https://doi.org/10.1142/9789811278631_0001

Generators of the C.E. Degrees and Strongly Meet Inaccessible Degrees

Klaus Ambos-Spies

Institut für Informatik
University of Heidelberg
Im Neuenheimer Feld 205
D-69120 Heidelberg, Germany
ambos@math.uni-heidelberg.de

Ding Decheng

Department of Mathematics
Nanjing University
Nanjing, Jiangsu Province 210093, P. R. of China
dcding@nju.edu.cn

Peter Fejer

Department of Computer Science
University of Massachusetts Boston
Boston, MA 02125, USA
peter.fejer@umb.edu

We show that any set of computably enumerable (c.e.) degrees which generates the c.e. degrees under join and meet generates the high c.e. degrees under join. This result is obtained by showing that any high c.e. degree is the join of two strongly meet inaccessible (s.m.i.) degrees. Here a c.e. degree **a** is s.m.i. if any set of c.e. degrees which generates **a** under join and meet generates **a** under join. Moreover, by a further existence result for the s.m.i. degrees, we answer a question on the possible ranks of (definable) generators raised in Ambos-Spies [3] (where the rank of a generator is the supremum of the number of elements needed for generating any c.e. degree): there are definable generators of infinite rank and of rank n for any number $n \geq 1$.

Contents

1. Introduction

The partial ordering $(\mathbf{R}, \leq)$ of the computably enumerable degrees is an upper semilattice — i.e., for any c.e. degrees $\mathbf{a}$ and $\mathbf{b}$, their join $\mathbf{a} \vee \mathbf{b}$ exists — but not a lattice. In fact, as Lachlan [11] and Yates [21] have shown, for an incomparable pair of c.e. degrees, the meet may or may not exist. This asymmetry between joins and meets is further demonstrated by the fact that, by Sacks's splitting theorem [15], any nonzero c.e. degree is join-reducible (splits), i.e., the join of two lesser c.e. degrees, whereas, as shown by Lachlan [11] and Yates [21], an incomplete c.e. degree may be meet-reducible (branching), i.e., the meet of two greater c.e. degrees, or not. Presence and failure of meets are homogeneously distributed in the c.e. degrees: as Slaman [17] has shown, any nonempty interval of c.e. degrees contains an incomparable pair possessing a meet, hence a branching degree, whereas, by Ambos-Spies [2] and Fejer [10], respectively, any nonempty interval contains a pair without meet and a nonbranching degree. Despite this homogeneity, other results indicate that the lack of meets is more typical than their existence. So, as Ambos-Spies [2] and, independently, Harrington (unpublished) have shown, there is a degree $\mathbf{a} \neq \mathbf{0}, \mathbf{0}'$ such that $\mathbf{a}$ is not half of an incomparable pair with meet whereas any degree $\mathbf{a} \neq \mathbf{0}, \mathbf{0}'$ is half of an incomparable pair without meet.

The study of generators of the c.e. degrees sheds more light on the meet operator. Here a set $\mathbf{A}$ of c.e. degrees is a generator (of $\mathbf{R}$) if any c.e. degree is in the closure of $\mathbf{A}$ under join and meet, and $\mathbf{A}$ is a join (meet) generator if any nonzero (incomplete) c.e. degree is in the closure of $\mathbf{A}$ under join (meet). By Fejer's density theorem for the nonbranching degrees, any meet generator is dense whereas, by Sacks's splitting theorem, the set $\mathbf{L}$ of the low c.e. degrees is a join generator. Moreover, Ambos-Spies [3] has shown that any generator intersects any nonempty initial segment. So, for example, the set of the nonlow c.e. degrees is not a generator. These results together with the negative result on meets listed above, led Ambos-Spies to conjecture that, in the process of generating the c.e. degrees, the meet operator can be neglected, i.e., that any set which generates the c.e. degrees generates the nonzero c.e. degrees under join.

This conjecture was refuted in Ambos-Spies, Lempp and Slaman [6] where a generator is constructed which is not a join generator. Despite this negative answer, it is reasonable to ask whether there are some natural substantial parts of $\mathbf{R}$ which are generated under join by any generator. Our main result shows that this is indeed the case: any generator generates the high c.e. degrees under join (Corollary 2.9). In order to obtain this result we introduce and study the strongly meet inaccessible (s.m.i.) degrees. Here a c.e. degree $\mathbf{a}$ is s.m.i. if any set of c.e. degrees which generates $\mathbf{a}$ under join and meet generates $\mathbf{a}$ under join. So, in order to get our main result, it suffices to show that any high c.e. degree can be split into two s.m.i. degrees (Theorem 2.8).

Strong meet inaccessibility is a refinement of meet inaccessibility introduced in Ambos-Spies [3]. A c.e. degree $\mathbf{a}$ is meet inaccessible if no set $\mathbf{A}$ of c.e. degrees which does not intersect the lower cone of $\mathbf{a}$ generates $\mathbf{a}$. So, in order to show that any generator intersects any nontrivial initial segment of $\mathbf{R}$, it suffices to show that any nonzero c.e. degree bounds a meet inaccessible degree. In fact, Ambos-Spies [3] shows that any nonzero degree can be split into two meet inaccessible degrees, and this result was subsequently strengthened by Zhang [22] and Ding [8] who showed that the set $\mathbf{MI}$ of the meet inaccessible degrees is dense in $\mathbf{R}$. As we show here the corresponding results fail for strong meet inaccessibility (see Lemmas 2.4 and 2.6 below). So the construction of strongly meet inaccessible degrees is considerably more delicate than that of meet inaccessible degrees.

We demonstrate the usefulness of the s.m.i. degrees by some further results on generators related to their ranks. Ambos-Spies [3] defines the rank of a generator $\mathbf{G}$ to be the least number $n \geq 1$ such that any c.e. degree can be generated by a subset of $\mathbf{G}$ of cardinality at most n, and he defines the rank of $\mathbf{G}$ to be ω (infinite) if no such number n exists. The join rank of a join generator is defined similarly. Now Ambos-Spies [3] has shown that there are join generators of infinite rank and definable join generators of rank n for any $n \geq 1$. He raised the question whether the corresponding results for generators and their ranks in place of join generators and their join ranks hold (see Problem 3 in [3]). Here, by refining the approach in [3] and by proving an appropriate existence result for s.m.i. degrees (Theorem 2.12), we affirmatively answer this question. In fact we show that there is a *definable* generator of rank ω (Theorem 2.11) and that for, any $n \geq 1$, there are definable generators which have rank and join rank n (Theorem 2.10).

The outline of this chapter is as follows. In Sec. 2, we present our main results. Here we first introduce strong meet inaccessibility and study some of the basic properties of this notion and its relation to the meet inaccessibility notion introduced in [3]. In particular, we give more handy characterizations of, and sufficient conditions for, strong meet inaccessibility (which subsequently are used in the constructions of s.m.i. degrees). Then we present our main results and state the existence theorems for s.m.i. degrees needed for the proofs. The proofs of these existence theorems are deferred to the subsequent sections. In Sec. 3, we explain the ideas underlying our constructions of s.m.i. degrees by constructing a low s.m.i. degree. Then in Secs. 4 and 5, this construction is refined in order to prove the required high-splitting theorem and bounding theorem for s.m.i. degrees which we need for our results on generators. Finally, in Sec. 6 we pose some open problems.

We conclude this section by giving some notation. Although we use the term "c.e." here sometimes for emphasis, all degrees considered in this paper are c.e. So boldface lower case letters denote c.e. degrees and boldface capital letters denote sets of c.e. degrees. We let $\mathbf{R}(\leq \mathbf{a}) = \{\mathbf{b} : \mathbf{b} \leq \mathbf{a}\}$ be the lower cone of $\mathbf{a}$. Similarly, $\mathbf{R}(\not\leq \mathbf{a}) = \{\mathbf{b} : \mathbf{b} \not\leq \mathbf{a}\}$ denotes the complement of the lower cone of $\mathbf{a}$ (etc.).

