Name: ________________________________

Put all your answers on the test itself. Be sure to put your name above.

1. Using the method from class (this is the same method given in the footnote on page 46 of the text), give the state diagram of a DFA that recognizes the intersection of the language recognized by the first DFA below with the language recognized by the second DFA below.

Answer:

[22 points]
2. (a) Let $M$ be the NFA given below.

If you want to transform $M$ into an equivalent regular expression, you have to first modify $M$ into a GNFA $M'$ before you start eliminating states. Give the state diagram of this GNFA $M'$. (You are not being asked to eliminate any states. Just give the GNFA $M'$ that you first transform $M$ into. Do not give any transitions labeled with $\emptyset$.)

Solution:
(b) Suppose that at some point while transforming an NFA into a regular expression you have the following GNFA.

Show the GNFA you would get from this one by eliminating state \( r \). (You are not being asked to convert the GNFA into a regular expression. Just eliminate the state \( r \).)

Solution:

\[
ab^* \cup a^* (aa \cup b)^* (a \cup b) \\
\]

\[
a^* \cup (a \cup \varepsilon) b (aa \cup b)^* (a \cup b) \\
\]

\[
\varepsilon \cup a^* (aa \cup b)^* ba \\
\]

\[
(a \cup \varepsilon) b (aa \cup b)^* ba \\
\]

[22 points]
3. Let $G$ be the grammar

$$
S \rightarrow V a V a T \\
T \rightarrow \varepsilon | a W b T | b V a T \\
W \rightarrow \varepsilon | a W b W \\
V \rightarrow \varepsilon | b V a V 
$$

(This is the grammar given in the solutions to Problem 4 on Homework 4.)

(a) Using the method from class, give a PDA $P$ that recognizes the language generated by $G$. (Leave transitions that push more than one symbol onto the stack in $P$. Do not replace these transitions with ones that only push one symbol at a time.)

**Solution:**

![PDA Diagram]

- $q_{start} \rightarrow \varepsilon, \varepsilon \rightarrow S \$ 
- $\varepsilon, S \rightarrow V a V a T 
- \varepsilon, T \rightarrow \varepsilon 
- \varepsilon, T \rightarrow a W b T 
- \varepsilon, T \rightarrow b V a T 
- \varepsilon, W \rightarrow \varepsilon 
- \varepsilon, W \rightarrow a W b W 
- \varepsilon, V \rightarrow \varepsilon 
- \varepsilon, V \rightarrow b V a V 
- \varepsilon, \$ \rightarrow \varepsilon 

$q_{accept}$
(b) Show the accepting computation of your PDA $P$ on the input string $baaaab$ by filling in the chart on the next page showing the state, tape contents and stack contents after each step. There are 16 steps in the computation and the initial state, tape and stack are filled in. [You will probably want to work out a leftmost derivation in $G$ for the string $baaaab$ before you answer this question, but you are not being asked to show the derivation. You just need to give the accepting computation of $P$.]

<table>
<thead>
<tr>
<th>State</th>
<th>Tape</th>
<th>Stack</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q_{start}$</td>
<td>$baaaab$</td>
<td>$\varepsilon$</td>
</tr>
<tr>
<td>$q_{loop}$</td>
<td>$baaaab$</td>
<td>$S$</td>
</tr>
<tr>
<td>$q_{loop}$</td>
<td>$baaaab$</td>
<td>$VaVaTS$</td>
</tr>
<tr>
<td>$q_{loop}$</td>
<td>$baaaab$</td>
<td>$bVaVaTS$</td>
</tr>
<tr>
<td>$q_{loop}$</td>
<td>$aaab$</td>
<td>$VaVaTS$</td>
</tr>
<tr>
<td>$q_{loop}$</td>
<td>$aaab$</td>
<td>$aVaTS$</td>
</tr>
<tr>
<td>$q_{loop}$</td>
<td>$aab$</td>
<td>$VaTS$</td>
</tr>
<tr>
<td>$q_{loop}$</td>
<td>$aab$</td>
<td>$aTS$</td>
</tr>
<tr>
<td>$q_{loop}$</td>
<td>$ab$</td>
<td>$TS$</td>
</tr>
<tr>
<td>$q_{loop}$</td>
<td>$ab$</td>
<td>$aWbTS$</td>
</tr>
<tr>
<td>$q_{loop}$</td>
<td>$b$</td>
<td>$WbTS$</td>
</tr>
<tr>
<td>$q_{loop}$</td>
<td>$b$</td>
<td>$bTS$</td>
</tr>
<tr>
<td>$q_{loop}$</td>
<td>$\varepsilon$</td>
<td>$TS$</td>
</tr>
<tr>
<td>$q_{loop}$</td>
<td>$\varepsilon$</td>
<td>$S$</td>
</tr>
<tr>
<td>$q_{accept}$</td>
<td>$\varepsilon$</td>
<td>$\varepsilon$</td>
</tr>
</tbody>
</table>

[22 points]
4. Classify each of the following languages as being i) regular, or ii) context-free but not regular, or iii) not context-free. Justify your answers.

