Put all your answers on the test itself. Be sure to put your name above.

1. Using the method from class (which is the same as the method from Theorem 1.47), give the state diagram of a nondeterministic finite automaton that recognizes the concatenation of the language recognized by the first automaton below with the language recognized by the second automaton below.

Answer:

[Diagram of automaton]
2. Let $N$ be the NFA given by

![Diagram of NFA](image)

and let $M$ be the DFA equivalent to $N$ obtained by the method from class. Answer the following questions about $N$ and $M$. (Put your answers in the blank spaces.)

(a) Does $N$ accept the string $\varepsilon$? Yes
(b) Does $N$ accept the string $ab$? Yes
(c) Does $N$ accept the string $bb$? No
(d) Does $N$ accept the string $bab$? No
(e) What is the start state of $M$? $\{q_0, q_3, q_4\}$
(f) Is $\{q_1, q_2\}$ an accept state of $M$? No
(g) Is $\{q_2, q_3, q_4\}$ an accept state of $M$? Yes
(h) What state does $M$ go to from state $\{q_1, q_2\}$ reading an $a$?
   $\{q_3, q_4\}$
(i) What state does $M$ go to from state $\{q_1, q_2\}$ reading a $b$?
   $\{q_2, q_3, q_4\}$
(j) What state does $M$ go to from state $\{q_2, q_3, q_4\}$ reading an $a$?
   $\{q_3, q_4\}$
(k) What state does $M$ go to from state $\{q_2, q_3, q_4\}$ reading an $b$?
   $\{q_2\}$

[22 points]
3. (a) Suppose that you are in the second step of putting a grammar into Chomsky normal form (eliminate \( \varepsilon \)-rules). You have already eliminated \( Y \rightarrow \varepsilon \) and the current grammar is:

\[
\begin{align*}
S_0 & \rightarrow S \\
S & \rightarrow aYX|aX|XSXS \\
X & \rightarrow \varepsilon|aXb \\
Y & \rightarrow XX|aXY|aX
\end{align*}
\]

Show the grammar that you get by eliminating \( X \rightarrow \varepsilon \). (You are not being asked to eliminate all \( \varepsilon \)-rules, just \( X \rightarrow \varepsilon \).)

\[
\begin{align*}
S_0 & \rightarrow S \\
S & \rightarrow aYX|aX|XSXS|aY|a|XS|XS|SS \\
X & \rightarrow aXb|ab \\
Y & \rightarrow XX|aXY|aX|aY|a
\end{align*}
\]

(b) Suppose that you are in the third step of putting a grammar into Chomsky normal form (eliminate unit rules). You have already eliminated \( A \rightarrow C \) and the current grammar is:

\[
\begin{align*}
S_0 & \rightarrow S \\
S & \rightarrow aAb|bSS|aA \\
A & \rightarrow B|ABC|Aa|a \\
B & \rightarrow C|BaB|aB|SaS \\
C & \rightarrow Aa|a
\end{align*}
\]

Show the grammar that you get by eliminating \( A \rightarrow B \). (You are not being asked to eliminate all unit rules, just \( A \rightarrow B \).)

\[
\begin{align*}
S_0 & \rightarrow S \\
S & \rightarrow aAb|bSS|aA \\
A & \rightarrow ABC|Aa|a|BaB|aB|SaS \\
B & \rightarrow C|BaB|aB|SaS \\
C & \rightarrow Aa|a
\end{align*}
\]

(c) Suppose that in the fourth step of converting a grammar into Chomsky normal form, you have a rule \( Z \rightarrow abWV \). Show the collection of rules of the form \( A \rightarrow BC \) and \( A \rightarrow c \) that you use to replace this one rule in the fourth and fifth steps of converting the grammar.

\( Z \rightarrow AX, X \rightarrow BY, Y \rightarrow WV, A \rightarrow a, B \rightarrow b \)

[22 points]
4. Classify each of the following languages as being i) regular, or ii) context-free but not regular, or iii) not context-free. Justify your answers.

