CS 420, Spring 2019
Homework 4 Solutions

1. Let A be the language consisting of those strings w in $\{0, 1, \#\}^*$ such that either w starts with 0 or $w = u\#u$ for some $u \in \{0, 1\}^*$. A is not regular. In a Pumping Lemma proof of this, you are given p and you choose s. For each of the following possible choices of s, state whether or not the choice is a good one. If the choice is bad, provide the decomposition that allows the string to be pumped.

   (a) $s = 0^p1^p\#0^p1^p$

   Solution: This is bad choice when $p \geq 2$ because if we take $x = \varepsilon$, $y = 0$, $z = 0^{p-1}1^p\#0^p1^p$, then xy^iz starts with a 0 for all values of i, so is in the language.

   (b) $s = 1^p0^p\#1^p0^p$

   Solution: This is a good choice. Given x, y, z as in the Pumping Lemma, we choose $i = 2$. Then, xy^iz starts with a 1, and the part to the left of the # is different from the part to the right, so the string is not in the language.

   (c) $s = (10)^p\#(10)^p$

   Solution: This is a good choice for the same reason as in Part (b).

2. Read the discussion of minimum pumping length given in Problem 1.55 of the text (third US edition) and then give the minimum pumping length for the following languages. Justify your answers.

   (a) $\{w \in \{0, 1\}^* | w \text{ contains exactly three 1's}\}$.

   Solution: The minimum pumping length is 4. To see this, first note that $p = 3$ is not a pumping length because 111 is in the language and it cannot be pumped down to stay in the language.

   Now suppose that s is in the language and has length ≥ 4. Since s is in the language, it contains exactly three 1's, so since $|s| \geq 4$, at least one of the first four symbols in s is a 0. We let y be the first 0 in s, x be the part of s before y and z be the part of s after y. Then,$|xy| \leq 4$ and since $y = 0$, xy^iz is in the language foir all $i \geq 0$.

   (b) $\{w \in \{0, 1\}^* | w \text{ contains 001 as a substring}\}$.

   Solution: The minimum pumping length is 4. To see this, first note that $p = 3$ is not a pumping length because 001 is in the language and it cannot be pumped down to stay in the language.

   Now suppose that s is in the language and has length ≥ 4. Since s is in the language, it contains 001 as a substring. We consider two cases.

   Case 1: s starts with 001. Then, we let $x = 001$, y be the fourth symbol in s, and z be the rest of s.We have $|xy| = 4$ and since $x = 001$, xy^iz is in the language for all $i \geq 0$.

Case 2: s does not start with 001. Then, we let $x = \varepsilon$, y be the first symbol of s, and z be the rest of s. We have $|xy| = 1 \leq 4$ and since z contains 001, xy^iz is in the language for all $i \geq 0$.

(c) $\{11, 110\}^*$.

Solution: The minimum pumping length is 3. To see this, first note that $p = 2$ is not a pumping length because 110 is in the language and has length $3 \geq 2$, but if we try to pump 110 with $|xy| \leq 2$ and $|y| > 0$, then we must have $y = 1$ or $y = 11$, so xz is either 10 or 0, and these strings are not in the language.

Now suppose that s is in the language and has length ≥ 3. Since s is in the language and has length at least 3, $s = a_1 \cdots a_k$ where $k \geq 1$ and each a_i is either 11 or 110. We let $x = \varepsilon$, $y = a_1$ and $z = a_2 \cdots a_k$. Then, $|xy| \leq 3$ and xy^iz is in the language for all $i \geq 0$.

3. The Pumping Lemma says that every regular language has the Pumping Property. This means that if a language does not have the Pumping Property, then it is not regular. We said in class many times that if a language does have the Pumping Property, then it may or may not be regular. The purpose of this exercise is to give an example of a language that has the Pumping Property but is not regular, which shows that a language with the Pumping Poperty may or may not be regular.

   Define

   $L = \{a^iw | i \geq 0, w \in \{b, c\}^*$ and if $i = 1$, then w has the same number of b's and c's$\}$

   In other words, if $i \neq 1$, then a^iw is in L no matter what $w \in \{b, c\}^*$ is, but if $i = 1$, then w must have the same number of b's as c's in order for a^iw to be in L.

   (a) Show that L has the Pumping Property.

   Solution: Choose $p = 2$. We show that this is a pumping length for L. Given $s \in L$ with $|s| \geq 2$, we must have $s = a^iw$ for some $i \geq 0$ and $w \in \{b, c\}^*$. We consider cases depending on what i is.

   Case 1: $i = 0$. Then we must have $w \neq \varepsilon$ since $|s| \geq 2$. Let $x = \varepsilon$, y be the first symbol in w and z be the rest of w. Then $|xy| = 1 \leq 2$, $|y| > 0$ and for all $r \geq 0$, xy^rz is in L because it is in $\{b, c\}^*$

   Case 2: $i = 1$. Then w must contain the same number of b's and c's. Take $x = \varepsilon, y = a, z = w$. Then $|xy| = 1 \leq 2$, $|y| > 0$ and for all $r \geq 0$, $xy^rz = a^rw$, which is in L.

   Case 3: $i = 2$. Take $x = \varepsilon, y = aa, z = w$. Then $|xy| = 2$, $|y| > 0$ and for all $r \geq 0$, $xy^rz = a^{2r}w$, which is in L (since $2r \neq 1$).

   Case 4: $i > 2$. Take $x = \varepsilon, y = a, z = a^{i-1}w$. Then $|xy| = 1 \leq 2$, $|y| > 0$ and for all $r \geq 0$, $xy^rz = a^{i+r-1}w$, which is in L (since $i + r - 1 \neq 1$).

(Note that the special treatment in Case 3 is necessary because if $s = aab^jc^k$ we can't take $y = a$ since then $xz = ab^jc^k$ is not in L if $j \neq k$.)

(b) Show that L^R does not have the Pumping Propertry, so is not regular.

Solution: Note that $L^R = \{wa^i | i \geq 0,\, w \in \{b,c\}^*$ and if $i = 1$, then w contains the same number of b's and c's$\}$. . We show that L^R does not have the Pumping Property. Given p, choose $s = b^pc^pa$. Then $s \in L^R$ and $|s| \geq p$. Given x, y, z as in the Pumping Lemma, we must have $y = b^k$ for some $k > 0$. Choose $i = 2$. Then, $xyyz = b^{p+k}c^ka$, which is not in L^R.

(c) Use the result in Homework 3, Exercise 3 to show that L is not regular.

Solution: If L were regular, then, according to the result in Homework 3, Exercise 3, L^R would also be regular and this contradicts Part (b), so L is not regular.

4. Give context-free grammars for the following languages:

(a) $\{0^n1^n2^m3^m | n, m \geq 0\}$.

$$\begin{array}{rcl} S & \rightarrow & TU \\ T & \rightarrow & 0T1|\varepsilon \\ U & \rightarrow & 2U3|\varepsilon \end{array}$$

(b) $\{x\#y | x, y \in \{0,1\}^*$ and $|x| = 2|y|\}$.

$$\begin{array}{rcl} S & \rightarrow & BBSB|\# \\ B & \rightarrow & 0|1 \end{array}$$

(c) $\{x\#y | x, y \in \{0,1\}^*$ and $|x| \neq 2|y|\}$.

$$\begin{array}{rcl} S & \rightarrow & BBSB|T\#|\#T|B\#T \\ T & \rightarrow & BT|B \\ B & \rightarrow & 0|1 \end{array}$$