1. (a) Let $M$ be the special PDA given in the solutions to Problem 1 on Homework 7. Show an accepting computation for $M$ on the string 0000011 by giving a chart with the state, tape contents, and stack contents after each step.

**Solution:**

<table>
<thead>
<tr>
<th>State</th>
<th>Tape</th>
<th>Stack</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s$</td>
<td>000011</td>
<td>$\varepsilon$</td>
</tr>
<tr>
<td>$s$</td>
<td>00011</td>
<td>0</td>
</tr>
<tr>
<td>$s$</td>
<td>0011</td>
<td>00</td>
</tr>
<tr>
<td>$s$</td>
<td>011</td>
<td>000</td>
</tr>
<tr>
<td>$s$</td>
<td>1</td>
<td>0000</td>
</tr>
<tr>
<td>$t$</td>
<td>1</td>
<td>0000</td>
</tr>
<tr>
<td>$u$</td>
<td>1</td>
<td>000</td>
</tr>
<tr>
<td>$t$</td>
<td>$\varepsilon$</td>
<td>0</td>
</tr>
<tr>
<td>$u$</td>
<td>$\varepsilon$</td>
<td>0</td>
</tr>
<tr>
<td>$w$</td>
<td>$\varepsilon$</td>
<td>$#0$</td>
</tr>
<tr>
<td>$w$</td>
<td>$\varepsilon$</td>
<td>0</td>
</tr>
<tr>
<td>$w$</td>
<td>$\varepsilon$</td>
<td>$\varepsilon$</td>
</tr>
</tbody>
</table>

(b) Let $G$ be the CFG equivalent to $M$ that is obtained by using the method from class. Give a parse tree in $G$ for the string 0000011.

**Solution:**
2. Use the Pumping Lemma to show that the following languages are not context-free.

(a) \(\{u\#w\#|u, v, w \in \{0, 1\}^* \text{ and } |u| = |v| = |w|\}\).

Solution: Given \(p \geq 1\), chose \(s = 0^p\#0^p\#0^p\). Then, \(s\) is in the language and \(|s| = 3p + 2 \geq p\). Given \(u, v, x, y, z\) with \(s = uvxyz\), \(|vxy| \leq p\), and \(|vy| > 0\), we choose \(i = 2\). To see that \(uvxvyyz\) is not in the language, we consider cases.

Case 1: Either \(v\) or \(y\) contains a \(\#\). Then, \(uvxvyyz\) contains more than one \(\#\), so it cannot be in the language.

Case 2: Neither \(v\) nor \(y\) contains a \(\#\). Then in \(uvxvyyz\), one or two of the blocks of \(0\)'s is increased in length, but the other block does not change, so the three blocks do not have the same length and \(uvxvyyz\) is not in the language.

(b) \(\{a^n b^m c^n | n \leq r \text{ and } m \leq r\}\).

Solution: Given \(p \geq 1\), chose \(s = a^p b^p c^p\). Then, \(s\) is in the language and \(|s| = 3p \geq p\). Given \(u, v, x, y, z\) with \(s = uvxyz\), \(|vxy| \leq p\), and \(|vy| > 0\), how we choose \(i\) depends on \(v, x, y\).

Case 1: Either \(v\) or \(y\) contains a \(c\). Then, we choose \(i = 0\). In \(uxz\), the number of \(c\)'s is less than \(p\), but since \(|vxy| \leq p\), the number of \(a\)'s in \(uxz\) is still \(p\), so \(uxz\) is not in the language.
Case 2 Neither \( v \) nor \( y \) contains a \( c \). Then we choose \( i = 2 \). In \( uvvxyyz \), either the number of \( a \)'s is greater than \( p \) or the number of \( b \)'s is greater than \( p \), but the number of \( c \)'s is still \( p \), so \( uvvxyyz \) is not in the language.

(c) \( \{ w \in \{ a, b, c \}^* | n_a(w) \leq n_c(w) \text{ and } n_b(w) \leq n_c(w) \} \)

[Here \( n_x(w) \) means the number of occurrences of the symbol \( x \) in the string \( w \).]

Solution: The same proof as given in the previous problem works for this language. The string \( s \) chosen in that proof is in this language, and the pumped string in the previous proof is not in this language.

(d) \( \{ w_1cw_2cw_3cw_4 | w_1 = w_4 \text{ or } w_2 = w_3 \text{ and } w_i \in \{ a, b \}^+ \text{ for } i = 1, 2, 3, 4 \} \)

[This one is hard.]