Related to generators we will use the following notation. For any set $\mathbf{A}$ of c.e. degrees we let $\mathrm{CL}(\mathbf{A})$ denote the closure of $\mathbf{A}$ (under join and meet), i.e., $\mathrm{CL}(\mathbf{A})$ is the least set $\mathbf{B}$ of c.e. degrees such that

$$\mathbf{A} \subseteq \mathbf{B} \tag{1.1}$$

$$\forall\, n \geq 0\, \forall\, \mathbf{a}_0, ..., \mathbf{a}_n \in \mathbf{B}\ (\mathbf{a}_0 \vee ... \vee \mathbf{a}_n \in \mathbf{B}) \tag{1.2}$$

and

$$\forall\, n \geq 0\, \forall\, \mathbf{a}_0, ..., \mathbf{a}_n \in \mathbf{B}\ (\mathbf{a}_0 \wedge ... \wedge \mathbf{a}_n \downarrow \Rightarrow\ \mathbf{a}_0 \wedge ... \wedge \mathbf{a}_n \in \mathbf{B}). \tag{1.3}$$

The closure of $\mathbf{A}$ under join is obtained by omitting clause (1.3) and is denoted by $\mathrm{CL}_j(\mathbf{A})$; the closure of $\mathbf{A}$ under meet is obtained by omitting clause (1.2) and is denoted by $\mathrm{CL}_m(\mathbf{A})$. We say that $\mathbf{A}$ generates $\mathbf{B}$ if $\mathbf{B}$ is contained in the closure of $\mathbf{A}$, $\mathbf{A}$ generates $\mathbf{B}$ under join if $\mathbf{B}$ is contained in the closure of $\mathbf{A}$ under join, and $\mathbf{A}$ generates $\mathbf{B}$ under meet if $\mathbf{B}$ is contained in the closure of $\mathbf{A}$ under meet. Similarly, we say that $\mathbf{A}$ generates $\mathbf{b}$ (under join, under meet) if $\mathbf{A}$ generates $\{\mathbf{b}\}$ (under join, under meet). Finally, we call $\mathbf{A}$ a generator if $\mathbf{A}$ generates $\mathbf{R}$ and we call $\mathbf{A}$ a join generator if $\mathbf{A}$ generates $\mathbf{R} \setminus \{\mathbf{0}\}$ under join. (Note that, by the existence of minimal pairs, $\mathbf{A}$ is a generator iff $\mathbf{A}$ generates $\mathbf{R} \setminus \{\mathbf{0}\}$.)

Since $\mathbf{R}$ is an upper semi-lattice, the above definitions will not change if we consider in (1.2) the binary case only, i.e., fix $n = 1$. A similar change in (1.3), however, is not admissible (see Ambos-Spies [3]).

For any subset $\mathbf{A}$ of $\mathbf{R}$, it is clear that $\mathbf{B} = \mathbf{R}$ satisfies conditions (1.1) to (1.3) and further, the intersection of any non-empty family of subsets of $\mathbf{R}$, each of which satisfies these three conditions, again satisfies the three conditions, so for any subset $\mathbf{A}$ of $\mathbf{R}$, $\mathrm{CL}(\mathbf{A})$ exists, and similarly, $\mathrm{CL}_j(\mathbf{A})$ and $\mathrm{CL}_m(\mathbf{A})$ exist. It is useful, however, to have "bottom-up" descriptions of these closures. It is easy to see that for any subset $\mathbf{A}$ of $\mathbf{R}$, we have

$$\mathrm{CL}_j(\mathbf{A}) = \{\mathbf{a}_0 \vee \cdots \vee \mathbf{a}_n : n \geq 0 \;\&\; \mathbf{a}_0, ..., \mathbf{a}_n \in \mathbf{A}\}$$

and

$$\mathrm{CL}_m(\mathbf{A}) = \{\mathbf{a}_0 \wedge \cdots \wedge \mathbf{a}_n : n \geq 0 \;\&\; \mathbf{a}_0, ..., \mathbf{a}_n \in \mathbf{A} \;\&\; \mathbf{a}_0 \wedge \cdots \wedge \mathbf{a}_n \downarrow\}.$$

To give a bottom-up description of the closure of a subset $\mathbf{A}$ of $\mathbf{R}$, we define $\mathrm{CL}^0(\mathbf{A}) = \mathbf{A}$, and, for $k \geq 0$, $\mathrm{CL}^{2k+1}(\mathbf{A}) = \mathrm{CL}_j(\mathrm{CL}^{2k}(\mathbf{A}))$, and $\mathrm{CL}^{2k+2}(\mathrm{A}) = \mathrm{CL}_m(\mathrm{CL}^{2k+1}(\mathbf{A}))$. Then we have

$$\mathbf{A} = \mathrm{CL}^0(\mathbf{A}) \subseteq \mathrm{CL}^1(\mathbf{A}) \subseteq \mathrm{CL}^2(\mathbf{A}) \subseteq \cdots$$

and

$$\mathrm{CL}(\mathbf{A}) = \bigcup_{i \geq 0} \mathrm{CL}^i(\mathbf{A}).$$

2. Main Section

Here we present our new results on generators. They are derived from theorems on the distribution of the strongly meet inaccessible degrees. The proofs of the latter, which from a technical point of view form the core of this chapter, are given in the subsequent three sections. We first introduce strong meet inaccesibility and give some basic properties of this new notion.

Definition 2.1: A c.e. degree $\mathbf{a}$ is *strongly meet inaccessible* (s.m.i.) if

$$\forall \mathbf{A} \subseteq \mathbf{R}(\mathbf{a} \in \mathrm{CL}(\mathbf{A}) \Rightarrow \mathbf{a} \in \mathrm{CL}_j(\mathbf{A})). \tag{2.1}$$

The set of the s.m.i. degrees is denoted by **SMI**.

We obtain an alternative characterization of the s.m.i. degrees $\mathbf{a}$ by looking at the relation between the set of the degrees generated by the complement of the lower cone of $\mathbf{a}$ and the set of the degrees which cup to $\mathbf{a}$. Here we call a degree $\mathbf{b}$ $\mathbf{a}$-*cuppable* if $\mathbf{b} \leq \mathbf{a}$ and there is a degree $\mathbf{c} < \mathbf{a}$ such that $\mathbf{a} = \mathbf{b} \vee \mathbf{c}$ and we let

$$\mathrm{Cu}(\mathbf{a}) = \{\mathbf{b} \leq \mathbf{a} : \ \mathbf{b} \text{ is } \mathbf{a}\text{-cuppable}\} \ \text{ and } \ \mathrm{NCu}(\mathbf{a}) = \mathbf{R}(\leq \mathbf{a}) \setminus \mathrm{Cu}(\mathbf{a}).$$

Lemma 2.2: *Let* $\mathbf{a}$ *be a c.e. degree such that* $\mathbf{a} > \mathbf{0}$. *The following are equivalent.*

(i) $\mathbf{a}$ *is strongly meet inaccessible.*
(ii) $\mathbf{R}(\not\leq \mathbf{a}) \cup \mathrm{NCu}(\mathbf{a})$ *is closed under join and meet.*
(iii) $\mathrm{CL}(\mathbf{R}(\not\leq \mathbf{a})) \cap \mathrm{Cu}(\mathbf{a}) = \emptyset$.
(iv) $\mathrm{CL}_m(\mathbf{R}(\not\leq \mathbf{a})) \cap \mathrm{Cu}(\mathbf{a}) = \emptyset$.

Proof: The proof of the implication $(i) \Rightarrow (ii)$ is by contraposition. Assume that $\mathbf{R}(\not\leq \mathbf{a}) \cup \mathrm{NCu}(\mathbf{a})$ is not closed under join and meet. Fix $\mathbf{b}$ such that $\mathbf{b} \in \mathrm{CL}(\mathbf{R}(\not\leq \mathbf{a}) \cup \mathrm{NCu}(\mathbf{a}))$ and $\mathbf{b} \in \mathrm{Cu}(\mathbf{a})$. By the latter, take $\mathbf{c} < \mathbf{a}$ such that $\mathbf{a} = \mathbf{b} \vee \mathbf{c}$. Then, for $\mathbf{A} = \mathbf{R}(\not\leq \mathbf{a}) \cup \mathrm{NCu}(\mathbf{a}) \cup \{\mathbf{c}\}$, $\mathbf{a}$ is in the closure of $\mathbf{A}$. But since $\mathrm{NCu}(\mathbf{a})$ is an ideal, $\mathbf{a}$ is not in the join closure of $\mathbf{A}$.

The implications $(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (iv)$ are immediate.

Finally, for a proof of $(iv) \Rightarrow (i)$ assume

$$\mathrm{CL}_m(\mathbf{R}(\not\leq \mathbf{a})) \cap \mathrm{Cu}(\mathbf{a}) = \emptyset. \tag{2.2}$$

Since, by $\mathbf{a} > \mathbf{0}$, $\mathbf{a}$ is $\mathbf{a}$-cuppable, it follows that $\mathbf{a}$ cannot be expressed as a meet $\mathbf{c_0} \wedge \cdots \wedge \mathbf{c_n}$ with each $\mathbf{c_i} > \mathbf{a}$, i.e., $\mathbf{a}$ is nonbranching.

