1. Use the Pumping Lemma to show that the following languages are not regular:

(a) \( \{0^n1^m0^{2n} | n, m \geq 0 \} \):

Solution: Given \( p \geq 1 \), choose \( s = 0^p1^{2p} \). Then, \( s \) is in the language and \( |s| = 3p + 1 \geq p \). Given \( x, y, z \) with \( s = xyz \), \( |xy| \leq p \) and \( |y| > 0 \), we must have \( y = 0^k \) for some \( k \) with \( 1 \leq k \leq p \). Choose \( i = 2 \). Then \( xy^iz = 0^{p+k}1^{2p} \). Since \( k > 0 \), \( 2p \neq 2(p + k) \), so \( xy^iz \) is not in the language.

(b) \( \{0^n10^m | n \leq m \} \)

Solution: Given \( p \geq 1 \), choose \( s = 0^p1^p \). Then, \( s \) is in the language and \( |s| = 2p + 1 \geq p \). Given \( x, y, z \) with \( s = xyz \), \( |xy| \leq p \) and \( |y| > 0 \), we must have \( y = 0^k \) for some \( k \) with \( 1 \leq k \leq p \). Choose \( i = 2 \). Then \( xy^iz = 0^{p+k}1^{2p} \). Since \( k > 0 \), \( p + k \neq p \), so \( xy^iz \) is not in the language.

(c) \( \{c^5a^n b^m | n > m \} \)

Solution: Given \( p \geq 1 \), choose \( s = c^5a^{p+1}b^p \). Then, \( s \) is in the language and \( |s| = 2p + 6 \geq p \). Given \( x, y, z \) with \( s = xyz \), \( |xy| \leq p \) and \( |y| > 0 \), we choose \( i = 0 \). To show that \( xy^iz = xz \) is not in the language, we consider two cases.

Case 1: \( y \) contains one or more \( c \)'s. Then \( xz \) does not contain exactly five \( c \)'s, so it is not in the language.

Case 2: \( y \) does not contain any \( c \)'s. Then, since \( |xy| \leq p \), \( y \) consists only of \( a \)'s and \( xz = c^5a^{p+1-|y|}b^p \). Since \( |y| > 0 \), \( p + 1 - |y| \neq p \), so \( xz \) is not in the language.

(d) \( \{a^n \# b^m \# c^m | n, m \geq 0 \} \)

Solution: Given \( p \geq 1 \), choose \( s = \# b^p \# c^p \). [Note that when you choose the string \( s \), you must choose the number of \( a \)'s to be 0 because otherwise the adversary can choose \( y = a \) and no matter how the string is pumped, it stays in the language.] Then, \( s \) is in the language and \( |s| = 2p + 2 \geq p \). Given \( x, y, z \) with \( s = xyz \), \( |xy| \leq p \) and \( |y| > 0 \), we choose \( i = 2 \). To show that \( xy^iz = xyyz \) is not in the language, we consider two cases.

Case 1: \( y \) contains a \( \# \). Then \( xyyz \) contains more than two \( \# \)'s, so it is not in the language.

Case 2: \( y \) does not contain \( \# \). Then, since \( |xy| \leq p \), \( y \) consists only of \( b \)'s and \( xyyz = \# b^{p+|y|} \# c^p \). Since \( |y| > 0 \), \( p + |y| \neq p \), so \( xyyz \) is not in the language.

2. Let \( A \) be the language consisting of those strings \( w \) in \( \{0, 1, \#\}^* \) such that either \( w \) starts with 0 or \( w = u\#u \) for some \( u \in \{0, 1\}^* \). \( A \) is not regular.
In a Pumping Lemma proof of this, you are given \( p \) and you choose \( s \). For each of the following possible choices of \( s \), state whether or not the choice is a good one. If the choice is bad, provide the decomposition that allows the string to be pumped.

(a) \( s = 0^p1^p \#0^p1^p \)

**Solution:** This is a bad choice when \( p \geq 2 \) because if we take \( x = \varepsilon \), \( y = 0 \), \( z = 0^{p-1}1^p \#0^p1^p \), then \( xy^iz \) starts with a 0 for all values of \( i \), so is in the language.

(b) \( s = 1^p0^p \#1^p0^p \)

**Solution:** This is a good choice. Given \( x, y, z \) as in the Pumping Lemma, we choose \( i = 2 \). Then, \( xy^iz \) starts with a 1, and the part to the left of the \( \# \) is different from the part to the right, so the string is not in the language.

(c) \( s = (10)^p \#(10)^p \)

**Solution:** This is a good choice for the same reason as in Part (b).

3. Read the discussion of minimum pumping length given in Problem 1.55 of the text (third US edition) and then give the minimum pumping length for the following languages. Justify your answers.

(a) \( \{ w \in \{0,1\}^* \mid w \text{ starts with } 0 \text{ and has odd length} \} \)

**Solution:** The minimum pumping length is 3. To see this, first note that \( p = 2 \) is not a pumping length because 011 is in the language and it cannot be pumped to stay in the language because if 011 = \( xyz \) with \( |y| > 0 \) and \( |xy| \leq p \), then either \( y = 0 \) or \( y = 1 \) or \( y = 01 \). In the first two cases, pumping down gives a string of length 2, which is not in the language, and in the third case, pumping down gives a string that does not start with 0, so is not in the language.

