1. Use the Pumping Lemma to show that the following languages are not context-free.

(a) \( \{a^n b^{3n} c^{2n} | n \geq 0 \} \).

**Solution:** Given \( p \geq 1 \), chose \( s = a^p b^{3p} c^{2p} \). Then, \( s \) is in the language and \( |s| = 6p \geq p \). Given \( u, v, x, y, z \) with \( s = uvxyz \), \( |vxy| \leq p \), and \( |vy| > 0 \), we choose \( i = 2 \). Since \( |vxy| \leq p \), \( vy \) contains at most two types of symbols in consecutive order.

**Case 1:** Either \( v \) or \( y \) contains an \( a \). Then, in \( uv^2xyyz \), the number of \( a \)'s is more than \( p \), but the number of \( c \)'s is \( 2p \), so \( uv^2xyyz \) is not in the language.

**Case 2:** Either \( v \) or \( y \) contains a \( c \). Then in \( uv^2xyyz \), the number of \( c \)'s is more than \( 2p \), but the number of \( a \)'s is \( p \), so \( uv^2xyyz \) is not in the language.

**Case 3:** Both \( v \) and \( y \) consist only of \( b \)'s. Then, in \( uv^2xyyz \), the number of \( b \)'s is more than \( 3p \), but the number of \( a \)'s is \( p \), so \( uv^2xyyz \) is not in the language.

(b) \( \{u \# w \# u | u, v, w \in \{0, 1\}^* \text{ and } |u| < |v| < |w| \} \).

**Solution:** Given \( p \geq 1 \), chose \( s = 0^p \# 0^{p+2} \# 0^{p+1} \). Then, \( s \) is in the language and \( |s| = 3p + 5 \geq p \). Given \( u, v, x, y, z \) with \( s = uvxyz \), \( |vxy| \leq p \), and \( |vy| > 0 \), how we chose \( i \) depends on \( v \) and \( y \).

**Case 1:** Either \( v \) or \( y \) contains a \( \# \). Then, choose \( i = 2 \). Since \( uv^2xyyz \) contains more than two \( \# \)'s, it is not in the language.

**Case 2:** Neither \( v \) nor \( y \) contains a \( \# \).

**Case 2.1:** Either \( v \) or \( y \) contains a symbol to the right of the second \( \# \). Choose \( i = 0 \). Since \( |vxy| \leq p \), \( uxz \) contains fewer than \( p + 1 \) symbols to the right of the second \( \# \), but contains \( p \) symbols to the left of the first \( \# \), so \( uxz \) is not in the language.

**Case 2.2:** Either \( v \) or \( y \) contains a symbol to the left of the first \( \# \). Choose \( i = 2 \). Then, since \( |vxy| \leq p \), \( uv^2xyyz \) contains more than \( p \) symbols to the left of the first \( \# \) and \( p + 1 \) symbols to the right of the second \( \# \), so \( uv^2xyyz \) is not in the language.

**Case 2.3:** \( v \) and \( y \) are both between the two \( \# \)'s. Choose \( i = 0 \). Then, \( uxz \) contains fewer than \( p + 2 \) symbols between the two \( \# \)'s, but contains \( p + 1 \) symbols to the right of the second \( \# \), so \( uxz \) is not in the language.

(c) \( \{u \# w \# u^R | u, w \in \{0, 1\}^* \text{ and } |u| = |w| \} \).

**Solution:** Given \( p \geq 1 \), chose \( s = 0^p \# 0^p \# 0^p \). Then, \( s \) is in the language and \( |s| = 3p + 2 \geq p \). Given \( u, v, x, y, z \) with \( s = uvxyz \), \( |vxy| \leq p \), and \( |vy| > 0 \), we chose \( i = 2 \).
Case 1: Either $v$ or $y$ contains a #. Then, since $uvvxyyz$ contains more than two #’s, it is not in the language.

