_{\alpha'}$.
  - $\alpha < \alpha'$: This is impossible, since, at step 1 of stage $s+1$, all followers of $R_{\alpha'}$ are cancelled and any followers assigned later are $\geq s+1$.
  - $\alpha' < \alpha$: This is impossible, since then k_i^α would be cancelled at stage $t+1$.
  - $\alpha' = \alpha$: If $|\alpha|$ is odd, then R_α has no followers. If $|\alpha|$ is even, then followers of R_α go into B_{1-i}.
- $y = 2a_t + 1$: Since k_i^α is not cancelled after stage s, k_i^α is defined at step 2 of stage $t+1$ and is $\geq s+1 > y = 2a_t + 1$. (Any value assigned to k_i^α after stage $s+1$ will be $> s+1$.) If $2a_t + 1 < k_{i'}^{\alpha'}$ for any $\alpha' < \alpha$ and i' at step 2 of stage $t+1$ or if $i = 1$ and k_0^α has a value $> 2a_t + 1$, then k_i^α is cancelled at stage $t+1$, contradicting assumption. Thus, $2a_t + 1$ is put into B_{1-i}, not B_i.
- $y = \gamma_x^{\alpha'}(t)$ for some x.
  - $\alpha < \alpha'$: This is impossible, since $N_{\alpha'}$ is initialized at step 1 of stage $s+1$ and any values assigned to markers after stage s are $\geq s+1$.
  - $\alpha' <_L \alpha$: This is impossible, since then k_i^α would be cancelled at stage $t+1$.
  - $\alpha' \subseteq \alpha$: Since $y = \gamma_x^{\alpha'}(t) \leq s$, $\gamma_x^{\alpha'}$ must have been given this value before stage $s+1$, so $\gamma_x^{\alpha'}(t) = \gamma_x^{\alpha'}(s)$. Since $\alpha' \subseteq \alpha \subseteq \delta_s$, by Lemma 9, $\gamma_x^{\alpha'}$ is i-controllable at stage s, and, by convention, the use must be $\leq s$. By choice of y, $B_{i,s} \restriction s+1 = B_{i,t} \restriction s+1$ and $C_s \restriction s+1 = C_t \restriction s+1$, so, by Lemma 10, $\gamma_x^{\alpha'}$ is $(i$-)controllable at stage t (by s), so $y = \gamma_x^{\alpha'}(t)$ is not put into C at stage $t+1$.

∎

Lemma 12 $(\forall n)(R_n$ *requires attention only finitely often*$)$.

Proof. Let $n \geq 0$ and assume that the result is true for all $n' < n$. Since a value can be assigned to k_i^α only at stages at which R_α receives attention, we can let M be the maximum value assigned to any k_i^α with $\alpha < \mu_n$, and, by Lemma 6 and inductive hypothesis, we can take s_0 so that:

- $(\forall s > s_0)(\forall \alpha < \mu_n)(R_\alpha$ does not receive attention at stage $s)$;
- $(\forall s > s_0)(\forall \alpha <_L \mu_n)(\forall x)(\gamma_x^\alpha$ is not cancelled at stage $s)$;
- $(\forall s > s_0)(\mu_n \leq \delta_s)$;
- $A_{s_0} \restriction M = A \restriction M$.

Then, at no stage $> s_0$ is R_{μ_n} initialized and if R_{μ_n} requires attention at a stage $> s_0$, then R_{μ_n} receives attention at this stage.

First, suppose that n is even. Suppose that R_{μ_n} requires attention infinitely often. Then R_{μ_n} cannot be satisfied at stage s for any $s > s_0$, else R_{μ_n} would remain satisfied throughout the construction and never require attention again. Thus, every time that R_{μ_n} requires attention at a stage $t > s_0$, $2t$ is appointed a follower of R_{μ_n} and $2t$ will never be cancelled.

Let $s_0 < t_0 < t_1 < \cdots$ be all the stages $> s_0$ at which R_{μ_n} requires attention. Then, we claim that for all j, $A_{t_{j+1}} \restriction 2t_j = A \restriction 2t_j$. Suppose not, say for some j, $a_s < 2t_j$, for some $s \geq t_{j+1}$. Then

- R_{μ_n} is not satisfied at stage s.
- $\mu_n \leq \delta_s$.
- $2t_j$ is a follower of R_{μ_n} at the end of stage s.
- $2t_j$ is not the largest follower of R_{μ_n} at the end of stage s (since $2t_{j+1} > 2t_j$ is also a follower).
- $a_s < 2t_j$.

Thus, R_{μ_n} requires attention through (A1) at stage $s+1$, so an attack is begun and R_{μ_n} is satisfied at stage $s+1$, a contradiction. Thus $A_{t_{j+1}} \restriction 2t_j = A \restriction 2t_j$. Since $\{t_j\}$ is a recursive sequence, A is recursive, a contradiction. Thus R_{μ_n} requires attention only finitely often.

Now, suppose that n is odd. At each stage $s+1 > s_0$ at which R_{μ_n} requires attention, either $k_0^{\mu_n}$ or $k_1^{\mu_n}$ is assigned a value. We first show that $k_0^{\mu_n}$ is assigned a value only finitely often, then that $k_1^{\mu_n}$ is assigned a value only finitely often, and, hence, that R_{μ_n} receives attention only finitely often.

For no $s > s_0$ is $k_0^{\mu_n}$ cancelled at stage s. Thus, $k_0^{\mu_n}$ can be assigned a value via (B2) only once after stage s_0. If $k_0^{\mu_n}$ is assigned a value via (B1) at stage $s+1 > s_0$, then $k_0^{\mu_n}(s+1) = s+1$ and $(\forall s' \geq s+1)(k_0^{\mu_n}(s') \downarrow \geq s+1)$. By Lemma 11, $C_s \restriction s+1 = C \restriction s+1$ and $B_{0,s} \restriction s+1 = B_0 \restriction s+1$. Let α' and then x be minimal satisfying (B1) at stage s. Then $\gamma_x^{\alpha'}$ is 0-controllable at stage s by s, so, for all $s' > s$, $k_0^{\mu_n}(s') \downarrow$ and, by Lemma 10, $\gamma_x^{\alpha'}$ is 0-controllable at stage s' by $s < k_0^{\mu_n}(s+1) \leq k_0^{\mu_n}(s')$. Thus, if $s+1$ and $t+1$ are two stages $> s_0$ such that $k_0^{\mu_n}$ is assigned a value via (B2) at stages $s+1$ and $t+1$, then we cannot have the corresponding α', x be the same at both stages. Since there are only finitely many α', x with $\alpha' <_L \mu_n$ such that $(\exists s)(\gamma_x^{\alpha'}(s) \downarrow)$, there can be only finitely many stages $> s_0$ such that a value is assigned to $k_0^{\mu_n}$ via (B1) at the stage.

Let M' be the maximum value assigned to $k_0^{\mu_n}$ and take s_0' so that:

- $s_0' \geq s_0$;
- $(\forall s \geq s_0')(k_0^{\mu_n}$ is not assigned a value at stage $s)$;

- $A_{s_0'} \upharpoonright M' = A \upharpoonright M'$.

Then, at no stage $s > s_0'$ is $k_1^{\mu_n}$ cancelled. An argument similar to the one just given for $k_0^{\mu_n}$ shows that $k_1^{\mu_n}$ is assigned a value at only finitely many stages. ∎

Lemma 13 $(\forall n)(R_{2n}$ *is met*$)$.

Proof. Suppose that R_{2n} is $P_{e,i}$. We consider two cases.

Case 1: $R_{\mu_{2n}}$ is satisfied at stage s for infinitely many s.