Now, given $\mathbf{A} \subseteq \mathbf{R}$ and $\mathbf{a} \in \mathrm{CL}(\mathbf{A})$, we have to show that $\mathbf{a} \in \mathrm{CL}_j(\mathbf{A})$. Take p minimal with $\mathbf{a} \in \mathrm{CL}^p(\mathbf{A})$. By eliminating the other possibilities, we show that $p \leq 1$, and hence $\mathbf{a} \in \mathrm{CL}_j(\mathbf{A})$.

If $p = 2k+2$ for $k \geq 0$, then $\mathbf{a} = \mathbf{c_0} \wedge \cdots \wedge \mathbf{c_n}$ with each $\mathbf{c_i} \in \mathrm{CL}^{2k+1}(\mathbf{A})$. By minimality of p, none of the $\mathbf{c_i}$ is equal to $\mathbf{a}$. This contradicts the fact that $\mathbf{a}$ is nonbranching.

If $p = 2k + 1$ with $k > 0$, then $\mathbf{a} = \mathbf{b_0} \vee \cdots \vee \mathbf{b_n}$ with each $\mathbf{b_i} \in \mathrm{CL}^{2k}(\mathbf{A})$. Taking n minimal, each $\mathbf{b_i}$ is in $\mathrm{Cu}(\mathbf{a})$. Since $k > 0$, each $\mathbf{b_i}$ can be expressed as a meet of elements in $\mathrm{CL}^{2k-1}(\mathbf{A})$. By (2.2), for each i, at least one of these elements, say $\mathbf{c_i}$, is less than or equal to $\mathbf{a}$. But then we have $\mathbf{a} = \mathbf{c_0} \vee \cdots \vee \mathbf{c_n}$ with each $\mathbf{c_i}$ in $\mathrm{CL}^{2k-1}(\mathbf{A})$, which implies that $\mathbf{a} \in \mathrm{CL}^{2k-1}(\mathbf{A})$, contradicting minimality of p. □

The following lemma, which provides a more handy sufficient condition for strong meet inaccessibility, gives the method we will use to construct s.m.i. degrees.

Lemma 2.3: *Let* $\mathbf{a}$ *be a c.e. degree such that* $\mathbf{a} > \mathbf{0}$ *and*

$$\forall \mathbf{b} \in \mathrm{Cu}(\mathbf{a}) \forall \mathbf{c_0}, \mathbf{c_1} > \mathbf{b}(\mathbf{c_0}, \mathbf{c_1} \not\leq \mathbf{a} \Rightarrow \exists \mathbf{d} \leq \mathbf{c_0}, \mathbf{c_1}(\mathbf{d} \not\leq \mathbf{a})). \tag{2.3}$$

Then $\mathbf{a}$ *is strongly meet inaccessible.*

Proof: By Lemma 2.2, it suffices to show that (2.2) holds. For a contradiction, suppose that $\mathbf{b} \in \mathrm{Cu}(\mathbf{a})$ and $\mathbf{b} = \mathbf{c_0} \wedge \cdots \wedge \mathbf{c_n}$ with each $\mathbf{c_i} \not\leq \mathbf{a}$. We assume that n is minimal and rule out all possibilities. Since $\mathbf{b} \leq \mathbf{a}$, $n = 0$ is impossible. Our assumption (2.3) rules out $n = 1$. If $n > 1$, then by (2.3), there is $\mathbf{d}$ with $\mathbf{d} \leq \mathbf{c_0}, \mathbf{c_1}$ and $\mathbf{d} \not\leq \mathbf{a}$. Letting $\mathbf{d_1} = \mathbf{d} \vee \mathbf{a}$, we have $\mathbf{d_1} \not\leq \mathbf{a}$ and $\mathbf{b} = \mathbf{d_1} \wedge \mathbf{c_2} \wedge \cdots \wedge \mathbf{c_n}$, contradicting minimality of n. □

Next, we look at some necessary (but not sufficient) conditions for strong meet inaccessibility. Obviously, any strongly meet inaccessible degree is nonbranching. In fact, strong meet inaccessibility is a proper strengthening of meet inaccessibility introduced in Ambos-Spies [3]. A c.e. degree $\mathbf{a}$ is *meet accessible* if there is a set $\mathbf{A}$ of c.e. degrees generating $\mathbf{a}$ such that $\mathbf{A}$ does not intersect the lower cone of $\mathbf{a}$; and $\mathbf{a}$ is *meet inaccessible* otherwise. As shown in [3], a c.e. degree $\mathbf{a}$ is meet inaccessible iff $\mathbf{a} > \mathbf{0}$ and $\mathbf{a} \notin \mathrm{CL}(\mathbf{R}(\not\leq \mathbf{a}))$. So, it is immediate by Lemma 2.2 that any s.m.i. degree is meet inaccessible. In [3], meet inaccessible degrees are used in order to show that, for any c.e. degree $\mathbf{a} > \mathbf{0}$ and any generator $\mathbf{G}$, $\mathbf{G}$ intersects the lower cone of $\mathbf{a}$. For this sake, it is observed that, for any degree $\mathbf{a}$ which bounds a meet inaccessible degree, the complement $\mathbf{R}(\not\leq \mathbf{a})$ of the lower cone of $\mathbf{a}$ does not generate $\mathbf{R}$, and it is shown that any nonzero degree bounds a meet inaccessible degree. In fact, Ambos-Spies [3] shows that any nonzero c.e. degree is the join of two meet inaccessible degrees whence the set $\mathbf{MI}$ of the meet inaccessible degrees is a join generator. Later on, the latter has been strengthened by Zhang [22] and Ding [8] who showed that $\mathbf{MI}$ is dense in $\mathbf{R}$. In contrast, it easily follows from results in the literature that the set of the strongly meet inaccessible degrees is not dense in $\mathbf{R}$. In fact, the following is true.

Lemma 2.4: *For any c.e. degree* $\mathbf{a} > \mathbf{0}$ *there are c.e. degrees* $\mathbf{b}$ *and* $\mathbf{c}$ *such that* $\mathbf{c} < \mathbf{b} \leq \mathbf{a}$ *and no c.e. degree in the closed interval* $[\mathbf{c}, \mathbf{b}]$ *is strongly meet inaccessible.*

Proof: Fix $\mathbf{a} > \mathbf{0}$. Since any nonzero c.e. degree bounds a contiguous degree (Ladner and Sasso [13]), w.l.o.g. we may assume that $\mathbf{a}$ is contiguous. Moreover, since any countable distributive lattice can be embedded into any nontrivial principal ideal of $\mathbf{R}$ by a map which preserves the greatest element (Ambos-Spies, Ding, Fejer [5]), we may fix pairwise incomparable c.e. degrees $\mathbf{a_0}, \mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}$ which generate (as atoms) the four-atom Boolean algebra in $\mathbf{R}(\leq \mathbf{a})$ with greatest element $\mathbf{a} = \mathbf{a_0} \vee \mathbf{a_1} \vee \mathbf{a_2} \vee \mathbf{a_3}$. So, since contiguous degrees have the covering property (below we come back to this in more detail when we discuss the possible ranks of generators), for any c.e. degree $\mathbf{d} \leq \mathbf{a}$ there are c.e. degrees $\mathbf{d_i} \leq \mathbf{a_i}$ $(i \leq 3)$ such that $\mathbf{d} = \mathbf{d_0} \vee \mathbf{d_1} \vee \mathbf{d_2} \vee \mathbf{d_3}$.

Now, let $\mathbf{b} = \mathbf{a_0} \vee \mathbf{a_1}$ and $\mathbf{c} = \mathbf{a_0}$. Then, given $\mathbf{d}$ such that $\mathbf{c} \leq \mathbf{d} \leq \mathbf{b}$, we have to show that $\mathbf{d}$ is not s.m.i. Note that $\mathbf{a_0} \leq \mathbf{d}$ and $\mathbf{a_0} = (\mathbf{a_0} \vee \mathbf{a_2}) \wedge (\mathbf{a_0} \vee \mathbf{a_3})$ where, by $\mathbf{d} \leq \mathbf{a_0} \vee \mathbf{a_1}$, it holds that $\mathbf{a_0} \vee \mathbf{a_2} \not\leq \mathbf{d}$ and $\mathbf{a_0} \vee \mathbf{a_3} \not\leq \mathbf{d}$. So, by Lemma 2.2, it suffices to show that $\mathbf{a_0}$ is $\mathbf{d}$-cuppable. Fix degrees $\mathbf{d_i} \leq \mathbf{a_i}$ $(i \leq 3)$ such that $\mathbf{d} = \mathbf{d_0} \vee \mathbf{d_1} \vee \mathbf{d_2} \vee \mathbf{d_3}$. Since $\mathbf{d} \leq \mathbf{a_0} \vee \mathbf{a_1}$, it follows that, for $j = 2, 3$, $\mathbf{d_j} \leq (\mathbf{a_0} \vee \mathbf{a_1}) \wedge \mathbf{a_j}$ hence $\mathbf{d_j} \leq \mathbf{a}_1$ by choice of the degrees $\mathbf{a_0}, \mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}$. So, for $\hat{\mathbf{d}} = \mathbf{d_1} \vee \mathbf{d_2} \vee \mathbf{d_3}$, $\hat{\mathbf{d}} \leq \mathbf{a}_1$ hence $\hat{\mathbf{d}} < \mathbf{d}$. Since, by $\mathbf{d_0} \leq \mathbf{a_0} \leq \mathbf{d}$, $\mathbf{d} = \mathbf{a_0} \vee \hat{\mathbf{d}}$, this implies the claim. □