Case 2: Neither $v$ nor $y$ contains a #. Then, in $uvvxyyz$, the number of symbols to the left of the first #, the number of symbols between the two #’s, and the number of symbols to the right of the second # are not all the same, so $uvvxyyz$ is not in the language.

(d) \{w_1cw_2cw_3cw_4\mid w_1 = w_3 \text{ or } w_2 = w_4 \text{ and } w_i \in \{a, b\}^+ \text{ for } i = 1, 2, 3, 4\}.

[This one is hard.]

Solution: Given $p \geq 1$, choose $s = a^p b^p c a c a b^p c b$. Then, $s$ is in the language and $|s| = 4p + 5 \geq p$. Given $u, v, x, y, z$ with $s = uvxyz$, $|vxy| \leq p$, and $|vy| > 0$, we choose $i = 0$. To see that $uvvxyyz$ is not in the language, we consider cases,

Case 1: Either $v$ or $y$ contains a $c$. Then, $uxz$ contains fewer than 3 $c$’s, so is not in the language.

Case 2: Neither $v$ nor $y$ contains a $c$ and either $v$ or $y$ is between the first two $c$’s, or after the third $c$. Then, in $uxz$, the string between the first two $c$’s or after the third $c$ is $\epsilon$, so $uxz$ is not in the language.

Case 3: $v$ and $y$ are both to the left of the first $c$, or are both between the second and third $c$’s. Then in $uxz$, the string to the left of the first $c$ is different than the string between the second and third $c$’s, and $a \neq b$, so $uxz$ is not in the language.

Case 4: $v$ is to the left of the first $c$ and $y$ is in between the second and third $c$’s. Then, since $|vxy| \leq p$, $v$ consists of $b$’s and $y$ consists of $a$’s. Thus $uxz = a^p b^{p-|v|} c a c a b^{p-|y|} b^p c b$. Since $|vy| > 0$, $uxz$ is not in the language.

2. What is the minimum value of $p$ that works in the Pumping Lemma for the following context-free languages?

(a) \{0^n 10^m \mid n \leq m\}.

Solution:

The minimum pumping length is 3. To see that 2 is not a pumping length, let $s = 010$. Then, $s$ is in the language and $|s| = 3 \geq 2$. If we have $s = uvxyz$ with $|vxy| \leq 2$, then either $vy$ contains 1, in which case $uvvxyyz$ contains more than one 1, so is not in the language, or $v$ and $y$ are both to the left of the 1 (and one of them is empty), so $uvvxyyz = 0010$ which is also not in the language, or $v$ and $y$ are both to the right of the 1, and $uxz = 01$, which is not in the language.

To see that 3 is a pumping length, let $s$ be in the language with $|s| \geq 3$. Then, $s = 0^n 10^m$ with $n \leq m$. We consider cases to determine how to define $u, v, x, y, z$.

Case 1: $n = 0$. Then, since $|s| \geq 3$, $m > 0$ and we can pump $s$ by letting $u = 1, v = 0, x = y = \epsilon, z = 0^{m-1}$. We have $|vxy| = 1 \leq 3$. 

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Case 2: $n > 0$. Then, since $n \leq m$, we have $m > 0$ We pump $s$ by letting $u = 0^{n-1}, v = 0, x = 1, y = 0, z = 0^{m-1}$ We have $|vxy| = 3$.

(b) \{a^n#b^m#c^m|n, m \geq 0\}.

Solution:
The minimum pumping length is 3. To see that 2 is not a pumping length, let $s = \#\#$. Then, $s$ is in the language and $|s| = 2$, but $s$ cannot be pumped to stay in the language.

To see that 3 is a pumping length, let $s$ be in the language with $|s| \geq 3$. Then, $s = a^n#b^m#c^m$. We consider cases to determine how to define $u, v, x, y, z$.

Case 1: $n > 0$. Then, we can pump $s$ by letting $u = \varepsilon, v = a, x = y = \varepsilon, z = a^{n-1}#b^m#c^m$. We have $|vxy| = 1 \leq 3$.