(a) \{w_1\#w_2\#w_3|w_1, w_2, w_3 \in \{a, b\}^* \text{ and either } |w_1| = |w_2| \text{ or } |w_2| = |w_3|\}

Classification: ii

Explanation:
The language is context-free because it is given by the following grammar:

\[
\begin{align*}
S & \rightarrow T\#U\#T \\
T & \rightarrow DTD\# \\
U & \rightarrow DU\varepsilon \\
D & \rightarrow a\,b
\end{align*}
\]

We use the pumping lemma to show that the language is not regular. Given \( p \geq 1 \), we choose \( s = a^p\#a^p\# \). Then, \( s \) is in the language and \( |s| = 2p + 2 \geq p \). Given \( x, y, z \) with \( s = xyz \), \( |y| > 0 \) and \( |xy| \leq p \), we must have \( y = a^k \) for some \( k > 0 \). We choose \( i = 2 \). Then, \( xyyz = a^{p+k}\#a^p\# \), which is not in the language since \( p + k > p > 0 \).
(b) \{w \in \{a, b\}^* | w \text{ has even length or } w \text{ ends with an } a\}
Classification: \widetilde{\mathcal{L}}
Explanation:
The language is given by the regular expression
\[(a \cup b)(a \cup b)^* \cup (a \cup b)^*a\]
(c) \{w_1#w_2#w_3|w_1, w_2, w_3 \in \{a, b\}^* \text{ and } |w_1| = |w_2| = |w_3|\}

Classification: iii

Explanation:

We use the pumping lemma to show that the language is not context-free. Given \( p \geq 1 \), choose \( s = a^p#a^p#a^p \). Then, \( s \) is in the language and \( |s| = 3p + 2 \geq p \). Given \( u, v, x, y, z \) with \( s = uvxyz \), \( |vy| > 0 \) and \( |vxy| \leq p \), we choose \( i = 2 \). We consider cases to show that \( uvxyyz \) is not in the language.

**Case 1:** Either \( v \) or \( y \) contains a \#. Then, \( uvxyyz \) contains more than two \#'s, so is not in the language.

**Case 2:** Neither \( v \) nor \( y \) contains a \#. Then, in \( uvxyyz \), one of the two blocks of \( a \)'s has length more than \( p \), while at least one of the blocks still has length \( p \). Thus, \( uvxyyz \) is not in the language.

[24 points]
5. We defined a right regular grammar to be a context-free grammar such that every rule in the grammar has one of the forms $A \rightarrow aB$, or $A \rightarrow a$ or $A \rightarrow \varepsilon$, where $A$ and $B$ are variables and $a$ is a terminal. Let $G_1 = (V_1, \Sigma, R_1, S_1)$ and $G_2 = (V_2, \Sigma, R_2, S_2)$ be two right regular grammars with $V_1 \cap V_2 = \emptyset$. Describe a direct construction of a right regular grammar $G = (V, \Sigma, R, S)$ such that $L(G) = L(G_1) \circ L(G_2)$. (For partial credit, your grammar $G$ can have rules of the form $A \rightarrow B$ with $A$ and $B$ variables, even though these rules are not allowed in right regular grammars.)

**Solution:** We define $V = V_1 \cup V_2$ and $S = S_1$. The set of rules $R$ is defined by

- $R$ contains all rules in $R_2$.
- $R$ contains all rules in $R_1$ of the form $A \rightarrow aB$.
- For each rule of the form $A \rightarrow a$ in $R_1$, $R$ contains $A \rightarrow aS_2$.
- For each rule of the form $A \rightarrow \varepsilon$ in $R_1$ and each rule $S_2 \rightarrow w$ in $R_2$, $R$ contains $A \rightarrow w$.

Note that in the last clause in the definition of $R$, it would be simpler to put into $R$ the rule $A \rightarrow S_2$ for each rule $A \rightarrow \varepsilon$ in $R_1$, but $A \rightarrow S_2$ is not allowed in a right regular grammar, so we had to do the more complicated construction given above. [10 points]