Name: ________________________________

Put all your answers on the test itself. Be sure to put your name above.

1. Let $M_1$ be the Turing machine of Figure 3.10 on page 173 of the text. For each of the following configurations of $M_1$, show the configuration that the given configuration yields, i.e., the next configuration after the given one. (Do not try to trace the entire computation starting from the given configuration. You are only being asked for the next configuration.)

(a) $q_1\#$
    Next configuration: $\#q_4$

(b) $x11\#q_4011$
    Next configuration: $x11q_6\#x11$

(c) $x11\#q_4111$
    Next configuration: $x11\#1q_{\text{reject}}11$

(d) $xxx\#xxxq_8$
    Next configuration: $xxx\#xxx \downarrow q_{\text{accept}}$

[24 points]
2. Apply the method from class that decides $E_{DFA}$ to the following DFA and answer the questions below.

(a) List the states you mark in the order they get marked. 
$p, r, t, u$

(b) Does the DFA belong to $E_{DFA}$? Yes

(c) How does your answer to (b) follow from your answer to (a)?
No accept state is marked.

[22 points]
3. Is it possible to $m$-reduce $E_{CFG}$ to $EQ_{CFG}$?
   Yes X
   No ______

Explain your answer below. (You may use results proven in the book, in class, on the homework, and in the homework solutions without reproving them.)

**Solution:** In class, we showed that $ALL_{CFG} \leq_m EQ_{CFG}$ and the proof that $E_{CFG} \leq_m EQ_{CFG}$ is similar. Let $H$ be the grammar $S \rightarrow S$. Then, $L(H) = \emptyset$, so for any CFG $G$, $⟨G⟩ \in E_{CFG}$ if and only if $⟨G, H⟩ \in EQ_{CFG}$. Thus, if we define $f(⟨G⟩) = ⟨G, H⟩$ (and for completeness, we define $f(w)$ to be a junk string when $w$ is a junk string), then $f$ is an $m$-reduction of $E_{CFG}$ to $EQ_{CFG}$.

[16 points]
4. Let \( B = \{ \langle M \rangle | M \text{ is Turing machine and } L(M) = \{ \varepsilon \} \} \). Suppose that you want to show that \( \text{REJECT}_{TM} \leq_m B \) using a reduction \( f \) that maps \( \langle M, w \rangle \) to \( \langle M_1 \rangle \). (Recall that in the homework, we defined \( \text{REJECT}_{TM} = \{ \langle M, w \rangle | M \text{ is a Turing machine and } M \text{ rejects } w \} \).)

(a) Fill in the blanks in the following three statements in a way that states what you have to do to make the reduction work. (In all cases you will be writing down something about the behavior of the Turing machine \( M_1 \). Make your answers as general as possible.)

- If \( M \) accepts \( w \), then \( L(M_1) \neq \{ \varepsilon \} \).
- If \( M \) rejects \( w \), then \( L(M_1) = \{ \varepsilon \} \).
- If \( M \) loops on \( w \), then \( L(M_1) \neq \{ \varepsilon \} \).

(b) Give the definition of the desired Turing machine \( M_1 \), given \( M \) and \( w \).

\( M_1 = \) “On input \( x \)
1. If \( x \neq \varepsilon \), reject.
2. If \( x = \varepsilon \), run \( M \) on \( w \).
3. If \( M \) rejects, accept. IF \( M \) accepts, reject.”

[18 points]
5. Let $B$ be the set of infinite binary sequences $b = b(1)b(2) \cdots$ such that for all $n$, $b(n) \leq b(n + 1)$. (In other words, if an entry in $b$ is 1, then all later entries in $b$ are 1). Show that $B$ is countably infinite.

**Solution:** For $k \geq 1$, let $b_k$ be the infinite binary sequence where for $i < k$, the $i$th entry in $b_k$ is a 0, and for $i \geq k$, the $i$th entry in $b_k$ is a 1, and let $b_0$ be the infinite sequence all of whose entries are 0. Then, the elements of $B$ can be listed without repetition as $b_0, b_1, b_2, \ldots$, so $B$ is countably infinite.

[10 points]
6. Let $C = \{ \langle M \rangle | M$ is a Turing machine and for some string $w$, $M$ accepts both $w$ and $ww \}$. Prove that $C$ is Turing recognizable.

**Solution:** The proof that $C$ is Turing recognizable is similar to the proof given in class that $\overline{E_{TM}}$ is Turing recognizable.

$C$ is recognized by the following Turing machine $N$:

$N =$ “On input $\langle M \rangle$ where $M$ is a Turing machine

1. Let $s_1, s_2, \ldots$ be the strings over the input alphabet of $M$.
2. For $i = 1, 2, 3, \ldots$
3. Run $M$ on $s_1, s_2, \ldots, s_i$ for $i$ steps and also run $M$ on $s_1s_1, s_2s_2, \ldots, s_is_i$ for $i$ steps. If for any $j$ with $1 \leq j \leq i$, $M$ accepts both $s_j$ and $s_js_j$ in $i$ steps, accept, else next $i$.

[10 points]