Then, since by Lemma 12 only finitely many attacks are ever made on $R_{\mu_{2n}}$, there must be a stage $t+1$ such that an attack is made on $R_{\mu_{2n}}$ at stage $t+1$, say through x, and $R_{\mu_{2n}}$ is not initialized at any stage $\geq t+1$. By (A1.1), x is not the largest follower of $R_{\mu_{2n}}$ at the end of stage t, say $s < t$ and a follower $y > x$ is appointed to $R_{\mu_{2n}}$ at stage $s+1$. Since followers are appointed in increasing order, x must have been a follower of $R_{\mu_{2n}}$ at the end of stage s, so, by (A2), $\{e\}_s^{C_s}(x) = 0$. At stage $s+1$, $k_{1-i}^{\mu_{2n}}$ is set to $s+1$. If $R_{\mu_{2n}}$ were initialized at any stage v with $s+1 \leq v \leq t$, then the attack through x would not take place at stage $t+1$ and we have already noted that $R_{\mu_{2n}}$ is not initialized at any stage $\geq t+1$. Also, $k_i^{\mu_{2n}}$ is never assigned a value. Thus, $k_{1-i}^{\mu_{2n}}$ is never cancelled after stage s, so, by Lemma 11, $C_s \upharpoonright s+1 = C \upharpoonright s+1$. Thus, $\{e\}^C(x) = 0$. But $x \in B_{i,t+1}$, so $B_i \neq \{e\}^C$.

Case 2: $R_{\mu_{2n}}$ is satisfied at stage s for only finitely many s.

By Lemma 12, only finitely many followers are ever assigned to $R_{\mu_{2n}}$. If $\{e\}^C(x) = 0$ for each follower of $R_{\mu_{2n}}$ which is never cancelled, then $R_{\mu_{2n}}$ would require attention infinitely often (via (A2)), contradicting Lemma 12. Thus, there is a follower x of $R_{\mu_{2n}}$ such that x is never cancelled and $\neg(\{e\}^C(x) = 0)$. If $x \in B_i$, then, since x cannot be a follower of any other R_α, Case 1 would hold. Thus, $x \notin B_i$ and $B_i \neq \{e\}^C$. ∎

Lemma 14 $(\forall e)(N_e$ *is met*$)$.

Proof. Let $e = \langle e_0, e_1 \rangle$, and suppose that $\{e_0\}^{B_0 \oplus C} = \{e_1\}^{B_1 \oplus C} = f$, total. Then, by a standard argument, $\mu_{e+1}(e) = 0$, so there is a strategy $N_{\mu_{e+1}}$ to meet N_e. We will use Lemma 3, with $B_i \oplus C$ $(B_{i,s} \oplus C_s)$ in place of B_i $(B_{i,s})$ and C (C_s) in place of A (A_s), to show that $f \leq_T C$.

Take s_0 such that $N_{\mu_{e+1}}$ is not initialized at any stage $s \geq s_0$. (Such an s_0 exists by Lemma 12.) Let $\gamma(x,s) = \gamma_x^{\mu_{e+1}}(s_0+s)$ and $\Gamma(x,s) = \Gamma_x^{\mu_{e+1}}(s_0+s)$. If $\lambda x, s[\gamma(x,s)]$ and $\lambda x, s[\Gamma(x,s)]$ meet conditions (M1)-(M7) of Lemma 3, then, by that lemma, we are done.

Property (M4) follows from the fact that the only numbers which go into C are meet markers and each number can be the position of at most one meet marker throughout the construction.

To see that (M7) holds, note that there are infinitely many $s \geq s_0$ such that $\mu_{e+1} \subseteq \delta_s$ and $\ell(e,s) > 0$. For each such s, $\ell(e,s) > \ell(e,t)$ for all $t < s$ such that $\mu_e \subseteq \delta_s$, and hence $\gamma^{\mu_{e+1}}_{\ell(e,s)-1}(s)\uparrow$. Thus, for such s, if we let $s' = s - s_0$ and x is the least number with $\Gamma(x, s')\uparrow$, then $x < \ell(e, s)$, and hence $\Gamma(x, s' + 1)\downarrow$.

The remaining properties follow easily from the construction. ∎

All that remains is to show that $C \leq_T A$. To do this, we must show that there are infinitely many permanent configurations and that they can be recognized by A.

Lemma 15 *If x is a configuration at stage s and $A_s \upharpoonright x = A \upharpoonright x$, then:*

1. *$B_{i,s} \upharpoonright x = B_i \upharpoonright x$, for $i = 0, 1$;*
2. *$C_s \upharpoonright x = C \upharpoonright x$; and*
3. *x is a configuration at stage t for all $t \geq s$.*

Proof. Suppose that x is a configuration at stage s and $A_s \upharpoonright x = A \upharpoonright x$. Then (1) is immediate, since the B_i's are constructed by permitting on A. To show (2) and (3), we show by induction on t that for all $t \geq s$, $C_t \upharpoonright x = C_s \upharpoonright x$ and x is a configuration at stage t. The basis is immediate. Suppose that $t \geq s$, $C_t \upharpoonright x = C_s \upharpoonright x$, and x is a configuration at stage t. If a number $y < x$ enters C at stage $t + 1$, then $y = \gamma^\alpha_z(t)$ for some α and z. Since x is a configuration at stage t, γ^α_z is controllable at stage t, so y does not enter C at stage $t+1$. Thus, $C_{t+1} \upharpoonright x = C_t \upharpoonright x = C_s \upharpoonright x$.

We also have $x \leq t + 1$. If for some α, y, $\gamma^\alpha_y(t + 1)\downarrow < x \leq s$, then we must have $\gamma^\alpha_y(s) \downarrow = \gamma^\alpha_y(t + 1) < x$. Since x is a configuration at stage s, $(\exists i_0 < 2)(\gamma^\alpha_y$ is i_0-controllable at stage s by x). Since $B_{i_0,s} \upharpoonright x = B_{i_0,t+1} \upharpoonright x$ and $C_s \upharpoonright x = C_{t+1} \upharpoonright x$, by Lemma 10, γ^α_y is (i_0-)controllable at stage $t+1$ by x. Thus, x is a configuration at stage $t + 1$, completing the induction. ∎

Lemma 16 *There are infinitely many permanent configurations.*

Proof. Fix $n \geq 0$. We claim that there is an s such that for some $i < 2$, $k^{\mu_{2n+1}}_i$ is assigned a value at stage $s + 1$ and is never later cancelled. Suppose not. By Lemma 12, $R_{\mu_{2n+1}}$ receives attention only finitely often, so for each $i < 2$, either $k^{\mu_{2n+1}}_i$ is never defined or the final value assigned to it is later cancelled. But then, $R_{\mu_{2n+1}}$ requires attention infinitely often (via (B2)), contradicting Lemma 12.

Thus, since $R_{\mu_{2n+1}}$ receives attention only finitely often, we may take s to be the greatest stage such that some $k^{\mu_{2n+1}}_i$ is assigned a value at stage $s + 1$ and is never later cancelled, say $k^{\mu_{2n+1}}_{i_0}$ is assigned the value $s+1$ at stage $s+1$. Take t_0 so that:

(2.6) $t_0 > s$;

(2.7) $\mu_{2n+1} \subseteq \delta_{t_0}$;

(2.8) $(\forall s' > t_0)(\forall i < 2)(k^{\mu_{2n+1}}_i$ is not cancelled at stage $s')$;

(2.9) $R_{\mu_{2n+1}}$ does not require attention at stage t_0+1; and

(2.10) $A_{t_0}\restriction s+1 = A\restriction s+1$.

We claim that $s+1$ is a configuration at stage t_0. We have $s+1 \leq t_0$. Suppose that $\gamma_y^\alpha(t_0)\downarrow < s+1$. We consider cases.

- $\mu_{2n+1} < \alpha$: This is impossible since at step 1 of stage $s+1$ all N_α with $\mu_{2n+1} < \alpha$ are initialized and any values assigned to γ_y^α later are bigger than s.

