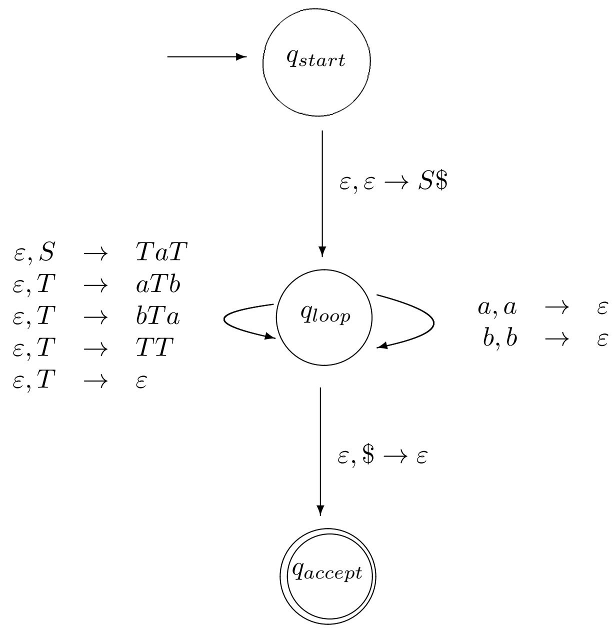
CS 420 Spring 2019
Homework 7 Solutions

1. Let G be the grammar

$$\begin{aligned} S &\rightarrow TaT \\ T &\rightarrow aTb|bTa|TT|\varepsilon \end{aligned}$$

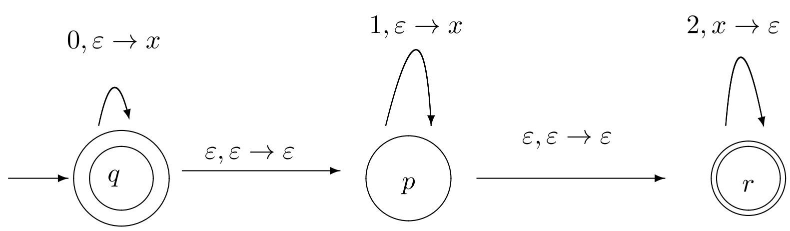
(a) Using the method from class, give a PDA M with $L(M) = L(G)$.



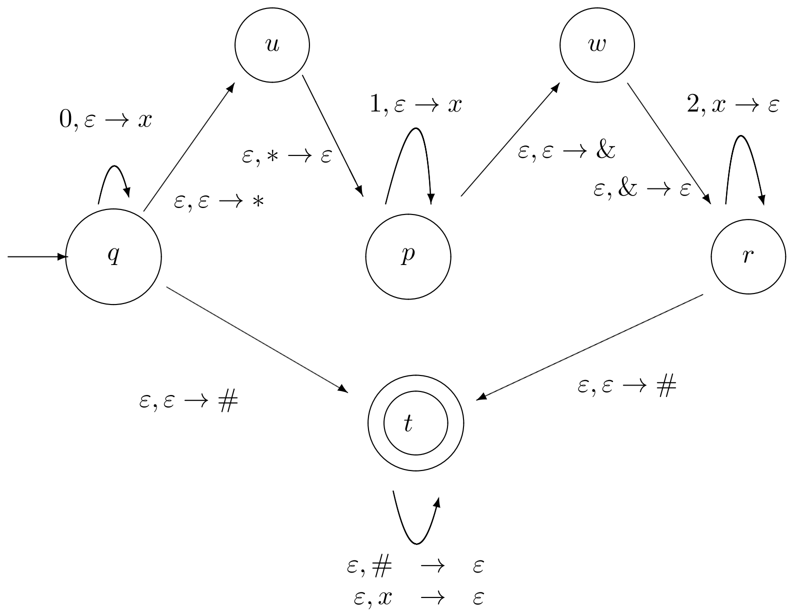
(b) Show an accepting computation for M on the string *ababbaa* by giving a chart with the state, tape contents, and stack contents after each step.

State	Tape	Stack
q_{start}	$ababbaa$	ε
q_{loop}	$ababbaa$	$S\$$
q_{loop}	$ababbaa$	$TaT\$$
q_{loop}	$ababbaa$	$aTbaT\$$
q_{loop}	$babbaa$	$TbaT\$$
q_{loop}	$babbaa$	$baT\$$
q_{loop}	$abbaa$	$aT\$$
q_{loop}	$bbaa$	$T\$$
q_{loop}	$bbaa$	$bTa\$$
q_{loop}	baa	$Ta\$$
q_{loop}	baa	$bTaa\$$
q_{loop}	aa	$Taa\$$
q_{loop}	aa	$aa\$$
q_{loop}	a	$a\$$
q_{loop}	ε	$\$$
q_{accept}	ε	ε

2. Let M be the following PDA:



   (a) Convert M into a "special" PDA M'.
   Solution:



(b) Give the Case 2b rules when you convert M' into a CFG G using the method from class.