The existence of generators which are not join generators, proven by Ambos-Spies, Lempp and Slaman [6], implies that the set of strongly meet inaccessible degrees is not a join generator. In order to show this, we use the following

Proposition 2.5: *Let* $\mathbf{A}$ *be a set of degrees generated by the set* $\mathbf{SMI}$ *of the strongly meet inaccessible degrees under join. Then, for any generator* $\mathbf{G}$*,* $\mathbf{G}$ *generates* $\mathbf{A}$ *under join.*

Proof: For a generator $\mathbf{G}$, $\mathbf{SMI} \subseteq \mathrm{CL}(\mathbf{G})$ hence $\mathbf{SMI} \subseteq \mathrm{CL}_j(\mathbf{G})$ by (2.1). Since $\mathbf{A} \subseteq \mathrm{CL}_j(\mathbf{SMI})$, it follows that $\mathbf{A} \subseteq \mathrm{CL}_j(\mathbf{G})$. □

Lemma 2.6: *The set* $\mathbf{SMI}$ *of the strongly meet inaccessible degrees is not a join generator.*

Proof: By Ambos-Spies, Lempp and Slaman [6], let $\mathbf{G}$ be a generator which does not generate $\mathbf{R} \setminus \{\mathbf{0}\}$ under join. Then, by Proposition 2.5, $\mathbf{SMI}$ does not generate $\mathbf{R} \setminus \{\mathbf{0}\}$ under join. □

Having given some basic properties of the s.m.i. degrees, we now turn to our main results. We first look at the question which degrees $\mathbf{a}$ are generated under join by *any* generator. By the existence of generators which are not join generators [6], there are degrees $\mathbf{a}$ which do not have this property. In fact, any join generator contains such a degree. So, for instance, there are a low degree $\mathbf{a}$ and a generator $\mathbf{G}$ such that $\mathbf{G}$ does not generate $\mathbf{a}$ under join. On the other hand, any strongly meet inaccessible $\mathbf{a}$ is generated under join by any generator. In fact, Proposition 2.5 shows that any degree which is generated under join by s.m.i. degrees has this property. A trivial example of a degree which is generated under join by any generator is the complete degree $\mathbf{0}'$ which, obviously is s.m.i. Our first technical main theorem gives a first nontrivial example of such a degree, namely a low degree.

Theorem 2.7: *There is a low c.e. degree* $\mathbf{a}$ *which is strongly meet inaccessible.*

The proof of Theorem 2.7 — which explains the basic construction of nontrivial s.m.i. degrees and which will be the basis for the other constructions of s.m.i. degrees — is given in Sec. 3.

Our main result in this direction, namely that any high degree is generated under join by any generator, is obtained along the same lines by proving the following.

Theorem 2.8: *Every high c.e. degree* $\mathbf{a}$ *is the join of two strongly meet inaccessible degrees.*

The proof of Theorem 2.8 is given in Sec. 4.

Corollary 2.9: *If* $\mathbf{G}$ *is a generator then* $\mathbf{G}$ *generates the set* $\mathbf{H}$ *of the high c.e. degrees under join.*

Proof: This is immediate by Theorem 2.8 and Proposition 2.5. □

In the remainder of this section we apply the strongly meet inaccessible degrees to some other questions on generators. We solve some open problems on the rank of (definable) generators raised in Ambos-Spies [3]. First, we recall the definition of the rank and join rank given in [3].

If a degree $\mathbf{a}$ is in the closure of a class $\mathbf{A}$ then

$$\text{rank}_{\mathbf{A}}(\mathbf{a}) = \min\{n : \exists\, \mathbf{a}_1, \dots, \mathbf{a}_n \in \mathbf{A}\ (\mathbf{a} \in \text{CL}(\{\mathbf{a}_1, \dots, \mathbf{a}_n\}))\},$$

and, similarly, if $\mathbf{a}$ is in the closure of $\mathbf{A}$ under join then

$$\text{j-rank}_{\mathbf{A}}(\mathbf{a}) = \min\{n : \exists\, \mathbf{a}_1, \dots, \mathbf{a}_n \in \mathbf{A}\ (\mathbf{a} \in \text{CL}_j(\{\mathbf{a}_1, \dots, \mathbf{a}_n\}))\}.$$

Then the *rank* of a generator $\mathbf{G}$, rank($\mathbf{G}$) for short, is the least number $n \geq 1$ such that $\text{rank}_{\mathbf{G}}(\mathbf{a}) \leq n$ for all $\mathbf{a} \in \mathbf{R}$ if such an n exists, and $\text{rank}(\mathbf{G}) = \omega$ otherwise. Similarly, the *join rank* (*j-rank* for short) of a join-generator $\mathbf{J}$, j-rank($\mathbf{J}$), is the least number $n \geq 1$ such that $\text{j-rank}_{\mathbf{J}}(\mathbf{a}) \leq n$ for all $\mathbf{a} \in \mathbf{R}$ if such an n exists, and $\text{j-rank}(\mathbf{J}) = \omega$ otherwise. In [3] it is shown that, for any $n \geq 1$, there is a definable join generator $\mathbf{J}_n$ of join rank n and that there is a join-generator $\mathbf{J}_\omega$ of infinite join rank, and the question is raised whether these results on the join rank of join generators carry over to ranks and generators (see Problem 3 in [3]). Here, we answer this question affirmatively. Moreover, we show that there is a *definable* generator of infinite rank and that the required generators can be chosen so that they are join generators and that their ranks and join ranks agree:

Theorem 2.10: *For any $n \geq 1$ there is a definable join generator $\mathbf{G}_n$ of rank n and join rank n.*

Theorem 2.11: *There is a definable join generator $\mathbf{G}_\omega$ of infinite rank (hence infinite join rank).*

For the proofs of Theorems 2.10 and 2.11, we need the following existence result for s.m.i. degrees which is proved in Sec. 5, as well as a simple observation relating the rank of a s.m.i. degree to its join rank.

Theorem 2.12: *Let $\mathbf{b}_0, \dots, \mathbf{b}_{n-1}$ ($n \geq 1$) be nonzero c.e. degrees. There are c.e. degrees $\mathbf{a}_0, \dots, \mathbf{a}_{n-1}$ such that $\mathbf{0} < \mathbf{a}_i \leq \mathbf{b}_i$ for $i < n$ and $\mathbf{a}_0 \vee \dots \vee \mathbf{a}_{n-1}$ is s.m.i.*

Proposition 2.13: *Assume that $\mathbf{a}$ is s.m.i. and $\mathbf{a} \in \text{CL}(\mathbf{A})$. Then $\mathbf{a} \in \text{CL}_j(\mathbf{A})$ and*

$$\text{rank}_{\mathbf{A}}(\mathbf{a}) = \text{j-rank}_{\mathbf{A}}(\mathbf{a}).$$

Proof: The first part of the claim is immediate by strong meet inaccessibility of $\mathbf{a}$. So, since, for $\mathbf{a} \in \text{CL}_j(\mathbf{A})$, $\text{rank}_{\mathbf{A}}(\mathbf{a}) \leq \text{j-rank}_{\mathbf{A}}(\mathbf{a})$, it suffices to show that $\text{j-rank}_{\mathbf{A}}(\mathbf{a}) \leq \text{rank}_{\mathbf{A}}(\mathbf{a})$. Assume that $\text{rank}_{\mathbf{A}}(\mathbf{a}) = n$ and fix $\mathbf{a}_0, \dots, \mathbf{a}_{n-1} \in \mathbf{A}$ such that $\mathbf{a} \in \text{CL}(\mathbf{a}_0, \dots, \mathbf{a}_{n-1})$. Then, by strong meet inaccessibility of $\mathbf{a}$, $\mathbf{a} \in \text{CL}_j(\mathbf{a}_0, \dots, \mathbf{a}_{n-1})$ hence, by $\{\mathbf{a}_0, \dots, \mathbf{a}_{n-1}\} \subseteq \mathbf{A}$, $\text{j-rank}_{\mathbf{A}}(\mathbf{a}) \leq n$. □

Just as the proof of the related results in [3], the proofs of Theorems 2.10 and 2.11 exploit some results on contiguous and nonbounding degrees in the literature, which we summarize first.