- $\alpha <_L \mu_{2n+1}$: Then, since $R_{\mu_{2n+1}}$ does not require attention at stage t_0+1 but $\mu_{2n+1} \subseteq \delta_{t_0}$, either γ_y^α is not controllable at stage t_0 or $(\exists i' < 2)(k_{i'}^{\mu_{2n+1}}(t_0)\downarrow \wedge \gamma_y^\alpha$ is controllable at stage t_0 by $k_{i'}^{\mu_{2n+1}}(t_0))$. In the former case, $R_{\mu_{2n+1}}$ would be initialized at stage $t_0+1 \geq s+1$, contradicting choice of s. In the latter case, since, by (2.8), $k_{i'}^{\mu_{2n+1}}$ is not cancelled at any stage $> t_0$, by choice of s, $k_{i'}^{\mu_{2n+1}}$ was given before stage $s+2$ the value it has at the end of stage t_0 and hence, γ_y^α is controllable at stage t_0 by $k_{i'}^{\mu_{2n+1}}(t_0) \leq s+1$.

- $\alpha \subseteq \mu_{2n+1}$: Since $\gamma_y^\alpha(t_0) < s+1$, γ_y^α must have been appointed this value before stage $s+1$, so $\gamma_y^\alpha(s)\downarrow = \gamma_y^\alpha(t_0)$. Since $\alpha \subseteq \mu_{2n+1} \subseteq \delta_s$, by Lemma 9, γ_y^α is i_0-controllable at stage s, so is i_0-controllable at stage s by s. But, by Lemma 11, $B_{i_0,s}\restriction s+1 = B_{i_0}\restriction s+1$, $C_s\restriction s+1 = C\restriction s+1$, so, by Lemma 10, γ_y^α is controllable by $s \leq s+1$ at stage t_0.

Thus, $s+1$ is a configuration at stage t_0.

By (2.10) and Lemma 15, $s+1$ is a permanent configuration. For different values of n, the corresponding stages s will be different, so there are infinitely many permanent configurations. ∎

Lemma 17 *$C \leq_T A$.*

Proof. Let $T = \{(x,s) : x$ is a configuration at stage s and $A_s\restriction x = A\restriction x\}$. Then $T \leq_T A$. By Lemma 16, there are infinitely many x such that $(\exists s)((x,s) \in T)$. By Lemma 15, if $(x,s) \in T$, then $C_s\restriction x = C\restriction x$. Thus, $C \leq_T A$. ∎

Now, by Lemma 13, the $P_{e,i}$ requirements are all met, so (2.1) holds; by Lemma 14, the N_e requirements are met, so (2.2) holds; and Lemmas 8, 17, and 7 show that (2.3), (2.4) and (2.5) hold. This completes the proof of the Lachlan Splitting Theorem.

3 The Main Result

In this section, we give the proof of our main result. Our presentation parallels that of the previous section whenever possible.

Theorem 18 *If* $\mathbf{a}$ *is a nonzero r.e. degree, then every countable distributive lattice can be embedded into* $\mathcal{R}(\leq \mathbf{a})$ *preserving 1.*

Since every countable distributive lattice can be (lattice) embedded into the countable atomless Boolean algebra preserving top element, it suffices to show that for every nonzero r.e. $\mathbf{a}$, the countable atomless Boolean algebra can be embedded into $\mathcal{R}(\leq \mathbf{a})$ preserving 1.

Let $\Phi = \{\eta_n\}_{n\in\omega}$ be a uniformly recursive collection of subsets of ω closed under union, intersection, and complement, that forms a countable atomless Boolean algebra under these operations and such that $\eta_n = \omega$ if and only if $n = 0$. Let A be an r.e. set of a given nonzero r.e. degree $\mathbf{a}$ with effective enumeration $(a_0, a_1, \ldots)$ and let $A_s = \{a_0, \ldots, a_{s-1}\}$.

We will construct a uniformly r.e. sequence $\{B_i\}_{i\in\omega}$ of disjoint sets and an r.e. set C. For each $\gamma \in \Phi$, we define $B_\gamma = \bigcup_{i\in\gamma} B_i$ and we map Φ into $\mathcal{R}(\leq \mathbf{a})$ by sending γ to $\deg_T(B_\gamma \oplus C)$. We will construct these sets so that for all $\gamma, \delta \in \Phi$ and all i we have:

(3.1) $B_i \not\leq_T C$,

(3.2) for all sets E, if $E \leq_T B_\gamma \oplus C, B_\delta \oplus C$, then $E \leq_T B_{\gamma\cap\delta} \oplus C$,

(3.3) $B_\omega \leq_T A$,

(3.4) $C \leq_T A$, and

(3.5) $A \leq_T B_\omega$.

We show that if conditions (3.1) through (3.5) are met, then the mapping is the desired embedding. First note that since the B_i's are disjoint and uniformly r.e. and each element of Φ is recursive, we have for all $\gamma, \delta \in \Phi$:

(3.6) $\gamma \subseteq \delta \to B_\gamma \oplus C \leq_T B_\delta \oplus C$.

Fixing now $\gamma, \delta \in \Phi$, we get from (3.6):

(3.7) $\deg_T(B_\gamma \oplus C) \cup \deg_T(B_\delta \oplus C) \leq \deg_T(B_{\gamma\cup\delta} \oplus C)$,

(3.8) $\deg_T(B_{\gamma\cap\delta} \oplus C) \leq \deg_T(B_\gamma \oplus C), \deg_T(B_\delta \oplus C)$.

We also have $B_{\gamma\cup\delta} = B_\gamma \cup B_\delta$ and from this it is easy to see that

(3.9) $\deg_T(B_{\gamma\cup\delta} \oplus C) \leq \deg_T(B_\gamma \oplus C) \cup \deg_T(B_\delta \oplus C)$.

It follows from (3.7) and (3.9) that

$$\deg_T(B_{\gamma\cup\delta} \oplus C) = \deg_T(B_\gamma \oplus C) \cup \deg_T(B_\delta \oplus C),$$

which shows that the mapping preserves joins. Condition (3.2), together with (3.8) gives

$$\deg_T(B_{\gamma\cap\delta} \oplus C) = \deg_T(B_\gamma \oplus C) \cap \deg_T(B_\delta \oplus C),$$

which ensures that meets are preserved as well.

To see that the mapping is an injection, suppose that $\gamma, \delta \in \Phi$ with $\gamma \neq \delta$, wlog $i \in \gamma - \delta$. We must show that $B_\gamma \oplus C \not\leq_T B_\delta \oplus C$. Suppose to the contrary that $B_\gamma \oplus C \leq_T B_\delta \oplus C$. Then, using (3.6), we get, since $i \in \gamma$, $B_i \leq_T B_\gamma \leq_T B_\gamma \oplus C \leq_T B_\delta \oplus C$ and similarly, since $i \in \bar{\delta}$, $B_i \leq_T B_{\bar{\delta}} \oplus C$. But $\deg_T(B_\delta \oplus C) \cap \deg_T(B_{\bar{\delta}} \oplus C) = \deg_T(B_{\delta\cap\bar{\delta}} \oplus C)$, so $B_i \leq_T B_{\delta\cap\bar{\delta}} \oplus C = \emptyset \oplus C \equiv_T C$, contradicting condition (3.1).

Finally, conditions (3.3), (3.4) and (3.5) ensure that $B_\omega \oplus C \equiv_T A$, so the embedding preserves 1.

Requirements and Techniques

In order to meet (3.1), we meet as before, for each $e \in \omega$ and $i < 2$, the requirement:

$$P_{e,i} : \{e\}^C \neq B_i$$

using the Friedberg-Muchnik technique. In order to preserve the computation against which it has diagonalized, $P_{e,i}$ restrains not just C, but all B_j with $j \neq i$ as well.

To meet (3.2), for each n_0, n_1, e_0, e_1 in ω, we meet the requirement:

$$N_{(n_0,n_1,e_0,e_1)} :$$
$$\{e_0\}^{B_{\eta_{n_0}} \oplus C} = \{e_1\}^{B_{\eta_{n_1}} \oplus C} = f \text{ total} \rightarrow f \leq_T B_{\eta_{n_0}\cap\eta_{n_1}} \oplus C.$$

These requirements are met as before, with a system of markers and values. Meet markers for requirement $N_{(n_0,n_1,e_0,e_1)}$ need not be put into C, but can also be put into any B_i with $i \in \eta_{n_0} \cap \eta_{n_1}$. Requirement $N_{(n_0,n_1,e_0,e_1)}$ chooses the place to put its meet markers so as to respect, if possible, the restraint of the highest priority $P_{e',i'}$ whose restraint it might injure.

