1. (a) $\text{REJECT}_{TM}$ is defined as $\{\langle M, w \rangle | M$ is a Turing machine, and $M$ rejects $w\}$. Prove that $\text{REJECT}_{TM}$ is Turing recognizable.

**Solution:** $\text{REJECT}_{TM}$ is recognized by the following Turing machine $V$:

$V =$ “On input $\langle M, w \rangle$

1. Run $M$ on $w$.
2. If $M$ rejects, accept. If $M$ accepts, reject.”

(b) Show that $\text{REJECT}_{TM}$ is undecidable using diagonalization. Your proof should be similar to, but not the same as, the proof that $A_{TM}$ is undecidable.

**Solution:** We assume that $\text{REJECT}_{TM}$ is decidable and obtain a contradiction. Suppose that the Turing machine $H$ decides $\text{REJECT}_{TM}$.

This means that

$$H(\langle M, w \rangle) = \begin{cases} 
\text{reject} & \text{if } M \text{ accepts } w \\
\text{accept} & \text{if } M \text{ rejects } w \\
\text{reject} & \text{if } M \text{ loops on } w
\end{cases}$$

Using $H$, we define another Turing machine $D$

$D =$ “On input $\langle M \rangle$ where $M$ is a Turing machine

1. Run $H$ on $\langle M, \langle M \rangle \rangle$.
2. If $H$ accepts, accept. If $H$ rejects, reject.”

We have

$$D(\langle M \rangle) = \begin{cases} 
\text{reject} & \text{if } M \text{ accepts } \langle M \rangle \\
\text{accept} & \text{if } M \text{ rejects } \langle M \rangle \\
\text{reject} & \text{if } M \text{ loops on } \langle M \rangle
\end{cases}$$

Applying this to $M = D$, we get

$$D(\langle D \rangle) = \begin{cases} 
\text{reject} & \text{if } D \text{ accepts } \langle D \rangle \\
\text{accept} & \text{if } D \text{ rejects } \langle D \rangle \\
\text{reject} & \text{if } D \text{ loops on } \langle D \rangle
\end{cases}$$

No matter what $D$ does on $\langle D \rangle$, we get a contradiction, so $D$ can’t exist, which means that $H$ can’t exist and $\text{REJECT}_{TM}$ is undecidable.

(c) Give a second proof that $\text{REJECT}_{TM}$ is undecidable by reducing $A_{TM}$ to $\text{REJECT}_{TM}$.

[This will involve some creativity because the technique we used to reduce $A_{TM}$ to $\text{HALT}_{TM}$ will not work here.]

**Solution:** Suppose that the Turing machine $R$ decides $\text{REJECT}_{TM}$. Then we define a Turing machine $S$ that decides $A_{TM}$ as follows:

$S =$ “On input $\langle M, w \rangle$
1. Construct a TM $M'$ from $M$ by reversing the accept and reject states.
2. Run $R$ on $\langle M', w \rangle$.
3. If $R$ accepts, accept. If $R$ rejects, reject.”

Note that $\langle M, w \rangle \in A_{TM}$ if and only if $M$ accepts $w$ if and only if $M'$ rejects $w$ if and only if $\langle M', w \rangle \in \text{REJECT}_{TM}$ if and only if $R$ accepts $\langle M, w \rangle$ if and only if $S$ accepts $\langle M, w \rangle$, so $S$ decides $A_{TM}$. Since $A_{TM}$ is undecidable, no such Turing machine $S$ exists, so $R$ cannot exist and $\text{REJECT}_{TM}$ is undecidable.