Recall that a c.e. degree is *contiguous* if it contains only one c.e. wtt-degree. Ladner and Sasso [13] have shown that any nonzero c.e. degree bounds a nonzero contiguous degree, and this has been extended by Ambos-Spies [1] as follows.

Let $\mathbf{c}_0, \ldots, \mathbf{c}_{n-1}$ be nonzero c.e. degrees ($n \geq 1$). There are c.e. degrees $\mathbf{b}_0, \ldots, \mathbf{b}_{n-1}$ such that, for $i < n$, $\mathbf{0} < \mathbf{b}_i \leq \mathbf{c}_i$, and, for $\emptyset \subset \alpha \subseteq \{0, \ldots, n-1\}$, $\bigvee_{i \in \alpha} \mathbf{b}_i$ is contiguous. (2.4)

(This is a special case of Theorem 4.2 in [1].) As Stob [19] has shown, contiguous degrees have a local distributivity property. To make this more precise, call a c.e. degree $\mathbf{b}$ *n-covering* ($n \geq 1$) if, for any c.e. degrees $\mathbf{a}, \mathbf{b}_0, ..., \mathbf{b}_{n-1}$,

$$\mathbf{a} \leq \mathbf{b}_0 \vee .... \vee \mathbf{b}_{n-1} = \mathbf{b} \Rightarrow \exists\, \mathbf{a}_0 \leq \mathbf{b}_0, ..., \mathbf{a}_{n-1} \leq \mathbf{b}_{n-1}\ [\mathbf{a} = \mathbf{a}_0 \vee ... \vee \mathbf{a}_{n-1}] \tag{2.5}$$

holds. Then Stob [19] has shown that contiguous degrees are 2-covering and, as observed in [1] (see Corollary 1.5 there), this can be easily extended to show

Let $\mathbf{b}$ be contiguous. Then $\mathbf{b}$ is n-covering for all $n \geq 1$. (2.6)

The existence of nonzero *nonbounding* degrees, i.e., nonzero c.e. degrees which do not bound a minimal pair of c.e. degrees, has been established by Lachlan [12], and this result has been strengthened by Ambos-Spies and Soare [4], who have shown that there is an infinite sequence of nonbounding degrees which pairwise form minimal pairs.

There are c.e. degrees $\mathbf{c}_n > \mathbf{0}$ such that, for any $n \geq 0$, $\mathbf{c}_n$ is nonbounding, and, for any $n \neq n' \geq 0$, $\mathbf{c}_n \wedge \mathbf{c}_{n'} = \mathbf{0}$. (2.7)

Now, following Ambos-Spies [3], say that a c.e. degree $\mathbf{b}$ is of *type* $n \geq 2$ *via* $\mathbf{b}_0, \ldots, \mathbf{b}_{n-1}$ if the degrees $\mathbf{b}_0, \ldots, \mathbf{b}_{n-1}$ are nonbounding and there is an embedding of the n-atom Boolean algebra $\mathcal{B}_n = (2^{\{1,\ldots,n\}}, \cup, \cap)$ into the initial segment $\mathbf{R}(\leq \mathbf{b})$ which preserves joins, meets and the least and greatest elements and which maps the atoms of $\mathcal{B}_n$ to the degrees $\mathbf{b}_0, \ldots, \mathbf{b}_{n-1}$; and say that $\mathbf{b}$ is of *type* n if $\mathbf{b}$ is of type n via some degrees $\mathbf{b}_0, \ldots, \mathbf{b}_{n-1}$.

Examples of degrees of type n are provided by the following fact.

Let $\mathbf{b}, \mathbf{b}_0, \ldots, \mathbf{b}_{n-1}$ $(n \geq 2)$ be c.e. degrees such that $\mathbf{b} = \mathbf{b}_0 \vee \cdots \vee \mathbf{b}_{n-1}$, $\mathbf{b}_0, \ldots, \mathbf{b}_{n-1}$ are nonbounding and pairwise minimal pairs, and, for any α s.t. $\emptyset \subset \alpha \subseteq \{0, \ldots, n-1\}$, $\bigvee_{i\in\alpha} \mathbf{b}_i$ is contiguous. Then $\mathbf{b}$ is of type n via $\mathbf{b}_0, \ldots, \mathbf{b}_{n-1}$. (2.8)

(Since the proof of (2.8), which exploits (2.6), is quite straightforward and implicit in the proof of Corollary 5.5 in [1], we omit it here.) Note that, by (2.4) and (2.7), for any $n \geq 2$ there are degrees $\mathbf{b}, \mathbf{b}_0, \ldots, \mathbf{b}_{n-1}$ as in the hypothesis of (2.8). So (by (2.6))

For $n \geq 2$ there is a contiguous (hence n-covering) degree $\mathbf{b}$ of type n (2.9)

holds. Finally, we observe that

Let $\mathbf{b}$ be n-covering and of type n via $\mathbf{b}_0, \ldots, \mathbf{b}_{n-1}$ $(n \geq 2)$. Then, for any c.e. degree $\mathbf{a} \leq \mathbf{b}$, $\mathbf{a}$ is nonbounding iff $\mathbf{a} \leq \mathbf{b}_i$ for some $i < n$ (2.10)

holds. (Namely, for a proof of the nontrivial implication, let $\mathbf{a} \leq \mathbf{b}$ be nonbounding. Then, since $\mathbf{b}$ is n-covering and $\mathbf{b} = \mathbf{b}_0 \vee \ldots \mathbf{b}_{n-1}$, there are c.e. degrees $\mathbf{a}_i \leq \mathbf{b}_i$ $(i < n)$ such that $\mathbf{a} = \mathbf{a}_0 \vee \cdots \vee \mathbf{a}_{n-1}$. But, since the degrees $\mathbf{b}_i$ are pairwise minimal pairs and since $\mathbf{a}$ is nonbounding, this implies that, for some $i < n$, $\mathbf{a} = \mathbf{a}_i$ hence $\mathbf{a} \leq \mathbf{b}_i$.)

Some of the above results together with Theorem 2.12 imply the following sufficient condition for a generator to have rank $\geq n$ (for given $n \geq 2$).

Lemma 2.14: *Let* $\mathbf{b}, \mathbf{b}_0, \ldots, \mathbf{b}_{n-1}$ *be c.e. degrees such that* $\mathbf{b}$ *is of type* n *via* $\mathbf{b}_0, \ldots, \mathbf{b}_{n-1}$ $(n \geq 2)$*, and let* $\mathbf{G}$ *be a generator such that*

$$\mathbf{G} \cap \mathbf{R}(\leq \mathbf{b}) \subseteq \mathbf{R}(\leq \mathbf{b}_0) \cup \cdots \cup \mathbf{R}(\leq \mathbf{b}_{n-1}) \tag{2.11}$$

holds. Then $\mathrm{rank}(\mathbf{G}) \geq n$.

Proof: By Theorem 2.12, fix c.e. degrees $\mathbf{a}, \mathbf{a}_0, \ldots, \mathbf{a}_{n-1}$ such that $\mathbf{a}$ is s.m.i., $\mathbf{0} < \mathbf{a}_i \leq \mathbf{b}_i$ for $i < n$, and $\mathbf{a} = \mathbf{a}_0 \vee \cdots \vee \mathbf{a}_{n-1}$. Then, by Proposition 2.13, it suffices to show that $\text{j-rank}_{\mathbf{G}}(\mathbf{a}) \geq n$.

For a contradiction, assume that $\text{j-rank}_{\mathbf{G}}(\mathbf{a}) = p < n$ and fix $\mathbf{c}_0, \ldots, \mathbf{c}_{p-1} \in \mathbf{G}$ such that $\mathbf{a} = \mathbf{c}_0 \vee \cdots \vee \mathbf{c}_{p-1}$. Since $\mathbf{a} \leq \mathbf{b}$, it follows by (2.11) and by $p < n$ that there is a number $i < n$ such that

$$\mathbf{a} \leq \bigvee_{j\in\{0,\ldots,n-1\}\setminus\{i\}} \mathbf{b}_j.$$

On the other hand, by $\mathbf{b}$ having type n via $\mathbf{b}_0, \dots, \mathbf{b}_{n-1}$,

$$\left(\bigvee_{j \in \{0,\dots,n-1\} \setminus \{i\}} \mathbf{b}_j \right) \wedge \mathbf{b}_i = \mathbf{0}.$$

So $\mathbf{a} \wedge \mathbf{b}_i = \mathbf{0}$. But this is impossible since, by choice of $\mathbf{a}_i$, $\mathbf{a}_i \leq \mathbf{a}, \mathbf{b}_i$ and $\mathbf{a}_i \neq \mathbf{0}$. □

The preceding observations suffice to prove Theorem 2.10.