Unlike in the Lachlan Splitting Theorem construction, even after it is no longer initialized, a given $P_{e,i}$ requirement may not be able to hold all meet markers of higher priority requirements. This is because B_i might be in the infimum of one of these higher priority requirements, and hence, when $P_{e,i}$ diagonalizes and puts an element into B_i, this could simultaneously destroy both sides of an agreement for the meet requirement, causing that requirement to put a marker into its infimum. However, in such a case, the meet requirement will respect $P_{e,i}$'s restraint, and put the marker into B_i. This action may require further meet markers to go in, but they will also go into B_i, and $P_{e,i}$ still successfully restrains C and all B_j with $j \neq i$, thereby preserving the computation it diagonalizes against.

To meet (3.3) and (3.4), we again use permitting for followers of $P_{e,i}$ requirements and configurations to allow A to recognize meet markers which will never go into the B_i's or C. The action which a single configuration producing strategy takes is similar to that in the Lachlan Splitting Theorem proof, although there are now infinitely many potential restraints for such a strategy (only finitely many of which ever get assigned a value), each trying to freeze markers by directing numbers into one fixed B_i. The analysis that these strategies succeed, however, is made more complicated by the fact that by directing numbers into a given B_i, a configuration producing requirement does not necessarily freeze a meet marker which is controllable on both sides, since B_i could be in the infimum. Yet, if a B_i change causes a meet marker to be uncontrollable, then the marker becomes undefined, and no longer of concern to the requirement. To make this part of the argument formally, we are led to the definition of "B_i-destroyable", given later.

We meet (3.5) by direct coding.

The Tree and Strategies

Since $\eta_0 = \omega$, if either $n_0 = 0$ or $n_1 = 0$, then $N_{(n_0,n_1,e_0,e_1)}$ is automatically met. Let $T = 2^{<\omega}$, let $\{R_{2n}\}_{n\in\omega}$ be some effective listing of the $P_{e,i}$'s, and let $\{N_n\}_{n\in\omega}$ be an effective listing of the $N_{(n_0,n_1,e_0,e_1)}$'s with $n_0, n_1 \neq 0$. For each $\alpha \in T$ with $|\alpha|$ even, there is a Friedberg-Muchnik type strategy R_α to meet $R_{|\alpha|}$ and for each $\alpha \in T_0$ (the elements of T which end with a 0), there is a strategy N_α which tries to meet $N_{|\alpha|-1}$, using markers γ_x^α and values Γ_x^α.

Definitions 4 and 5 apply here as well, except that the first part of Definition 4 is replaced by:

Definition 19 *Let $\alpha \in T_0$, $x, s, u \in \omega$, $i < 2$. Then γ_x^α is i-*controllable *at stage s by u if*

(a) $\gamma_x^\alpha(s)\downarrow$,

(b) $\{\widehat{e_i}\}_s^{B_{\eta_{n_i},s}\oplus C_s} = \Gamma_x^\alpha(s)$, *and*

(c) $u(B_{\eta_{n_i},s} \oplus C_s; e_i, x, s) \leq u$,

where $N_{|\alpha|-1} = N_{(n_0,n_1,e_0,e_1)}$.

Note the use of $\widehat{\ }$ in Part (b) of this definition, which is crucial for the proof of the second part of Lemma 25.

For each α with $|\alpha|$ odd, R_α is a strategy which tries to build a permanent configuration.

For each $i \in \omega$, let $\widehat{B_i} = \bigcup_{j\neq i} B_j$. For each $\alpha \in T$, strategy R_α can define numbers k_i^α, $i \in \omega$, which can be cancelled and redefined. In contrast with the previous section, where k_i^α, $i < 2$, was a restaint to preserve B_i, here k_i^α, $i \in \omega$, will be a restraint of priority α to preserve $\widehat{B_i}$ by directing numbers into B_i. If $|\alpha|$ is even and $R_{|\alpha|}$ is $P_{e,i}$, then $k_j^\alpha(s) \uparrow$ for all s if $j \neq i$, and k_i^α is set to preserve a computation against which R_α diagonalizes. If $|\alpha|$ is odd, then values of k_i^α are set to freeze meet markers and create configurations.

Definitions

Definition 20 *Let $\alpha \in T_0$, $x, s, u, i \in \omega$.*

1. *γ_x^α is $\widehat{B_i}$-*controllable at stage s by u *if for some $j < 2$, γ_x^α is j-controllable at stage s by u and $i \notin \eta_{n_j}$, where $N_{|\alpha|-1} = N_{(n_0,n_1,e_0,e_1)}$.*

2. *γ_x^α is $\widehat{B_i}$-*controllable at stage s *if it is $\widehat{B_i}$-controllable at stage s by u for some u.*

3. *γ_x^α is B_i-*destroyable *if $i \in \eta_{n_0} \cap \eta_{n_1}$ (where $N_{|\alpha|-1} = N_{(n_0,n_1,e_0,e_1)}$).*

The definitions of the Definitions subsection of the previous section carry over here with the following changes:

- For $e, s \in \omega$, we define
$$\ell(e,s) = \max\{x : (\forall y < x)(\{\widehat{e_0}\}_s^{B_{\eta_{n_0},s} \oplus C_s}(y) = \{\widehat{e_1}\}_s^{B_{\eta_{n_1},s} \oplus C_s}(y))\},$$
where $N_e = N_{(n_0,n_1,e_0,e_1)}$.

- To initialize R_α means to cancel all k_i^α and (if $|\alpha|$ is even) all followers of R_α.

- In the definition of "R_α requires attention at stage $s+1$", we replace (B1) with

  B1$'$) $(\exists \alpha' \in T_0)(\exists x)(\alpha' <_L \alpha \wedge \gamma_x^{\alpha'}$ is controllable at stage s $\wedge$
  $\neg(\exists i)(k_i^\alpha(s) \downarrow \wedge\, \gamma_x^{\alpha'}$ is $\widehat{B_i}$-controllable at stage s by $k_i^\alpha(s)))$.

Construction

The construction follows the format of the one given in the previous section.
Stage 0: Do nothing.
Stage $s+1$: The stage consists of 4 steps.

Step 1 (Action for R_α's): Find the least α such that R_α requires attention at stage $s+1$. Then R_α receives attention at stage $s+1$. Initialize all $R_{\alpha'}$, $N_{\alpha'}$ with $\alpha < \alpha'$. Distinguish the following two cases according to the length of α.

Case 1: $|\alpha|$ is even, say $R_{|\alpha|} = P_{e,i}$. Then if (A1) holds, put the least x satisfying (A1.1),(A1.2) into B_i. If (A1) does not hold (so (A2) holds), then appoint $3(s+1)$ to be a follower of R_α and set k_i^α to $s+1$.

Case 2: $|\alpha|$ is odd. Then, if (B1$'$) holds, proceed as follows: Take the least α' and then the least x satisfying (B1$'$). Take $i < 2$ minimal so that $\gamma_x^{\alpha'}$ is i-controllable at stage s and then take i_0 minimal so that $i_0 \notin \eta_{n_i}$, where $N_{|\alpha'|-1} = N_{(n_0,n_1,e_0,e_1)}$. ($i_0$ exists since $\eta_{n_0}, \eta_{n_1} \neq \omega$.) Set $k_{i_0}^\alpha$ to be $s+1$ and cancel all $k_{i'}^\alpha$ with $i_0 < i'$.

If (B1$'$) does not hold (so (B2) holds), then set k_0^α to be $s+1$.