Now suppose that \( s \) is in the language and has length \( \geq 2 \). Then, we let \( s = xyz \) with \( x = 0 \), \( y \) equal to the second and third symbols in \( s \) and \( z \) be all symbols in \( s \) after the third. Then, \( |y| = 1 > 0 \), \( |xy| = 3 \) and \( xy^iz \) is in the language for all \( i \geq 0 \), so 3 is a pumping length for the language.

(b) \( \{ w \in \{0,1\}^* \mid w \text{ contains } 011 \text{ as a substring} \} \)

**Solution:** The minimum pumping length is 4. To see this, first note that \( p = 3 \) is not a pumping length because 011 is in the language and it cannot be pumped down to stay in the language.

Now suppose that \( s \) is in the language and has length \( \geq 4 \). If \( s \) does not start with 011, then we let \( x = \varepsilon \), \( y \) be the first symbol in \( s \) and \( z \) be the rest of \( s \). Then, \( |y| = 1 > 0 \), \( |xy| = 1 \leq 4 \) and \( xy^iz \) is in the language for all \( i \geq 0 \). If \( s \) does start with 011, then we let \( x = 011 \), \( y \) be the fourth symbol in \( s \) and \( z \) be the rest of \( s \). We have \( |y| = 1 > 0 \), \( |xy| = 4 \) and \( xy^iz \) is in the language for all \( i \geq 0 \). Thus, 4 is a pumping length for the language.
(c) \( \{w \in \{0, 1\}^* | w \text{ does not contain 011 as a substring}\} \).

**Solution:** The minimum pumping length is 1. To see this, first note that \( p = 0 \) is not a pumping length because \( \varepsilon \) is in the language and it cannot be expressed as \( xyz \) with \( |y| > 0 \).

Now suppose that \( s \) is in the language and has length \( \geq 1 \). We let \( x = \varepsilon, y \) be the first symbol of \( s \) and \( z \) be the rest of \( s \). Then \( |y| = 1 > 0 \) and \( |xy| = 1 \). No matter whether \( y \) is 0 or 1, no matter how many times we repeat \( y \), this cannot create an occurrence of 011 in \( xy^iz \) if \( s = xyz \) does not have an occurrence, so 1 is a pumping length for the language.

4. **Problem 1.54**

(a) To see that \( F \) is not regular, by Problem 1.31, it suffices to show that \( F^R \) is not regular. We do this by showing that \( F^R \) does not have the Pumping Property. Given \( p \), choose \( s = c^rb^pa \). Then \( s \in F^R \) and \( |s| \geq p \). Given \( x, y, z \) as in the Pumping Lemma, we must have \( y = c^k \) for some \( k > 0 \). Choose \( i = 2 \). Then, \( xyyz = c^{p+k}b^ka \), which is not in \( F^R \).

(b) Choose \( p = 2 \). We show that this is a pumping length for \( F \). Given \( s \in F \) with \( |s| \geq 2 \), we must have we \( s = a^ib^jc^k \). We consider cases depending on what \( i, j, k \) are.

Case 1: \( i = 0, j = 0 \). Then we must have \( k > 0 \). Take \( x = \varepsilon, y = c, z = c^{k-1} \). Then \( |xy| \leq p, |y| > 0 \) and for all \( r \geq 0 \), \( xy^rz = c^{k+r-1} \), which is in \( F \).

Case 2: \( i = 0, j > 0 \). Take \( x = \varepsilon, y = b, z = b^{j-1}c^k \). Then \( |xy| \leq p, |y| > 0 \) and for all \( r \geq 0 \), \( xy^rz = b^{j+r-1}c^k \), which is in \( F \).

Case 3: \( i = 1 \). Then we must have \( j = k \). Take \( x = \varepsilon, y = a, z = b^jc^j \). Then \( |xy| \leq p, |y| > 0 \) and for all \( r \geq 0 \), \( xy^rz = a^{i+r-1}b^jc^j \), which is in \( F \).

Case 4: \( i = 2 \). Take \( x = \varepsilon, y = aa, z = b^jc^k \). Then \( |xy| \leq p, |y| > 0 \) and for all \( r \geq 0 \), \( xy^rz = a^{2r}b^jc^k \), which is in \( F \) (since \( 2r \neq 1 \)).

Case 5: \( i > 2 \). Take \( x = \varepsilon, y = a, z = a^{i-1}b^jc^k \). Then \( |xy| \leq p, |y| > 0 \) and for all \( r \geq 0 \), \( xy^rz = a^{i+r-1}b^jc^k \), which is in \( F \) (since \( i + r - 1 \neq 1 \)).

(Note that the special treatment in Case 4 is necessary because if \( s = aab^jc^k \) we can’t take \( y = a \) since then \( xz = ab^jc^k \) is not in \( F \) if \( j \neq k \).)

(c) The Pumping Lemma states that every regular language has the Pumping Property. It does not state that every language with the Pumping Property is regular, so the result in this problem does not contradict the Pumping Lemma.