(a) \{w_1cw_2cw_3|w_1, w_2, w_3 \in \{a,b\}^*\}

Classification: i

Explanation:
The language is given by the regular expression:

\[(a \cup b)^*c(a \cup b)^*c(a \cup b)^*\]
(b) \{w_1cw_2cw_3|w_1, w_2, w_3 \in \{a, b\}^* \text{ and } w_1 = w_3\}

Classification: $iii$

Explanation:
Given $p \geq 1$, choose $s = a^pb^ppccapb^p$. Then, $s$ is in the language and $|s| = 4p + 2 \geq p$. Given $u, v, x, y, z$ with $s = uvxyz$, $|vxy| \leq p$ and $|vy| > 0$, choose $i = 2$. We show that $uvvxyyyz$ is not in the language by considering cases.

**Case 1:** $v$ or $y$ contains a $c$. Then $uvvxyyyz$ contains more than two $c$’s, so is not in the language.

**Case 2:** Both $v$ and $y$ are to the left of the first $c$, or both $v$ and $y$ are to the right of the second $c$. Then, in $uvvxyyyz$, the string to the left of the first $c$ does not have the same length as the string to the right of the second $c$, so the two strings cannot be the same and $uvvxyyyz$ is not in the language.

**Case 3:** $v$ is to the left of the first $c$ and $y$ is to the right of the second $c$. Then, since $|vxy| \leq p$, $v$ contains only $b$’s and $y$ contains only $a$’s, so $uvvxyyyz = a^pb^p|v|ccapb^p|y|b^p$. Since either $|v| > 0$ or $|y| > 0$ (or both), $a^pb^p|v| \neq a^p+b|b^p$, so $uvvxyyyz$ is not in the language.
(c) \( \{w_1cw_2cw_3|w_1, w_2, w_3 \in \{a, b\}^* \text{ and } w_1 = w_3^R\} \)

Classification: II

Explanation:

The language is context-free because it is given by the following grammar:

\[
S \rightarrow aSa | bSb | cTc \\
T \rightarrow aT | bT | \varepsilon
\]

We use the Pumping Lemma to show that the language is not regular. Given \( p \geq 1 \), choose \( s = a^pccapa^p \). Then, \( s \) is in the language and \(|s| = 2p + 2 \geq p\). Given \( x, y, z \) with \( s = xyz \), \(|xy| \leq p\), and \(|y| > 0\), \( y \) must be \( a^k \) for some \( k \), \( 1 \leq k \leq p \). Choose \( i = 2 \). Then, \( xyyz = a^{p+k}ccapa^p \) which is not in the language.

[24 points]
5. Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA that recognizes a language $A$. Suppose that $x, y, z$ are strings in $\Sigma^*$ such that $xz \in A$ and $yz \notin A$. Let $p_x$ be the state that $M$ reaches after starting in the initial state and reading $x$, and let $p_y$ be the state that $M$ reaches after starting in the initial state and reading $y$. Explain why $p_x$ and $p_y$ cannot be the same state.

Solution: Let $p_{xz}$ be the state that $M$ reaches starting from $q_0$ and reading $xz$ and let $p_{yz}$ be the state that $M$ reaches starting from $q_0$ and reading $yz$. Since $xz \in A$, $p_{xz} \in F$ and since $yz \notin A$, $p_{yz} \notin F$, so, in particular, $p_{xz} \neq p_{yz}$. Since $p_{xz}$ is the state that $M$ reaches starting from $p_x$ and reading $z$, and $p_{yz}$ is the state that $M$ reaches starting from $p_y$ and reading $z$, if $p_x = p_y$, we would have $p_{xz} = p_{yz}$. Since $p_{xz} \neq p_{yz}$, we must have $p_x \neq p_y$.

[10 points]