Proof: (of Theorem 2.10) Since $\mathbf{R}$ is a join generator of join rank 1 (hence rank 1), w.l.o.g. we may assume that $n \geq 2$. Let

$$\mathbf{G}_n = \{\mathbf{a} : \text{ there is no } \mathbf{b} \geq \mathbf{a} \text{ which is both } n\text{-covering and of type } n\}$$
$$\cup \{\mathbf{a} : \mathbf{a} \text{ is nonbounding}\}.$$

Obviously, $\mathbf{G}_n$ is definable, and in [3] it has been shown that $\mathbf{G}_n$ is a join generator of join rank n. So it suffices to show that $\text{rank}(\mathbf{G}_n) \geq n$.

By (2.9), fix c.e. degrees $\mathbf{b}, \mathbf{b}_0, \dots, \mathbf{b}_{n-1}$ such that $\mathbf{b}$ is n-covering and $\mathbf{b}$ is of type n via $\mathbf{b}_0, \dots, \mathbf{b}_{n-1}$. Then, by definition of $\mathbf{G}_n$,

$$\mathbf{G}_n \cap \mathbf{R}(\leq \mathbf{b}) = \{\mathbf{a} \leq \mathbf{b} : \mathbf{a} \text{ is nonbounding}\}.$$

So, by (2.10), (2.11) holds for $\mathbf{G} = \mathbf{G}_n$. Hence, $\text{rank}(\mathbf{G}_n) \geq n$ by Lemma 2.14. □

For the proof of Theorem 2.11, in addition to the above observations we need two definability results: First, Downey and Lempp [9] have shown that the class of the contiguous degrees is definable (namely, a c.e. degree $\mathbf{a}$ is contiguous if and only if $\mathbf{a}$ is 2-covering). Second, Nies [14] has shown that, for any definable class $\mathbf{D}$ of c.e. degrees, the ideal generated by $\mathbf{D}$ is definable too. By inspecting the proof, one immediately gets the corresponding result for the join closure.

Lemma 2.15: *(Nies) For any definable class* $\mathbf{D}$ *of c.e. degrees, the join closure* $\text{CL}_j(\mathbf{D})$ *of* $\mathbf{D}$ *is definable too.*

Proof: (of Theorem 2.11) Let

$$\mathbf{G}_\omega = \overline{\{\mathbf{a} : \exists\, \mathbf{b} \geq \mathbf{a}\, (\mathbf{b} \in \text{CL}_j(\mathbf{NB}) \cap \mathbf{CONT})\}} \;\cup\; \mathbf{NB}$$

where $\mathbf{NB}$ and $\mathbf{CONT}$ are the classes of the nonbounding degrees and contiguous degrees, respectively. Since, obviously, $\mathbf{NB}$ is definable hence

$\mathrm{CL}_j(\mathbf{NB})$ is definable by Lemma 2.15, it follows by definability of **CONT** (Downey and Lempp [9]) that $\mathbf{G}_\omega$ is definable.

To show that $\mathbf{G}_\omega$ is a join generator, given $\mathbf{a} \in \mathbf{R} \setminus \mathbf{G}_\omega$, it suffices to show that $\mathbf{a} \in \mathrm{CL}_j(\mathbf{G}_\omega)$. By choice of $\mathbf{a}$ and by definition of $\mathbf{G}_\omega$, $\mathbf{a} \notin \mathbf{NB}$ and there is a degree $\mathbf{b} \in \mathrm{CL}_j(\mathbf{NB}) \cap \mathbf{CONT}$ such that $\mathbf{a} \leq \mathbf{b}$. So, by downward closure of $\mathbf{NB}$ and by (2.6), there is a number $n \geq 2$ and nonzero nonbounding degrees $\mathbf{a}_i$ $(i < n)$ such that $\mathbf{a} = \mathbf{a}_0 \vee \cdots \vee \mathbf{a}_{n-1}$. So, in particular, $\mathbf{a} \in \mathrm{CL}_j(\mathbf{NB})$ which, by $\mathbf{NB} \subseteq \mathbf{G}_\omega$, implies the claim.

It remains to show that $\mathrm{rank}(\mathbf{G}_\omega) \geq n$ for all numbers $n \geq 2$. Fix $n \geq 2$, and, by (2.9), fix c.e. degrees $\mathbf{b}, \mathbf{b}_0, \dots, \mathbf{b}_{n-1}$ such that $\mathbf{b}$ is contiguous and $\mathbf{b}$ is of type n via $\mathbf{b}_0, \dots, \mathbf{b}_{n-1}$. Then, by Lemma 2.14, it suffices to show

$$\mathbf{G}_\omega \cap \mathbf{R}(\leq \mathbf{b}) \subseteq \mathbf{R}(\leq \mathbf{b}_0) \cup \cdots \cup \mathbf{R}(\leq \mathbf{b}_{n-1}).$$

So fix $\mathbf{a} \in \mathbf{G}_\omega \cap \mathbf{R}(\leq \mathbf{b})$. Since $\mathbf{a} \leq \mathbf{b}$ and $\mathbf{b} \in \mathrm{CL}_j(\mathbf{NB}) \cap \mathbf{CONT}$, it follows by definition of $\mathbf{G}_\omega$ that $\mathbf{a}$ is nonbounding. So, by (2.10), $\mathbf{a} \leq \mathbf{b}_i$ for some $i < n$, which completes the proof. □

3. Proof of Theorem 2.7

Before we give the proof of Theorem 2.7, we give some notation and facts used not only in this proof but also in the proofs of Theorems 2.8 and 2.12 given in the following two sections.

Let $(\varphi_e)_{e\geq 0}$, $(W_e)_{e\geq 0}$ and $(\{e\}_e^\sigma)_{e\geq 0}$ be computable numberings of the unary partial computable functions, c.e. sets, and Turing functionals, respectively, where W_e is the domain of φ_e, and fix uniformly computable enumerations $(\varphi_{e,s})_{s\geq 0}$, $(W_{e,s})_{s\geq 0}$ and $(\{e\}_{e,s}^\sigma)_{s\geq 0}$ of φ_e, W_e and $\{e\}^\sigma$, respectively, such that

$$\varphi_{e,s}(x)\downarrow \text{ (i.e. } x \in W_{e,s}) \;\Rightarrow\; e, x < s, \tag{3.1}$$

$$\{e\}_s^X(x)\downarrow \Rightarrow\; e, x, u(X; e, x, s) < s \tag{3.2}$$

and

$$\{e\}_s^X(x)\downarrow \Rightarrow\; x < u(X; e, x, s) \tag{3.3}$$

hold, where $u(X; e, x)$ and $u(X; e, x, s)$ denote the use of $\{e\}_e^X(x)$ and $\{e\}_{e,s}^{X_s}(x)$, respectively.

For modelling some of the required Turing reductions, we use (standard) markers. The standard marker of W_e (with respect to the given enumeration $(W_{e,s})_{s\geq 0}$) is denoted by γ_e and defined by

$$\gamma_{e,0}(x) \quad = \langle x, 0\rangle$$

$$\gamma_{e,s+1}(x) = \begin{cases} \langle x, s+1\rangle & \text{if } W_{e,s+1} \restriction x \neq W_{e,s} \restriction x, \\ \gamma_{e,s}(x) & \text{otherwise.} \end{cases}$$

Note that $\gamma_e(x) = \lim_{s\to\omega} \gamma_{e,s}(x)$ exists, that $\gamma_e(x)$ is computable in W_e and that

$$\gamma_{e,s}(x) \leq \gamma_{e,s+1}(x) \leq \gamma_e(x) \tag{3.4}$$

for all numbers e, x, s. The following observation on the growth of the standard markers will be crucial.

Lemma 3.1: *(Standard Marker Lemma) Let A be any set and e be any index such that $W_e \not\leq_{\mathrm{T}} A$. Then, for any infinite A-c.e. set D and any partial A-computable function ψ such that D is contained in the domain of ψ, there is an infinite A-computable subset $\hat{D}$ of D such that $\psi(x) < \gamma_e(x)$ for all $x \in \hat{D}$.*

This lemma is implicit already in Lachlan [11] and follows from the lemma to Theorem 2 of Chapter 18 of Shoenfield [16].