Step 2 (Splitting): Find the least α, if any, such that for some i, k_i^α is defined and $3a_s + 1$ is less than the current value of k_i^α. If no such α exists, put $3a_s + 1$ into B_0. Otherwise, take the least such α and initialize all $R_{\alpha'}$ with $\alpha < \alpha'$. Let i_0 be minimal so that $3a_s + 1 < k_{i_0}^\alpha$. Put $3a_s + 1$ into B_{i_0} and cancel all k_i^α with $i_0 < i$.

Step 3 (Cancel Meet Markers): For each $\alpha \in T_0$ and x such that $\gamma_x^\alpha(s)\downarrow$, γ_x^α was not cancelled at Step 1 and γ_x^α is not controllable at stage s, proceed as follows: Initialize all $R_{\alpha'}$ with $\alpha <_L \alpha'$ and cancel γ_x^α and Γ_x^α. Find the least β (if any) such that there is an i with $k_i^\beta(s)\downarrow$ and $\gamma_x^\alpha(s) < k_i^\beta(s)$. If β does not exist, put $\gamma_x^\alpha(s)$ into C. Otherwise, take the least i_0 with $k_{i_0}^\beta(s)\downarrow$ and $\gamma_x^\alpha(s) < k_{i_0}^\beta(s)$, initialize all $R_{\alpha'}$ with $\beta < \alpha'$ and cancel all k_j^β with $j > i_0$. If $i_0 \in \eta_{n_0} \cap \eta_{n_1}$, where $N_{|\alpha|-1} = N_{(n_0,n_1,e_0,e_1)}$, then put $\gamma_x^\alpha(s)$ into B_{i_0}. Otherwise, put $\gamma_x^\alpha(s)$ into C.

Step 4 (Assign Meet Markers): For each $e < s$ such that $\delta_s(e) = 0$, find the least x, if any, such that $x < \ell(e,s)$ and $\gamma_x^{\delta_s \upharpoonright e+1}(s)\uparrow$, and define $\gamma_x^{\delta_s \upharpoonright e+1}(s+1) = 3\langle e, s+1\rangle + 2$ and $\Gamma_x^{\delta_s \upharpoonright e+1}(s+1) = \{e_0\}_s^{B_{\eta_{n_0},s} \oplus C_s}(x)$ where $N_e = N_{(n_0,n_1,e_0,e_1)}$. ∎

Correctness Proof

Let μ_n be defined as in the previous section.

Lemma 21 *$(\forall n)(\exists t)(\forall s \geq t)(\forall \alpha <_L \mu_n)(R_\alpha$ does not receive attention at stage $s \wedge (\forall x)(\gamma_x^\alpha$ is not cancelled at stage $s))$.*

Proof. The proof is the same as that of Lemma 6. ∎

Lemma 22 $A \leq_T B_\omega = \bigcup_{i\in\omega} B_i$.

Proof. We have $x \in A \leftrightarrow 3x + 1 \in B_\omega$. ∎

Lemma 23 *$\{B_i\}_{i\in\omega}$ is a disjoint family of sets.*

Proof. Each number $3x$ is assigned as a follower to at most one R_α, so can only be in one B_i. Each number $3x+1$ is put into no B_i if $x \notin A$ and into only one B_i if $x \in A$. Each number $3x+2$ is assigned to be γ_x^α for at most one α and x and is assigned only once, so can be put into at most one B_i. ∎

Lemma 24 *$(\forall \alpha)(\forall s)((\alpha \subseteq \delta_s \wedge \gamma_x^\alpha(s)\downarrow) \rightarrow (\forall i < 2)(\gamma_x^\alpha$ is i-controllable at stage s)).*

Proof. The proof is essentially the same as that of Lemma 9, with some slight changes due to the different form of the N_e requirements. ∎

Lemma 25 *Suppose that for some numbers i, u, s, t with $t \geq s$, we have*

$$\begin{aligned} B_{j,s}\restriction u &= B_{j,t}\restriction u, \text{ for all } j \neq i, \\ C_s\restriction u &= C_t\restriction u. \end{aligned}$$

Then,

1. *if γ_x^α is $\widehat{B_i}$-controllable at stage s by u, then γ_x^α is $\widehat{B_i}$-controllable at stage t by u; and*
2. *if γ_x^α is controllable at stage s by u and is B_i-destroyable, $\gamma_x^\alpha(t)\downarrow\leq s$, and γ_x^α is controllable at stage t, then γ_x^α is controllable at stage t by u.*

Proof. The proof of Part (1) is essentially the same as the proof of Lemma 10. To prove Part (2), suppose that γ_x^α is controllable at stage s by u and is B_i-destroyable. We show by induction on t', $s \leq t' \leq t$, that if $\gamma_x^\alpha(t')\downarrow\leq s$ and γ_x^α is controllable at stage t', then γ_x^α is controllable at stage t' by u. The basis is immediate. Suppose that $s \leq t' < t$ and the result holds for t'. Suppose also that $\gamma_x^\alpha(t'+1)\downarrow\leq s$. Then, since $s < t'+1$, we must have $\gamma_x^\alpha(t'+1) = \gamma_x^\alpha(t')$, so γ_x^α must be controllable at stage t' and by inductive hypothesis, γ_x^α is controllable at stage t' by u. Thus, if $N_{|\alpha|-1} = N_{(n_0,n_1,e_0,e_1)}$, there is a $j < 2$ such that γ_x^α is j-controllable at stage t' by u. If this computation is not destroyed at stage $t'+1$, then γ_x^α is still controllable at stage $t'+1$ by u. If the computation is destroyed, then $\{\widehat{e_j}\}_{t'+1}^{B_{\eta_{n_j},t'+1}\oplus C_{t'+1}}(x)\uparrow$ and a number z with $2z < u$ enters B_i at stage $t'+1$ (since by hypothesis, no number $< u$ enters C or any B_j with $j \neq i$ after stage s and before stage $t+1$). Since γ_x^α is controllable at stage $t'+1$, we must have $\{\widehat{e_{1-j}}\}_{t'+1}^{B_{\eta_{n_{1-j}},t'+1}\oplus C_{t'+1}}(x) = \Gamma_x^\alpha(t'+1)$, and, since $i \in \eta_{n_{1-j}}$, the use in the computation must be $< 2z < u$, so γ_x^α is $(1-j)$-controllable at stage $t'+1$ by u. ∎

Lemma 26 *Suppose that for some α and i, k_i^α is assigned a value at stage $s+1$ and is never later cancelled. Then,*

$$\begin{aligned} B_{j,s}\restriction s+1 &= B_j\restriction s+1, \text{ for all } j \neq i, \\ C_s\restriction s+1 &= C\restriction s+1. \end{aligned}$$

Proof. Suppose that k_i^α is assigned a value at stage $s+1$ and is never later cancelled, but that the desired conclusion does not hold, say that y is the first number $< s+1$ which enters $C \cup \bigcup_{j\neq i} B_j$ at a stage $> s$, say at stage $t+1$. We rule out all possibilities.

- y is a follower of some $R_{\alpha'}$: This is ruled out as in Lemma 11. (Note that if $|\alpha|$ is even, then followers of R_α go into B_i.)

- $y = 3c_t + 1$: By an argument similar to that of Lemma 11, if $y < k_{i'}^{\alpha'}$ at step 2 of stage $t+1$ for some α', i' with $\alpha' < \alpha$ or with $\alpha = \alpha'$ and $i' < i$, then k_i^α would be cancelled, so such a y must go into B_i and not a B_j with $j \neq i$.