Case 2: $n = 0$. Then, since $|s| \geq 3$, we have $m > 0$ We pump $s$ by letting $u = a^n#b^{m-1}, v = b, x = \#, y = c, z = c^{m-1}$. We have $|vxy| = 3$.

(c) \{a^n#a^m#a^q#a^r|n = q or m = r\}.

Solution:
The minimum pumping length is 4. To see that 3 is not a pumping length, let $s = \#\#\#$. Then, $s$ is in the language and $|s| = 3$, but $s$ cannot be pumped to stay in the language.

To see that 4 is a pumping length, let $s$ be in the language with $|s| \geq 4$. Then, $s = a^n#a^m#a^q#a^r$ with either $n = q$ or $m = r$. We consider cases to determine how to define $u, v, x, y, z$.

Case 1: $n = q$ and $m > 0$. Then, we can pump $s$ by letting $u = a^n#, v = a, x = y = \varepsilon, z = a^{n-1}#a^q#a^r$. We have $|vxy| = 1 \leq 4$.

Case 2: $n = q$ and $r > 0$. Similar to Case 1, but we pump one $a$ in the $a^r$ part.

Case 3: $n = q$ and $m = r = 0$. Then, since $|s| \geq 4$, we must have $n = q > 0$. We can pump $s$ by letting $u = a^{n-1}, v = a, x = \#, y = a, z = a^{n-1}#$. We have $|vxy| = 4$.

Case 4: $m = r$. This case breaks down into three cases similar to the three cases for $n = q$.

3. Problem 2.36 Let $A = \{a^i b^j c^k d^l|if \ i = 1 \ then \ j = k = l\}$.

To see that $A$ meets the conditions of the Pumping Lemma, we choose $p = 1$. If $s \in A$ and $|s| \geq p$, then we have $s = a^i b^j c^k d^l$ for some $i, j, k, l$. We consider cases in choosing $u, v, x, y, z$ as in the Pumping Lemma.

Case 1: $j = k = l = 0$. Then $i \neq 0$ since $s$ has length at least 1. We take $u = v = x = \varepsilon, y = a$ and $z = a^{i-1}$. We have $|vxy| = 1$ and for all $r \geq 0, uv^rxy^rz = a^{i+r-1}$ which is in $A$.

Case 2: $i = 1$. Then $j = k = l$. We take $u = v = x = \varepsilon, y = a$ and $z = a^{i-1}b^j c^k d^l$. We have $|vxy| = 1$ and for all $r \geq 0, uv^rxy^rz = a^{i+r-1}b^j c^k d^l$ which is in $A$. 3
Case 3: $i \neq 1$ and at least one of $j,k,l$ is not 0. Then we take $u = a^i$, $v = x = \varepsilon$, $y =$ the first symbol in $b^j c^k d^l$ and $z$ to be the rest of the string. Then $|vxy| = 1$ and for all $r \geq 0$, $uv^rxy^rz$ is a string of the form $a^ib^j c^k d^l$, which is in the language since $i \neq 1$.

To see that $A$ is not context-free, we suppose for a contradiction that $A$ is context-free. Then, according to Problem 2.18, so is the language $B = A \cap ab^*c^*d^*$. We will show that $B$ is not context-free and this will be the contradiction we are looking for. $B$ is the language $\{ab^j c^k d^l | j \geq 0 \}$. We use the Pumping Lemma to show that $B$ is not context-free. Given $p$, choose $s = ab^p c^p d^p$. Then $s \in B$ and $|s| \geq p$. Given $u,v,x,y,z$ as in the Pumping Lemma, we consider cases.

Case 1: Either $v$ or $y$ contains $a$. Then choose $i = 0$. Since $uxz$ does not contain an $a$, it is not in $B$.

Case 2: Neither $v$ nor $y$ contains $a$. Then choose $i = 0$. In $uxz$, one or two of the symbols $b,c,d$ has been reduced in number, but the third of these symbols has not been changed. Thus $uxz$ cannot be in $B$ since it does not have the same number of $bs$, $cs$ and $ds$. 