Solution: Given \( p \geq 1 \), choose \( s = a^pb^pcaacbcb \). Then, \( s \) is in the language and \( |s| = 4p + 5 \geq p \). Given \( u, v, x, y, z \) with \( s = uvxyz \), \( |vxy| \leq p \), and \( |vy| > 0 \), we choose \( i = 0 \). To see that \( uvvxyyz \) is not in the language, we consider cases,

Case 2: Neither \( v \) nor \( y \) contains a \( c \) and either \( v \) or \( y \) is between the first two \( c \)'s, or between the second and third \( c \)'s. Then, in \( uxz \), the string between the first two \( c \)'s or between the second and third \( c \)'s is \( \varepsilon \), so \( uxz \) is not in the language.

Case 3: \( v \) and \( y \) are both to the left of the first \( c \), or are both to the right of the last \( c \). Then in \( uxz \), the string to the left of the first \( c \) is different than the string to the right of the last \( c \), and \( a \neq b \), so \( uxz \) is not in the language.

Case 4: \( v \) is to the left of the first \( c \) and \( y \) to the right of the last \( c \). Then, since \( |vxy| \leq p \), \( v \) consists of \( b \)'s and \( y \) consists of \( a \)'s. Thus \( uxz = a^pb^{p−|v|}caacbcb^{p−|y|}b^p \). Since \( |vy| > 0 \), \( uxz \) is not in the language.

3. What is the minimum value of \( p \) that works in the Pumping Lemma for the following context-free languages?

(a) \( \{ w \in \{ a, b \}^* | n_a(w) = n_b(w) \} \)

Solution: The minimum pumping length is 2. To see that 1 is not a pumping length, consider the string \( s = ab \). This string is in the language and has length 2 \( \geq 1 \), but it cannot be pumped with \( |vxy| \leq 1 \), since then one of \( v \) and \( y \) is \( \varepsilon \) and the other one consists of a single symbol and pumping results in a string that does not have the same number of \( a \)'s as \( b \)'s.

To see that 2 is a pumping length, let \( s \) be a string in the language with \( |s| \geq 2 \). Then \( s \) must contain at least one \( a \) and at least one \( b \). Thus \( s \) must contain an occurrence of either \( ab \) or \( ba \). Letting \( v \) be
this substring and \(x = y = \varepsilon\), we have \(|vxy| = 2\), \(|vy| = 2 > 0\) and \(uv^iwy^iz\) is in the language for all \(i \geq 0\).

(b) \(\{a^n b^m c^k d^m | m \neq k\}\).

Solution:
The minimum pumping length is 3. To see that 2 is not a pumping length, let \(s = abd\). Then, \(s\) is in the language and \(|s| = 3 \geq 2\). If we have \(s = uvxyz\) with \(|vxy| \leq 2\), then \(v\) and \(y\) together cannot contain both \(a\)'s and \(d\)'s, so if either one contains \(a\)'s or \(d\)'s, then in \(uxz\), we do not have the same number of \(a\)'s as \(d\)'s, so \(uxz\) is not in the language. On the other hand, if neither \(v\) nor \(y\) contains \(a\)'s or \(d\)'s, then one of them contains the \(b\), and \(uxz\) contains no \(b\)'s and no \(c\)'s, so is not in the language.

To see that 3 is a pumping length, let \(s\) be in the language with \(|s| \geq 3\). Then, \(s = a^nb^mc^kd^n\) for some \(n, m, k\) with \(m \neq k\). We consider cases to determine how to define \(u, v, x, y, z\).

Case 1: \(m, k > 0\). Then, we pump \(s\) with \(u = a^n b^{m-1}, v = b, x = \varepsilon, y = c, z = c^{k-1} d^n\). We have \(|ux| = 2 \leq 3\).

Case 2: \(m > 1, k = 0\). Then, we pump \(s\) with \(u = a^n b^{m-1}, v = b, x = y = \varepsilon, z = a^n\). We have \(|vxy| = 1 \leq 3\).

Case 3: \(m = 1, k = 0\). Then, since \(|s| \geq 3\), we must have \(n > 0\) and we pump with \(u = a^{n-1}, v = a, x = b, y = a, z = a^{n-1}\). We have \(|vxy| = 3\).

Case 4: \(m = 0, k > 1\). Then, we pump \(s\) with \(u = a^n, v = c, x = y = \varepsilon, z = c^{k-1} d^n\). We have \(|vxy| = 1 \leq 3\).