General format of the proof

We now turn to the proof of Theorem 2.7. By Lemma 2.3, it suffices to show that there is a low c.e. degree $\mathbf{a}$ such that $\mathbf{a} > \mathbf{0}$ and

$$\forall\, \mathbf{b_0}, \mathbf{b_1}\ [\mathbf{b_0} \vee \mathbf{b_1} = \mathbf{a} \Rightarrow \mathbf{a} \leq \mathbf{b_0} \quad \text{or} \quad \forall\, \mathbf{c_0}, \mathbf{c_1} \not\leq \mathbf{a}\ \exists\, \mathbf{d}\ (\mathbf{d} \leq \mathbf{b_1} \vee \mathbf{c_0}, \mathbf{b_1} \vee \mathbf{c_1}\ \&\ \mathbf{d} \not\leq \mathbf{a})] \tag{3.5}$$

holds.

By a finite injury priority argument, we construct a c.e. set A such that $\mathbf{a} = deg(A)$ has the required properties. Together with A, we enumerate auxiliary c.e. sets C_n ($n \geq 0$). We let A_s and $C_{n,s}$ denote the finite parts of A and C_n, respectively, enumerated by the end of stage $s \geq 0$ of the construction. Moreover, we let $A_0 = C_{n,0} = \emptyset$, i.e., stage 0 of the construction is vacuous.

There is an infinite list of requirements R_e, $e \geq 0$, to be met. The requirements are of three different types, *noncomputability requirements* P_e, *lowness requirements* N_e, and *inaccessibility requirements* Q_e. The priority ordering is determined by $\mathrm{P}_e = \mathrm{R}_{3e}$, $\mathrm{N}_e = \mathrm{R}_{3e+1}$, and $\mathrm{Q}_e = \mathrm{R}_{3e+2}$.

At any stage $s+1$ of the construction, any requirement P_e, N_e and Q_e with $e \leq s$ may *require attention*, and the highest priority requirement which requires attention *becomes active* (or, in other words, *receives attention*). If R_e acts at stage $s+1$ then all lower priority requirements $\mathrm{R}_{e'}$, $e < e'$, are *initialized.* Moreover, if requirement R_e acts at stage $s+1$ then it may be declared to be *satisfied* at this stage. In this case, R_e remains satisfied unless it becomes initialized, i.e., if R_e is satisfied at stage s and R_e is not initialized at stage $s+1$ then R_e is satisfied at stage $s+1$ too. We say that R_e is *permanently satisfied at stage s* if R_e is satisfied at stage s and not initialized later, and R_e is *permanently satisfied* if there is a stage at which it is permanently satisfied.

We will argue that any requirement requires attention at most finitely often. So any requirement acts only finitely often and is initialized only finitely often.

Requirements

The requirements are as follows. In order to ensure that $\mathbf{a} > \mathbf{0}$ and $\mathbf{a}$ is low, we meet the *Noncomputability Requirements*

$$\mathrm{P}_e : A \neq \{e\}$$

and the *Lowness Requirements*

$$\mathrm{N}_e : \text{If } \{e\}_s^{A_s}(e)\downarrow \text{ for infinitely many } s \text{ then } \{e\}^A(e)\downarrow$$

respectively. In order to guarantee (3.5), for any numbers $n, m \geq 0$ and for the corresponding unique numbers $i_0, i_1, j_0, j_1, j_2, k_0, k_1$ and e such that $n = \langle i_0, i_1, j_0, j_1, j_2, k_0, k_1 \rangle$ and $e = \langle n, m \rangle$ we ensure

$$C_n \leq_{\mathrm{T}} W_{i_1} \oplus W_{k_0}, W_{i_1} \oplus W_{k_1} \tag{3.6}$$

and meet the *Inaccessibility Requirements*

$$\begin{aligned} \mathrm{Q}_e : \text{If} \quad & (i)\ \ W_{i_0} = \{j_0\}^A \ \&\ W_{i_1} = \{j_1\}^A \ \&\ A = \{j_2\}^{W_{i_0} \oplus W_{i_1}}, \\ & (ii)\ \ W_{k_0}, W_{k_1} \not\leq_{\mathrm{T}} A, \text{ and} \\ & (iii)\ A \not\leq_{\mathrm{T}} W_{i_0} \\ \text{then} \quad & (*)\ \ C_n \neq \{m\}^A. \end{aligned}$$

To see that the Q_e requirements, together with (3.6) ensure that (3.5) holds, suppose that $\mathbf{b_0} \vee \mathbf{b_1} = \mathbf{a}$, $\mathbf{a} \not\leq \mathbf{b_0}$, and $\mathbf{c_0}, \mathbf{c_1} \not\leq \mathbf{a}$. To show that (3.5) is met, we need to show that there is $\mathbf{d}$ with $\mathbf{d} \leq \mathbf{b_1} \vee \mathbf{c_0}, \mathbf{b_1} \vee \mathbf{c_1}$ and $\mathbf{d} \not\leq \mathbf{a}$. We may choose $i_0, i_1, j_0, j_1, j_2, k_1, k_2$ such that $W_{i_0} \in \mathbf{b_0}$, $W_{i_1} \in \mathbf{b_1}$, $W_{i_0} = \{j_0\}^A$, $W_{i_1} = \{j_1\}^A$, $A = \{j_2\}^{W_{i_0} \oplus W_{i_1}}$, $W_{k_0} \in \mathbf{c_0}$, and $W_{k_1} \in \mathbf{c_1}$. Let $n = \langle i_0, i_1, j_0, j_1, j_2, k_0, k_1 \rangle$ and define $\mathbf{d} = deg(C_n)$. Then, (3.6) implies $\mathbf{d} \leq \mathbf{b_1} \vee \mathbf{c_0}, \mathbf{b_1} \vee \mathbf{c_1}$. Assuming that we meet Q_e for all $e = \langle n, m \rangle$, then, since for such e all the hypotheses $(i), (ii), (iii)$ of Q_e hold, we must have $C_n \neq \{m\}^A$ for all m, which ensures that $\mathbf{d} \not\leq \mathbf{a}$.

The strategies for satisfying condition (3.6) and meeting the requirements R_e are described next. There — just as in the following in general — when discussing condition (3.6) or an inaccessibility requirement Q_e, we always tacitly assume that $n = \langle i_0, i_1, j_0, j_1, j_2, k_0, k_1 \rangle$ and $e = \langle n, m \rangle$.

Strategy for satisfying condition (3.6)

Condition (3.6) is satisfied by marker permitting. To be more precise, the construction guarantees

$$x \in C_{n,s+1} \setminus C_{n,s} \Rightarrow \exists\, z < \min\{\gamma_{k_0,s}(x), \gamma_{k_1,s}(x)\}\ (z \in W_{i_1,s+1} \setminus W_{i_1,s}) \tag{3.7}$$

where γ_{k_p} is the *standard marker* of W_{k_p} $(p = 0, 1)$. (Note that, by (3.4), (3.7) implies that a number x cannot enter C_n after a stage s such that $W_{i_1,s} \restriction \gamma_{k_p}(x) = W_{i_1} \restriction \gamma_{k_p}(x)$ (for $p = 0, 1$). Since γ_{k_p} is W_{k_p}-computable it follows that (3.7) ensures that (3.6) holds.)

Strategy for meeting P_e

The strategy for meeting the noncomputability requirements is standard. It uses *followers*.

Requirement P_e *requires attention* at stage $s+1$ if $e \leq s$, P_e is not satisfied at (the end of) stage s, and one of the following holds.

$$\mathrm{P}_e \text{ has no follower at the end of stage } s. \tag{3.8}$$

$$\begin{array}{c}\mathrm{P}_e \text{ has follower } y \text{ at the end of stage } s,\ y \text{ is } \textit{realized} \text{ at stage } s,\\ \text{i.e., } \{e\}_s(y) = 0, \text{ and } y \notin A_s.\end{array} \tag{3.9}$$

The corresponding action (if P_e receives attention at stage $s+1$) is as follows. If (3.8) holds then appoint $y = s+1$ as follower of P_e; and if (3.9) holds then put the follower y of P_e into A and declare P_e to be *satisfied.*

At any stage s there is at most one follower of P_e. We denote this follower by $y(e,s)$ and write $y(e,s)\uparrow$ if such a follower does not exist.

If P_e is *initialized* at stage $s+1$ then the follower of P_e at the end of stage s (if any) is cancelled (hence $y(e,s+1)\uparrow$). A follower which is never cancelled (i.e. which is appointed at a stage after which P_e is not initialized anymore) is called *permanent*. We will argue that there will be a (unique) permanent follower $y(e)$ of P_e and that this follower $y(e)$ witnesses that P_e is met, i.e., $A(y(e)) \neq \{e\}(y(e))$.

The noncomputability requirements are the only requirements which enumerate numbers into A. Since at any stage $s+1$ at most one requirement becomes active, it follows that

$$A_{s+1} \neq A_s \;\Rightarrow\; \exists!\, e \leq s\ (y(e,s)\downarrow\ \&\ A_{s+1} \setminus A_s = \{y(e,s)\}) \tag{3.10}$$

holds.