Solution: There are two transitions that push the stack symbol x and two transitions that pop x. Combining each pop with each push gives the following rules:

$$\begin{array}{rcl} A_{qr} & \to & 0A_{qr}2 \\ A_{qt} & \to & 0A_{qt} \\ A_{pr} & \to & 1A_{pr}2 \\ A_{pt} & \to & 1A_{pt} \end{array}$$

There are two transitions that push # and one transition that pops #. Combining each push with the pop gives the following rules:

$$\begin{array}{rcl} A_{qt} & \to & A_{tt} \\ A_{rt} & \to & A_{tt} \end{array}$$

There is one transition that pushes $*$ and one that pops $*$. This gives the following rule:

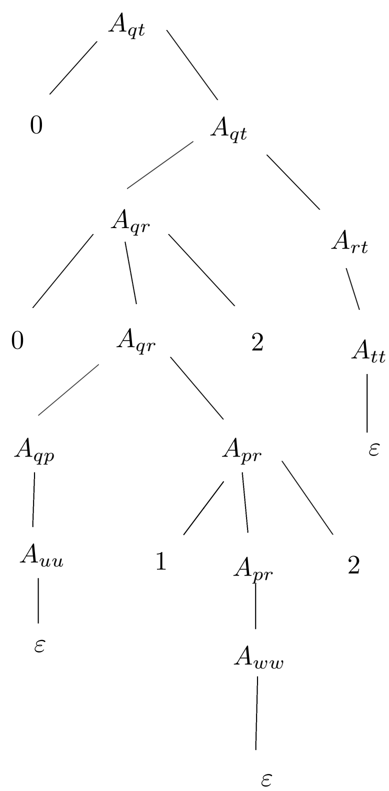
$$A_{qp} \to A_{uu}$$

Finally, the transtion that pushes & combined with the transition that pops & gives the following rule:

$$A_{pr} \to A_{ww}$$

(c) Give a parse tree in G for the string 00122.

Solution:



3. Use the Pumping Lemma to show that the following languages are not context-free.

   (a) $\{a^n b^m c^n d^m | n, m \geq 0\}$.

   Solution: Given $p \geq 1$, chose $s = a^p b^p c^p d^p$. Then, s is in the language and $|s| = 4p \geq p$. Given u, v, x, y, z with $s = uvxyz$, $|vxy| \leq p$, and $|vy| > 0$, we choose $i = 2$. Since $|vxy| \leq p$, vy contains at most two types of symbols in consecutive order.

   Case 1: Either v or y contains an a. Then, in $uvvxyyz$, the number of a's is more than p, but the number of c's is p, so $uvvxyyz$ is not in the language.

   Case 2: Either v or y contains a b. Then, in $uvvxyyz$, the number of b's is more than p, but the number of d's is p, so $uvvxyyz$ is not in the language.

   Case 3: Either v or y contains a c. Then in $uvvxyyz$, the number of c's is more than p, but the number of a's is p, so $uvvxyyz$ is not in the language.

Case 4: Either v or y contains a d. Then in $uvvxyyz$, the number of d's is more than p, but the number of b's is p, so $uvvxyyz$ is not in the language.

(b) $\{a^n c^m b^n | n > m \geq 0\}$.

Solution: Given $p \geq 1$, choose $s = a^{p+1} c^p b^{p+1}$. Then, s is in the language and $|s| = 3p + 2 \geq p$. Given u, v, x, y, z with $s = uvxyz$, $|vxy| \leq p$ and $|vy| > 0$, we choose $i = 2$.

Case 1: Either v or y contains an a. Then, since $|vxy| \leq p$, neither v nor y contains a b. Thus $uvvxyyz$ contains more a's than b's, so is not in the language.

Case 2: Either v or y contains a b. Then, since $|vxy| \leq p$, neither v nor y contains an a. Thus $uvvxyyz$ contains more b's than a's, so is not in the language.

Case 3: Both v and y consist only of c's. Then, $uvvxyyz$ contains at least $p + 1$ c's and exactly $p + 1$ a's and b's, so $uvvxyyz$ does not contain more a's than c's and is not in the language.

(c) $\{w \in \{a, b, c\}^* | n_a(w) = n_b(w) \text{ and } n_a(w) > n_c(w)\}$
[Here $n_x(w)$ means the number of occurrences of the symbol x in the string w.

Solution: The proof of the previous part works here as well to show that this language is not context-free.

(d) $\{w\#t\#w^R | w, t \in \{a, b\}^* \text{ and } |w| = |t|\}$.

Solution: Given $p \geq 1$, choose $s = a^p \# b^p \# a^p$. Then, s is in the language and $|s| = 3p + 2 \geq p$. Given u, v, x, y, z with $s = uvxyz$, $|vxy| \leq p$ and $|vy| > 0$, we choose $i = 2$.