- $y = \gamma_x^{\alpha'}(t)$ for some x: The cases $\alpha < \alpha'$ and $\alpha' <_L \alpha$ are ruled out as in Lemma 11. If $\alpha' \subseteq \alpha$, then, since $\gamma_x^{\alpha'}(t) = y \leq s$, $\gamma_x^{\alpha'}$ must have been given this value before stage $s+1$, so $\gamma_x^{\alpha'}(t) = \gamma_x^{\alpha'}(s)$. Since $\alpha' \subseteq \alpha \subseteq \delta_s$, by Lemma 24, $\gamma_x^{\alpha'}$ is both 0- and 1-controllable at stage s (by s), but $\gamma_x^{\alpha'}$ is not controllable at stage t, so $t > s$. If $\gamma_x^{\alpha'}$ were $\widehat{B_i}$-controllable at stage s, then it would be $\widehat{B_i}$-controllable at stage s by s. But, by assumption, no number $< s+1$ enters $\widehat{B_i} \cup C$ after stage s and before stage $t+1$, so, by Lemma 25, Part 1, $\gamma_x^{\alpha'}$ would still be controllable at stage t. Thus, $\gamma_x^{\alpha'}$ is not $\widehat{B_i}$-controllable at stage s. Since $\gamma_x^{\alpha'}$ is both 0- and 1-controllable at stage s by s, it must be that if $N_{|\alpha'|-1} = N_{(n_0,n_1,e_0,e_1)}$, then $i \in \eta_{n_0} \cap \eta_{n_1}$. Since $t > s$, $k_i^\alpha(t) \downarrow\geq s+1 > y = \gamma_x^{\alpha'}(t)$. If for some β and j with $\beta < \alpha$, $k_j^\beta(t) \downarrow> \gamma_x^{\alpha'}(t)$ or if for some $j < i$, $k_j^\alpha(t) \downarrow> \gamma_x^{\alpha'}(t)$, then k_i^α would be cancelled at stage $t+1$, contradicting assumption. Thus, $y = \gamma_x^{\alpha'}(t)$ is put into B_i at stage $t+1$ and not into $\widehat{B_i} \cup C$.

∎

Lemma 27 $(\forall n)(R_n$ *requires attention only finitely often*$)$.

Proof. Let $n \geq 0$ and assume that the result is true for all $n' < n$. Since a value can be assigned to k_i^α only at stages at which R_α receives attention, we can let M be the maximum value assigned to any k_i^α with $\alpha < \mu_n$, and, by Lemma 21 and inductive hypothesis, we can take s_0 so that:

- $(\forall s > s_0)(\forall \alpha < \mu_n)(R_\alpha$ does not receive attention at stage $s)$;
- $(\forall s > s_0)(\forall \alpha <_L \mu_n)(\forall x)(\gamma_x^\alpha$ is not cancelled at stage $s)$;
- $(\forall s > s_0)(\mu_n \leq \delta_s)$;
- $C_{s_0}\restriction M = C\restriction M$.
- $(\forall j)(B_{j,s_0}\restriction M = B_j\restriction M)$.

(It is possible to meet the last condition since the B_j's are disjoint.) Then, at no stage $> s_0$ is R_{μ_n} initialized and if R_{μ_n} requires attention at a stage $> s_0$, then R_{μ_n} receives attention at this stage.

If n is even, then the proof that R_{μ_n} requires attention only finitely often is as in Lemma 12, with $3t_j$ in place of $2t_j$.

Now suppose that n is odd. Each time that R_{μ_n} receives attention, some $k_i^{\mu_n}$ is assigned a value. Let

$$F_1 = \{\alpha \in T_0 : \alpha <_L \mu_n \wedge (\exists x)(\exists s)(\gamma_x^\alpha(s)\downarrow)\}.$$

Then F_1 is finite. Let

$$\begin{aligned} F_2 \quad = \quad & \{i : (\exists \alpha \in T_0)(\exists j < 2)(\exists n_0, n_1, e_0, e_1)(\alpha \in F_1 \\ & \wedge N_{|\alpha|-1} = N_{(n_0,n_1,e_0,e_1)} \wedge i = \min\{i' : i' \notin \eta_{n_j}\}\} \cup \{0\}. \end{aligned}$$

Then F_2 is finite and, if $(\exists s)(k_i^{\mu_n}(s)\downarrow)$, then $i \in F_2$. Thus, it suffices to show that for each i, $k_i^{\mu_n}$ is assigned a value at only finitely many stages. Suppose that $i \geq 0$ and that the result is true for all $i' < i$. Let M' be the maximum value which is ever assigned to any $k_{i'}^\alpha$ with $i' < i$ and take s_0' so that

- $s_0' \geq s_0$;
- $(\forall s > s_0')(\forall i' < i)(k_{i'}^{\mu_n}$ is not assigned a value at stage $s)$;
- $C_{s_0'} \restriction M' = C \restriction M'$;
- $(\forall j)(B_{j,s_0'} \restriction M' = B_j \restriction M')$.

Then at no stage $s > s_0'$ is $k_i^{\mu_n}$ cancelled. Thus, there can be only one stage $> s_0'$ at which $k_i^{\mu_n}$ is assigned a value via (B2) (and this one stage can exist only if $i = 0$). Suppose that $k_i^{\mu_n}$ is assigned a value via (B1$'$) at stage $s+1 > s_0'$. Then, $k_i^{\mu_n}(s+1) = s+1$ and $k_i^{\mu_n}$ is never cancelled after stage $s+1$, so, for all $s' \geq s+1$, $k_i^{\mu_n}(s')\downarrow \geq s+1$. By Lemma 26, $C_s \restriction s+1 = C \restriction s+1$ and $(\forall j \neq i)(B_{j,s} \restriction s+1 = B_j \restriction s+1)$. Let α' and then x be minimal to satisfy (B1$'$) at stage s. Then, by construction, $\gamma_x^{\alpha'}$ is $\widehat{B_i}$-controllable at stage s by s. By Lemma 25, for all $s' > s$, $k_i^{\mu_n}(s')\downarrow$ and $\gamma_x^{\alpha'}$ is $\widehat{B_i}$-controllable at stage s' by $s < k_i^{\mu_n}(s+1) \leq k_i^{\mu_n}(s')$. Thus, if $s+1$ and $t+1$ are two stages $\geq s_0'$ such that $k_i^{\mu_n}$ is assigned a value via (B1$'$) at stages $s+1$ and $t+1$, then we cannot have the corresponding α', x be the same at both stages. Since there are only finitely many α', x with $\alpha' <_L \mu_n$ such that $(\exists s)(\gamma_x^{\alpha'}(s)\downarrow)$, there can be only finitely many stages $> s_0'$ such that a value is assigned to $k_i^{\mu_n}$ via (B1$'$) at the stage, and hence there are only finitely many stages at which a value is assigned to $k_i^{\mu_n}$, as desired. ∎

Lemma 28 $(\forall n)(R_{2n}$ *is met*$)$.

Proof. The proof is essentially the same as that of Lemma 13, with $k_{1-i}^{\mu_{2n}}$ replaced by $k_i^{\mu_{2n}}$ and $k_i^{\mu_{2n}}$ replaced by all $k_j^{\mu_{2n}}$ with $j \neq i$. ∎

Lemma 29 $(\forall e)(N_e$ *is met*$)$.

Proof. The proof is the same as that of Lemma 14. ∎

Lemma 30 *If x is a configuration at stage s and $A_s \restriction x = A \restriction x$, then:*

1. *$B_{i,s} \restriction x = B_i \restriction x$, for all i;*
2. *$C_s \restriction x = C \restriction x$; and*
3. *x is a configuration at stage t for all $t \geq s$.*

Proof. Suppose that x is a configuration at stage s and $A_s \restriction x = A \restriction x$. Numbers of the forms $3y$ and $3y+1$ enter the B_i's via permitting on A, so no such number $< x$ can enter one of the B_i's after stage s. We show by induction on $t \geq s$ that for all i, $B_{i,t} \restriction x = B_{i,s} \restriction x$, $C_t \restriction x = C_s \restriction x$, and x is a configuration at stage t. The basis is immediate. Suppose that $t \geq s$ and the result holds for t. If a number $y < x$ enters C or one of the B_i's at stage $t+1$, then, as argued above, $y = \gamma_z^\alpha(t)$ for some α, z. But, since x is a configuration at stage t, such a γ_z^α is controllable at stage t, so y does not enter C or one of the B_i's at stage $t+1$, so $C_{t+1} \restriction x = C_t \restriction x = C_s \restriction x$ and for all i, $B_{i,t+1} \restriction x = B_{i,t} \restriction x = B_{i,s} \restriction x$.

