1. Using the method from class, give an NFA that recognizes $L_1^*$, where $L_1$ is the language from Exercise 3(a) of Homework 2.

Solution:

2. Give regular expressions for the following languages:

(a) $\{w \in \{0,1\}^* | w$ starts with either 010 or 11$\}$
   Solution: $(010 \cup 11)(0 \cup 1)^*$.

(b) $\{w \in \{0,1\}^* | w$ contains exactly one 1 and an even number of 0's$\}$
   Solution: $(00)^*1(00)^* \cup (00)^*01(00)^*0$.

(c) $\{w \in \{0,1\}^* | w$ does not contain 10$\}$
   Solution: $0^*1^*$.

(d) $\{w \in \{0,1\}^* | w$ does not start with 010 and does not start with 11$\}$
   Solution: $\varepsilon \cup 0 \cup 1 \cup 01 \cup (00 \cup 10 \cup 011)(0 \cup 1)^*$.

3. Convert the regular expression $a(b \cup c)^*$ into an NFA using the method from class (which is the same as the method from the book and is different from the method in JFLAP).

Solution:

First we have the following NFAs for $a$, $b$, and $c$.

$a$: 

$b$: 

$c$: 

Then, we obtain an NFA for $b \cup c$

and the following for $(b \cup c)^*$. 

Finally, we obtain the NFA below for $a(b \cup c)^*$. 

4. Convert the NFA in Figure 1.36 of the textbook into a regular expression using the method from class (which is the same as the method in the book, and is not the same as the method in JFLAP).

**Solution:** The NFA is

Turning this into a GNFA, we get
Eliminating state $q_2$ gives

$$\varepsilon \cup ba^*(a \cup b)$$

Eliminating state $q_3$ gives

$$((\varepsilon \cup ba^*(a \cup b))a)^*$$

Finally, eliminating state $q_1$ gives

$$((\varepsilon \cup ba^*(a \cup b))a)^*$$

and the regular expression is $((\varepsilon \cup ba^*(a \cup b))a)^*$. 

4
5. Problem 1.31 Show that if A is regular so is $A^R$.

Proof: Since A is regular, there is some NFA $M = (Q, \Sigma, \delta, q_0, F)$ that recognizes A. We construct an NFA $M'$ such that $L(M') = A^R$. The idea is that the start state of $M'$ is the accept state of M, the accept state of $M'$ is the start state of M, and for every transition in M, there is a transition with the same label in $M'$, but going in the reverse direction. The problem with this is that M may have more than one accept state. Since $M'$ can have only one start state, we add a new start state to $M'$ and $\varepsilon$ transitions from the new start state to all the accept states of M.

The formal construction is:

$M' = (Q \cup \{q_s\}, \Sigma, \delta', q_s, \{q_0\})$ where $q_s$ is a new state and

$$\delta'(q, a) = \begin{cases} \{p : q \in \delta(p, a)\} & \text{if } q \neq q_s \\ F & \text{if } q = q_s, a = \varepsilon \\ \emptyset & \text{if } q = q_s, a \neq \varepsilon \end{cases}$$

6. Use the Pumping Lemma to show that the following languages are not regular:

(a) $\{0^n1^{2n} | n \geq 0\}$

Solution: Given $p \geq 1$, choose $s = 0^p1^{2p}$. Then, s is in the language and $|s| = 3p \geq p$. Given $x, y, z$ with $s = xyz$, $|xy| \leq p$ and $|y| > 0$, we must have $y = 0^k$ for some $k$ with $1 \leq k \leq p$. Choose $i = 2$. Then $xy^iz = 0^{p+k}1^{2p}$. Since $k > 0$, $2p \neq 2(p+k)$, so $xy^iz$ is not in the language.

(b) $\{0^n1^m | n < m\}$

Solution: Given $p \geq 1$, choose $s = 0^p1^{p+1}$. Then, s is in the language and $|s| = 2p + 1 \geq p$. Given $x, y, z$ with $s = xyz$, $|xy| \leq p$ and $|y| > 0$, we must have $y = 0^k$ for some $k$ with $1 \leq k \leq p$. Choose $i = 2$. Then $xy^iz = 0^{p+k}1^{p+1}$. Since $k > 0$, $p + k \neq p + 1$, so $xy^iz$ is not in the language.

(c) $\{w\#u | w, u \in \{0, 1\}^* \text{ and } |w| > 2|u|\}$

Solution: Given $p \geq 1$, choose $s = 0^{2p+1}\#0^p$. Then, s is in the language and $|s| = 3p + 2 \geq p$. Given $x, y, z$ with $s = xyz$, $|xy| \leq p$ and $|y| > 0$, we must have $y = 0^k$ for some $k$ with $1 \leq k \leq p$. Choose $i = 0$. Then $xy^iz = 0^{2p+1-k}\#0^p$. Since $k > 0$, $2p + 1 - k \neq 2p$, so $xy^iz$ is not in the language.

(d) $\{x_1\#x_2\#x_3 | x_1, x_2, x_3 \in \{a, b\}^* \text{ and either } x_3 = x_1^R \text{ or } x_3 = x_2^R\}$

Solution: Given $p \geq 1$, choose $s = a^p\#a^p$. Then, s is in the language and $|s| = 2p + 2 \geq p$. Given $x, y, z$ with $s = xyz$, $|xy| \leq p$, $|y| > 0$, we must have $y = a^k$ for some $k$ with $1 \leq k \leq p$. Choose $i = 2$. Then $xy^iz = a^{p+k}\#a^p$, which is not in the language since $k = |y| > 0$. 

5
7. (a) Let $M$ be the NFA given in the solution to Problem 3(b) on Homework 2. Give an NFA $N$ with two states and no $\varepsilon$-transitions that recognizes the same language.

**Solution:**

\[ \begin{array}{c}
0 \\
\uparrow \\
4 \\
\downarrow \\
1, 2 \\
\uparrow \\
2 \\
\uparrow \\
\rightarrow 5
\end{array} \]

(b) Generalize what you did in Part (a) of this problem by proving the following theorem:

**Theorem:** If $M$ is an NFA, then there is an NFA $N$ with the following properties

1. $N$ has the same number of states as $M$.
2. $N$ has no $\varepsilon$-transitions.
3. $L(N) = L(M)$.

**Solution:** We obtain $N$ from $M$ by doing three things:

1. We remove all $\varepsilon$-transitions.
2. If in $M$ there is a series of $\varepsilon$-transitions that leads from a state $p$ to an accept state $q$, then in $N$, $p$ is also an accept state.
3. If in $M$ there is a series of $\varepsilon$-transitions from a state $p$ to a state $q$, and $M$ can go from $q$ to a state $r$ reading a symbol $a$, then $N$ can go directly from $p$ to $r$ reading $a$.

Formally, let $M = (Q, \Sigma, \delta, q_0, F)$. We define $N = (Q, \Sigma, \delta', q_0, F')$. If $q$ is a state in $Q$, we use the notation $E_M(q)$ for the set of states that can be reached from $q$ using 0 or more $\varepsilon$-transitions in $M$. The definition of $N$ is now given by

\[ F' = \{ q \in Q | E_M(q) \cap F \neq \emptyset \} \]

and

\[ \delta'(q, a) = \begin{cases} 
\emptyset & \text{if } a = \varepsilon \\
\bigcup \{ \delta(p, a) \mid p \in E_M(q) \} & \text{if } a \neq \varepsilon
\end{cases} \]