Case 1: Either v or y contains a $\#$. Then, $uvvxyyz$ contains more than two $\#$'s, so is not in the language.

Case 2: Neither v nor y contains a $\#$ and either v or y contains an a. Since $|vxy| \leq p$, v and y do not contain both a's to the left of the first $\#$ and to the right of the second $\#$, so in $uvvxyyz$, the string to the left of the first $\#$ is not the reversal of the string to the right of the second $\#$.

Case 2: Both v and y consist only of b's. Then, in $uvvxyyz$, the string between the two $\#$'s is longer than the string to the left of the first $\#$, so $uvvxyyz$ is not in the language.

4. What is the minimum value of p that works in the Pumping Lemma for the following context-free languages?

(a) $\{0^n 1^n 2^m 3^m | n, m \geq 0\}$.

Solution:

The minimum pumping length is 2. To see that 1 is not a pumping length, let $s = 01$. Then, s is in the language and $|s| = 2 \geq 1$, but the only way to pump the string is to let $v = 0$, $x = \varepsilon$ and $y = 1$ and then $|vxy| = 2 \not\leq 1$.

To see that 2 is a pumping length, let s be in the language with $|s| \geq 2$. Then, $s = 0^n 1^n 2^m 3^m$ fo some n, m.

Case 1: $n > 0$. Then, we can pump s by letting $u = 0^{n-1}, v = 0, x = \varepsilon, y = 1, z = 1^{n-1}2^m3^m$ and we have $|vxy| = 2$.

Case 2: $n = 0$. Then, since $|s| \geq 2$, we must have $m > 0$ and we can pump s by letting $u = 2^{m-1}, v = 2, x = \varepsilon, y = 3, z = 3^{m-1}$ and we have $|vxy| = 2$.

(b) $\{x\#y | x, y \in \{0,1\}^* \text{ and } |x| = 2|y|\}$.

Solution:

The minimum pumping length is 4. To see that 3 is not a pumping length, let $s = 00\#0$. Then, s is in the language and $|s| = 4 \geq 3$, but the only way to pump the string is to let $v = 00$, $x = \#$ and $y = 0$ and then $|vxy| = 4 \not\leq 3$.

To see that 4 is a pumping length, let s be in the language with $|s| \geq 4$. Then, $s = x'\#y'$ for some $x', y' \in \{0,1\}^*$ with $|x'| = 2|y'|$. Since $|s| \geq 4$, we must have $|y'| > 0$ and so $|x'| \geq 2$. We can pump s with u being all but the last two symbols in x', v the last two symbols of x', $x = \#$, y the first symbol of y' and z the rest of y'. We have $|vxy| = 4$, so 4 is a pumpimg length.

5. **Problem 2.18**

(a) Let C be a context-free language and R be a regular language. Prove that the language $C \cap R$ is context-free.

Proof: Let $N = (Q_N, \Sigma, \delta_N, q_0, F_N)$ be a DFA that recognizes R and $M = (Q_M, \Sigma, \Gamma, \delta_M, p_0, F_M)$ be a PDA that recognizes C. The machines N and M are combined to construct a PDA M' that recognizes $C \cap R$. This will show that $C \cap R$ is context-free. A state of M' will be a pair of states (p, q) with p a state of M and q a state of N. M' will simultaneously keep track of a state that M could be in after reading the symbols seen so far and a state that N could be in after reading these symbols. The formal definition is:

$$M' = (Q_M \times Q_N, \Sigma, \Gamma, \delta_{M'}, (p_0, q_0), F_M \times F_N)$$

The transition function $\delta_{M'}$ is defined by

$$\delta_{M'}((p,q), a, x) = \{((p', q'), y) | (p', y) \in \delta_M(p, a, x) \text{ and } \delta_N(q, a) = q'\}$$

for all $p \in Q_M$, $q \in Q_N$, $a \in \Sigma$ and $x \in \Gamma_\varepsilon$ and

$$\delta_{M'}((p,q), \varepsilon, x) = \{((p', q), y) | (p', y) \in \delta_M(p, \varepsilon, x)\}$$

for all $p \in Q_M$, $q \in Q_N$ and $x \in \Gamma_\varepsilon$.

(Every transition of a DFA processes an input symbol while a PDA may contain transitions that do not process input. The transitions just introduced simulate the action of a PDA transition that does not process an input symbol.)

(b) Use the part (a) to show that the language $A = \{w | w \in \{a, b, c\}^*$ and contains equal numbers of a's, b's, and c's$\}$ is not a CFL.

Proof: Assume A is a CFL. Let B be the regular language $a^*b^*c^*$. Then, by Part a, $A \cap B$ is context-free. However, it is easy to see that $A \cap B = \{a^n b^n c^n | n \geq 0\}$, and we know that this language is not context-free. Thus, A is not a CFL.