Case 5: \(m = 0, k = 1\). Then, since \(|s| \geq 3\), we must have \(n > 0\) and we pump \(s\) with \(u = a^{n-1}, v = a, x = c, y = a, z = a^{n-1}\). We have \(|vxy| = 3\).

Since we can’t have \(m = k = 0\), these cases cover all possibilities and 3 is a pumping length.

(c) \(\{w_1 \# w_2 | w_1, w_2 \in \{a, b\}^* \text{ and } n_a(w_1) = n_a(w_2)\}\).

Solution:
The minimum pumping length is 3. To see that 2 is not a pumping length, let \(s = a\#a\). Then, \(s\) is in the language and \(|s| = 3 \geq 2\). If we have \(s = uvxyz\) with \(|vxy| \leq 2\), then \(uxz\) is one of the strings \(a, aa, a\#, #a\), none of which are in the language.

To see that 3 is a pumping length, let \(s\) be in the language with \(|s| \geq 3\). Then, \(s = w_1 \# w_2\), where \(w_1\) and \(w_2\) have the same number of \(a\)'s. We consider cases to determine how to define \(u, v, x, y, z\).

Case 1: Either \(w_1\) or \(w_2\) contains a \(b\). Then we pump \(s\) by letting \(v\) be one of the \(b\)'s in \(s\) and letting both \(x\) and \(y\) be \(\varepsilon\). We have \(|ux| = 1 \leq 3\).

Case 2: Neither \(w_1\) nor \(w_2\) contains a \(b\). Then, we have \(s = a^n \# a^n\) for some \(n\). Since \(|s| \geq 3\), \(n > 0\) and we pump \(s\) by letting \(u = a^{n-1}\), \(v = a, x = \#, y = a\), and \(z = a^{n-1}\). We have \(|vxy| = 3\).
4. **Problem 2.36** Let $A = \{a^i b^j c^k d^l | i = 1 \text{ if } j = k = l \}$.

To see that $A$ meets the conditions of the Pumping Lemma, we choose $p = 1$. If $s \in A$ and $|s| \geq p$, then we have $s = a^i b^j c^k d^l$ for some $i, j, k, l$. We consider cases in choosing $u, v, x, y, z$ as in the Pumping Lemma.

**Case 1:** $j = k = l = 0$. Then $i \neq 0$ since $s$ has length at least 1. We take $u = v = x = \varepsilon$, $y = a$ and $z = a^{i-1}$. We have $|vxy| = 1$ and for all $r \geq 0$, $uv^rxy^rz = a^{i+r-1}$ which is in $A$.

**Case 2:** $i = 1$. Then $j = k = l$. We take $u = v = x = \varepsilon$, $y = a$ and $z = a^{i-1} b^j c^l d^l$. We have $|vxy| = 1$ and for all $r \geq 0$, $uv^rxy^rz = a^{i+r-1} b^j c^l d^l$ which is in $A$.

**Case 3:** $i \neq 1$ and at least one of $j, k, l$ is not 0. Then we take $u = a^i$, $v = x = \varepsilon$, $y = a$ and $z$ to be the rest of the string. Then $|vxy| = 1$ and for all $r \geq 0$, $uv^rxy^rz$ is a string of the form $a^i b^j' c^k' d^l'$, which is in the language since $i \neq 1$.

To see that $A$ is not context-free, we suppose for a contradiction that $A$ is context-free. Then, according to Problem 2.18, so is the language $B = A \cap ab^* c^* d^*$. We will show that $B$ is not context-free and this will be the contradiction we are looking for. $B$ is the language $\{ab^j c^k d^l | j \geq 0\}$. We use the Pumping Lemma to show that $B$ is not context-free. Given $p$, choose $s = ab^pc^kd^l$. Then $s \in B$ and $|s| \geq p$. Given $u, v, x, y, z$ as in the Pumping Lemma, we consider cases.

**Case 1:** Either $v$ or $y$ contains $a$. Then choose $i = 0$. Since $uxz$ does not contain an $a$, it is not in $B$.

**Case 2:** Neither $v$ nor $y$ contains $a$. Then choose $i = 0$. In $uxz$, one or two of the symbols $b, c, d$ has been reduced in number, but the third of these symbols has not been changed. Thus $uxz$ cannot be in $B$ since it does not have the same number of $bs$, $cs$ and $ds$. 