We also have $x \leq t+1$. If, for some α, y, $\gamma_y^\alpha(t+1) \downarrow < x \leq s$, then we must have $\gamma_y^\alpha(s) \downarrow = \gamma_y^\alpha(t+1) < x$. Since x is a configuration at stage s, $(\exists j < 2)(\gamma_y^\alpha$ is j-controllable at stage s by $x)$, so, for some i, γ_y^α is $\widehat{B_i}$-controllable at stage s by x. Since $B_{j,s} \restriction x = B_{j,t+1} \restriction x$ for all j and $C_s \restriction x = C_{t+1} \restriction x$, by Lemma 25, γ_y^α is $(\widehat{B_i}$-) controllable at stage $t+1$ by x. Thus, x is a configuration at stage $t+1$, completing the induction. ∎

Lemma 31 *There are infinitely many permanent configurations.*

Proof. Fix $n \geq 0$. Arguing as in the proof of Lemma 16, we may take s to be the greatest stage such that some $k_i^{\mu_{2n+1}}$ is assigned a value at stage $s+1$ and is never later cancelled, say $k_{i_0}^{\mu_{2n+1}}$ is assigned the value $s+1$ at stage $s+1$. Take t_0 to meet the five conditions of Lemma 16 (with "$\forall i < 2$" replaced by "$\forall i$" in the third condition) and to satisfy the additional condition:

$$(\forall \alpha)(\forall x)(\gamma_x^\alpha(t_0) \downarrow < s+1 \to \gamma_x^\alpha \text{ is controllable at stage } t_0).$$

(Note that if a marker γ_x^α is not controllable at stage t, then the marker is cancelled at stage $t+1$. Thus, since there are only finitely many markers ever assigned to values $< s+1$, the condition given above holds for all large enough t_0.)

We claim that $s+1$ is a configuration at stage t_0. We have $s+1 \leq t_0$. Suppose that $\gamma_y^\alpha(t_0) \downarrow < s+1$. The cases $\mu_{2n+1} < \alpha$ and $\alpha <_L \mu_{2n+1}$ are handled much as in the proof of Lemma 16. We handle the case $\alpha \subseteq \mu_{2n+1}$ as follows: Let $N_{|\alpha|-1} = N_{(n_0,n_1,e_0,e_1)}$. Since $\gamma_y^\alpha(t_0) \downarrow < s+1$, γ_y^α must have been appointed to this value before stage $s+1$, so $\gamma_y^\alpha(s) \downarrow = \gamma_y^\alpha(t_0)$. Since $\alpha \subseteq \mu_{2n+1} \subseteq \delta_s$, by Lemma 24, γ_y^α is both 0 and 1-controllable at stage s. By Lemma 26, $(\forall j \neq i_0)(B_{j,s} \restriction s+1 = B_j \restriction s+1)$ and $C_s \restriction s+1 = C \restriction s+1$. We consider two subcases:

- $i \notin \eta_{n_0} \cap \eta_{n_1}$: Then, γ_y^α is $\widehat{B_i}$-controllable at stage s by s, so, by Lemma 25, Part 1, γ_y^α is ($\widehat{B_i}$-) controllable at stage t_0 by $s \leq s+1$.
- $i \in \eta_{n_0} \cap \eta_{n_1}$: Then γ_y^α is controllable at stage s by s and is B_i-destroyable. Also, $\gamma_y^\alpha(t_0) \downarrow \leq s$ and γ_y^α is controllable at stage t_0 (by the assumption on t_0 given above), so, by Lemma 25, Part 2, γ_y^α is controllable at stage t_0 by $s \leq s+1$.

Thus, $s+1$ is a configuration at stage t_0. By Lemma 30 and our assumption that $A_{t_0} \restriction s+1 = A \restriction s+1$, $s+1$ is a permanent configuration. For different values of n, the corresponding stage s will be different, so there are infinitely permanent configurations. ∎

Lemma 32 *$C \leq_T A$, $B_\omega \leq_T A$.*

Proof. Let $T = \{(x,s) : x$ is a configuration at stage s and $A_s \restriction x = A_x\}$. Then, $T \leq_T A$. By Lemma 31, there are infinitely many x such that $(\exists s)((x,s) \in T)$. By Lemma 30, if $(x,s) \in T$, then $C_s \restriction x = C \restriction x$ and $B_{\omega,s} \restriction x = B_\omega \restriction x$. Thus, $C, B_\omega \leq_T A$. ∎

Now, by Lemma 28, the $P_{e,i}$ requirements are all met, so (3.1) holds; by Lemma 29, the N_e requirements are met, so (3.2) holds; and Lemmas 32 and 22 show that (3.3), (3.4) and (3.5) hold. This completes the proof of our main result.

4 Related Results and Open Problems

Our result, together with other recent and older results, allows us to provide a solution to all questions of the form:

> Which finite distributive lattices can be embedded into A preserving B?

where A and B are as specified in the introduction. In fact, for the more general question:

> Which distributive lattices can be embedded into A preserving B?

we can give the answer for all combinations of A and B except one. The answers are specified in the following chart, where A is specified by the row labels (with the word nontrivial omitted in the last three labels) and B is specified by the column labels (with preservation of meets and joins understood)

	—	0	1	0 and 1
$\mathcal{R}$	every	every	every	?
every initial segment	every	M	every	MN
every final segment	every	M	J	JMN
every interval	every	M	J	JMN

The key to this table is:

every	=	every countable distributive lattice,
M	=	every countable distributive lattice which does not have a meet reducible least element,
MN	=	every lattice in M which has at least two elements,
J	=	every countable distributive lattice which does not have a join reducible greatest element,
JMN	=	every lattice in both J and MN.

For the case of lattice embeddings into $\mathcal{R}$ preserving both 0 and 1, we have the result of Ambos-Spies, Lempp and Lerman [4] that a finite distributive lattice is embeddable into $\mathcal{R}$ preserving 0 and 1 if and only if it has more than one element and contains a join-irreducible, noncappable element; a characterization of all the distributive lattices so embeddable is not known.

We justify the entries in the table, first in the positive direction, by showing that every lattice in the given class can be embedded in the given way, then negatively, by showing that no other distributive lattices can be so embedded. Work of both Downey [9] and Ambos-Spies, Lempp and Soare shows that every countable distributive lattice can be embedded into every nontrivial interval of $\mathcal{R}$. This justifies the first column of the table. (The fact that every countable distributive lattice can be embedded into every nontrivial initial segment of $\mathcal{R}$ was shown earlier by Ambos-Spies [2], and the first entry in the first column follows from the next fact as well.)

Lachlan and, independently, Lerman showed that every countable distributive lattice can be embedded into $\mathcal{R}$ preserving 0. (See Soare [21, Theorem IX.2.1].) (The corresponding result for finite distributive lattices was shown earlier by Thomason [22].) Our main result justifies the two "every"'s in the third column. (As mentioned in the introduction, Ambos-Spies [1] earlier showed that every countable distributive lattice can be embedded into $\mathcal{R}$ preserving 1.)

This justifies each "every" entry in the table. Suppose that a lattice $\mathcal{L}$ belongs to the class M and we are given a nontrivial interval I of $\mathcal{R}$. We show that $\mathcal{L}$ can be embedded into I preserving 0. As we just discussed, $\mathcal{L}$ can be embedded into I. If $\mathcal{L}$ has no least element, we are done. If $\mathcal{L}$ has a least element, we can redefine the embedding so that it carries this least element to the bottom of the interval; since $\mathcal{L}$ belongs to M, the modified map is still a lattice embedding and it preserves least element. This provides the positive part of the justification for the remaining entries in the second column. The positive parts of the justifications of the entries in the last two columns is similar.

We turn now to the negative parts of the justifications. Lachlan [12] has shown the existence of a nonbranching degree, i.e., an r.e. degree $\mathbf{a} \neq \mathbf{0}'$ such that $\mathbf{a}$ is meet-irreducible. If a distributive lattice $\mathcal{L}$ can be embedded into every nontrivial final segment of $\mathcal{R}$ preserving 0, then $\mathcal{L}$ must be countable and $\mathcal{L}$ can be embedded into $\mathcal{R}(\geq \mathbf{a})$ preserving least element, so $\mathcal{L}$ cannot have a meet-reducible least element. Thus, $\mathcal{L} \in M$.

Lachlan [15] has also shown that there is a nonzero r.e. degree **a** such that **a** does not bound any minimal pairs, i.e., in $\mathcal{R}(\leq \mathbf{a})$, $\mathbf{0}$ is meet-irreducible. It follows, by an argument similar to that of the previous paragraph, that if a distributive lattice can be embedded into every nontrivial initial segment of $\mathcal{R}$ preserving least element, then the lattice belongs to M.

By the arguments of either of the last two paragraphs, if a distributive lattice $\mathcal{L}$ can be embedded into every nontrivial interval of $\mathcal{R}$ preserving 0, then $\mathcal{L} \in M$. This completes the justification of the second column of the table.

For the third column, we use the result of Harrington that there is an r.e. degree $\mathbf{a} \neq \mathbf{0}'$ such that $\mathbf{0}'$ does not split over **a** (i.e., $\mathbf{0}'$ is join-irreducible in $\mathcal{R}(\geq \mathbf{a})$). (Harrington's original proof is unpublished. A proof of his result can be found in [11] where Jockusch and Shore show how the result can be obtained from one in [14].) It follows that if a distributive lattice $\mathcal{L}$ is embeddable into all nontrivial final segments (or all nontrivial intervals) of $\mathcal{R}$, then $\mathcal{L} \in J$. (The result for arbitrary intervals follows also from Lachlan's result [14] that there are r.e. $\mathbf{a} < \mathbf{b}$ such that **b** does not split over **a**.)

The negative part of the justification for the fourth column of the table follows from the previous columns and the fact that if a lattice can be embedded into a nontrivial interval of $\mathcal{R}$ preserving 0 and 1, then it must have at least two elements.

The obvious generalization of the type of questions we have been discussing is to drop the word distributive and ask about embeddings of arbitrary lattices. Here, knowledge is fragmentary and there are many open problems. Even the most basic question, the so-called Embedding Question, of which finite lattices can be embedded into $\mathcal{R}$, is unsettled. Lachlan [13] has shown that M_5 and N_5, the two five element nondistributive lattices, are embeddable into $\mathcal{R}$ preserving least element, while Lachlan and Soare [17] showed that a lattice S_8 is not embeddable into $\mathcal{R}$. Ambos-Spies and Lerman ([6] and [7]) have a general condition under which a lattice is embeddable into $\mathcal{R}$ and another condition under which a lattice is not embeddable. It is not known whether the two conditions are complementary.

One might hope that by asking which lattices can be embedded into all nontrivial initial segments, final segments, or intervals, possibly preserving 0, 1, or both, one might get a more tractable version of the Embedding Question, but this has not proven to be the case so far. What is known is that N_5 ([1]) and M_5 ([5]) are embeddable into $\mathcal{R}$ preserving 1, that N_5 is embeddable into every nontrivial interval of $\mathcal{R}$ ([3], using [20]) and that there is a nontrivial initial segment of $\mathcal{R}$ into which M_5 cannot be embedded ([8]). (So the class of lattices which can be embedded into all nontrivial initial segments of $\mathcal{R}$ is strictly smaller than the class of lattices which can be embedded into $\mathcal{R}$. In fact, since a modular lattice is nondistributive if and only if it contains M_5 as a sublattice, a modular lattice is embeddable into every nontrivial initial segment of $\mathcal{R}$ if and only if it is a countable distributive lattice.) Using a result in [11], it follows from the result of Downey just mentioned that there is a nontrivial final segment of $\mathcal{R}$ into which M_5 cannot be embedded and hence that a modular lattice is embeddable into every nontrivial final segment of $\mathcal{R}$ if and only if it is countable and distributive.

There are connections between the different problems that arise in the way we have been considering. For instance, because of the density of $\mathcal{R}$, a lattice $\mathcal{L}$ is embeddable into every nontrivial initial segment of $\mathcal{R}$ if and only if $\mathcal{L}_{+1}$ is embeddable into every nontrivial initial segment of $\mathcal{R}$ preserving 1. Thus, the generalization to arbitrary lattices of the problem solved in this paper for distributive lattices is at least as hard as the question of which lattices are embeddable into every nontrivial initial segment of $\mathcal{R}$. Study of this latter question was initiated for nondistributive lattices by Downey in [8]. Although he made good progress, the question looks hard.

For embeddings into arbitrary nontrivial intervals of $\mathcal{R}$, the four problems one gets by insisting or not insisting on the preservation of 0 and 1 are all reducible to each other. This is shown by the following chain of reductions for $\mathcal{L}$ an arbitrary lattice, which holds because of the density of $\mathcal{R}$, the existence of nonbranching degrees, and the nonsplitting result of Lachlan:

- $\mathcal{L}$ is embeddable into every nontrivial interval of $\mathcal{R}$ preserving both 0 and 1 if and only if $\mathcal{L} \in MN$ and $\mathcal{L}_{-0}$ can be embedded into every nontrivial interval of $\mathcal{R}$ preserving 1.

- $\mathcal{L}$ is embeddable into every nontrivial interval of $\mathcal{R}$ preserving 1 if and only if $\mathcal{L} \in J$ and $\mathcal{L}_{+0}$ is embeddable into every nontrivial interval of $\mathcal{R}$ preserving 0.

- $\mathcal{L}$ is embeddable into every nontrivial interval of $\mathcal{R}$ preserving 0 if and only if $\mathcal{L} \in M$ and if $|\mathcal{L}| > 1$, then $\mathcal{L}_{-0}$ is embeddable into every nontrivial interval of $\mathcal{R}$.

- $\mathcal{L}$ is embeddable into every nontrivial interval of $\mathcal{R}$ if and only if $\mathcal{L}_{+0+1}$ is embeddable into every nontrivial interval of $\mathcal{R}$ preserving 0 and 1.

Thus, a solution to any one of these problems gives a solution for all the others.

We close by describing what is known about necessary conditions if we wish to generalize the result of this paper by determining which lattices can be embedded into every nontrivial initial segment of $\mathcal{R}$ preserving 1. If $\mathbf{a}$ is contiguous (i.e., $\mathbf{a}$ is r.e. and contains a single r.e. weak truth-table degree), then the r.e. degrees are "distributive at $\mathbf{a}$", that is, if for some r.e. $\mathbf{b}, \mathbf{c}, \mathbf{d}$, $\mathbf{a} = \mathbf{b} \cup \mathbf{c}$ and $\mathbf{d} \leq \mathbf{a}$, then there are r.e. $\mathbf{b}_0, \mathbf{c}_0$ with $\mathbf{b}_0 \leq \mathbf{b}$, $\mathbf{c}_0 \leq \mathbf{c}$ and $\mathbf{d} = \mathbf{b}_0 \cup \mathbf{c}_0$. (See [2, Corollary 1.5].) (Thus, if $\mathbf{a}$ is contiguous, then M_5 and N_5 cannot be embedded into $\mathcal{R}(\leq \mathbf{a})$ preserving 1.) The existence of nonzero contiguous degrees is shown in [18]. Also, Downey [8] has shown that there is an r.e. degree $\mathbf{a}$ such that $\mathcal{R}(\leq \mathbf{a})$ contains no critical triple, i.e., no degrees $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ with $\mathbf{b} \cup \mathbf{c} = \mathbf{b} \cup \mathbf{d}$, $\mathbf{c} \not\leq \mathbf{b}$ and $\mathbf{c} \cap \mathbf{d} \leq \mathbf{b}$. Thus, if $\mathcal{L}$ can be embedded into every nontrivial initial segment of $\mathcal{R}$ preserving 1, then $\mathcal{L}$ must be distributive at the top and can contain no critical triples. How close these necessary conditions are to being sufficient is unclear.

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