In the actual construction, the strategy for meeting the inaccessibility requirements $\mathrm{Q}_{e'}$ will require that the followers of the noncomputability requirements P_e have certain properties. For this sake, below we have to add another clause for P_e requiring attention.

Strategy for meeting N_e

The strategy for meeting the lowness requirements is standard too.

Requirement N_e *requires attention* at stage $s+1$ if $e \leq s$, N_e is not satisfied at (the end of) stage s, and the following holds.

$$\{e\}_s^{A_s}(e)\downarrow . \tag{3.11}$$

The corresponding action (if N_e receives attention at stage $s+1$) is as follows. Declare N_e to be *satisfied*.

Note that, by N_e becoming active, all followers of lower priority noncomputability requirements are cancelled. Since any follower appointed later will be greater than $s+1$ hence, by our convention (3.2), greater than $u(A_s; e, e, s)$, it follows by (3.10) that $\{e\}^A(e) = \{e\}_s^{A_s}(e)\downarrow$ and therefore that N_e is met unless N_e becomes initialized later. So if N_e is permanently satisfied then N_e is met.

Strategy for meeting Q_e

For the discussion of the strategy for meeting Q_e, fix e and the numbers n, m, i_0, i_1, j_0, j_1, j_2, k_0, k_1, such that $e = \langle n, m\rangle$ and $n = \langle i_0, i_1, j_0, j_1, j_2, k_0, k_1\rangle$.

There are two strategies for meeting Q_e. The first strategy attempts to refute hypothesis (i) of the requirement (see (3.12) and (3.13) below). This strategy is purely A-negative: it waits for a disagreement in one of the equations in (i) which can be preserved by putting a restraint on A. The second strategy attempts to meet the requirement by satisfying $(*)$ (see (3.14) below). The latter strategy is A-negative too (in order to preserve a computation with oracle A) and, in addition, C_n-positive.

Requirement Q_e *requires attention* at stage $s+1$ if $e \le s$, Q_e is not satisfied at (the end of) stage s, and one of the following holds.

$$\exists\, x \left(\begin{array}{l} A_s(x) \neq \{j_2\}_s^{W_{i_0,s} \oplus W_{i_1,s}}(x)\downarrow, \\ \text{and, for } u = u(W_{i_0,s} \oplus W_{i_1,s}; j_2, x, s) \text{ and for } p \le 1, \\ W_{i_p,s} \restriction u = \{j_p\}_s^{A_s} \restriction u \end{array} \right) \tag{3.12}$$

$$\exists\, x\, \exists\, p \le 1\ (W_{i_p,s}(x) = 1 \neq \{j_p\}_s^{A_s}(x)\downarrow) \tag{3.13}$$

$$\exists\, x \in \omega^{[e]} \tag{3.14}$$
$$\Big(\{m\}_s^{A_s}(x) = 0\ \&\ \exists\, z < \min\{\gamma_{k_0,s}(x), \gamma_{k_1,s}(x)\}(z \in W_{i_1,s+1} \setminus W_{i_1,s})\Big).$$

(Note that, by our convention (3.2), the quantifier $\exists\ x$ in (3.12)–(3.14) can be replaced by the bounded quantifier $\exists\ x \le s$ whence it is decidable whether these conditions hold or not.)

The corresponding action (if Q_e receives attention at stage $s+1$) is as follows. Declare Q_e to be *satisfied.* Moreover if (3.14) holds then, for the least x as there, put x into C_n. Note that the latter action is consistent with (3.7). (Also note that the set $\omega^{[e]}$ is reserved for Q_e, i.e., numbers from $\omega^{[e]}$ can be enumerated into C_n only by the Q_e-strategy.)

If Q_e is permanently satisfied then Q_e is met. This is immediate by the following claim since, if Q_e becomes active at stage $s+1$ and is not initialized later, then $A \restriction s+1 = A_s \restriction s+1$.

Claim 3.2: Assume that Q_e requires and receives attention at stage $s+1$ and that $A \upharpoonright s+1 = A_s \upharpoonright s+1$. Then either clause (i) in Q_e fails or clause $(*)$ in Q_e holds. So, in particular, Q_e is met.

Proof: If (3.12) holds, fix the least x as there and the corresponding u. Then, by (3.2), $x < s$ and $u(A_s; j_p, y, s) < s$ for all $y < u$ and $p \leq 1$. So, by $A \upharpoonright s+1 = A_s \upharpoonright s+1$, $A(x) = A_s(x)$ and, for $p \leq 1$, $\{j_p\}^A \upharpoonright u = \{j_p\}_s^{A_s} \upharpoonright u$. It follows, by choice of x and u, that $A(x) \neq \{j_2\}^{W_{i_0} \oplus W_{i_1}}(x)$ or, for some $p \leq 1$, $W_{i_p} \upharpoonright u \neq \{j_p\}^A \upharpoonright u$. In either case, this implies that hypothesis (i) of Q_e fails, hence Q_e is met.

If (3.13) or (3.14) holds then the argument that Q_e is met is similar. If (3.13) holds then, for any x and $p \leq 1$ as there, $1 = W_{i_p}(x) \neq \{j_p\}^A(x)$. So, again, hypothesis (i) of Q_e fails. Finally, if (3.14) holds then, for the least x as there, $C_n(x) = 1 \neq 0 = \{m\}^A(x)$. So the conclusion $(*)$ of Q_e holds. □

The above finitary strategy attempts to refute hypothesis (i) of Q_e by diagonalization or to guarantee the conclusion $(*)$ by diagonalization. It does not suffice to meet Q_e, and an additional infinitary strategy is needed which is based on the assumption that hypotheses (i) and (ii) of Q_e hold. Very roughly speaking, the idea of this strategy is as follows. By (i), $A = \{j_2\}^{W_{i_0} \oplus W_{i_1}}$. So a number y which enters A sufficiently late triggers a change of W_{i_0} or W_{i_1}. Now any follower y of a lower priority coding requirement which does not enter A sufficiently promptly is chosen so that a change of W_{i_1} gives the necessary permission to make an attack on Q_e via clause (3.14). So if the finitary strategy fails then we may argue that any "critical" number which enters A is "permitted" by W_{i_0}, hence $A \leq_{\mathrm{T}} W_{i_0}$ (so hypothesis (iii) of Q_e fails and the requirement is met).

This infinitary strategy interferes with the strategies of the noncomputability requirements but the impact on a fixed requirement is finitary (hence compatible with the finite injury framework). To implement the strategy, we use the following e-eligibility notion.

A number y is *e-eligible* ($e = \langle n, m \rangle$, $n = \langle i_0, i_1, j_0, j_1, j_2, k_0, k_1 \rangle$) *via* numbers x, u, v and r at stage $s+1$ if the following hold.

$$x \in \omega^{[e]} \tag{3.15}$$

$$\{m\}_s^{A_s}(x) = 0 \tag{3.16}$$

$$u = u(A_s; m, x, s) \tag{3.17}$$

$$y \geq u \tag{3.18}$$

$$y \notin A_s \tag{3.19}$$

$$A_s \upharpoonright y+1 = \{j_2\}_s^{W_{i_0,s} \oplus W_{i_1,s}} \upharpoonright y+1 \tag{3.20}$$

$$v = \max\{u(W_{i_0,s} \oplus W_{i_1,s}; j_2, z, s) : z \leq y\} \tag{3.21}$$

$$\forall\, p \leq 1\ (W_{i_p,s} \upharpoonright v+1 = \{j_p\}_s^{A_s} \upharpoonright v+1) \tag{3.22}$$

$$r = \max\{u(A_s; j_p, z, s) : p \leq 1\ \&\ z \leq v\} \tag{3.23}$$

and

$$v < \gamma_{k_0,s}(x), \gamma_{k_1,s}(x). \tag{3.24}$$

Call y *e-preeligible via* x, u, v and r at stage $s+1$ if (3.15)–(3.23) (but not necessarily (3.24)) hold. Call y *e-(pre)eligible* at stage $s+1$ if there are numbers x, u, v, and r such that y is *e-(pre)eligible* via x, u, v, and r at stage $s+1$. Note that, if y is e-(pre)eligible via x, u, v and r at stage $s+1$, then, by our conventions (3.2) and (3.3),

$$x < u \leq y < v < r < s. \tag{3.25}$$

(Also note that if y is e-(pre)eligible via x, u, v and r at stage $s+1$ then the numbers u, v and r are determined by y and x. So, in particular, since $x < y$, it is decidable whether y is e-(pre)eligible at stage $s+1$.)

The following claim explains how e-eligibility is used in meeting requirement Q_e.

Claim 3.3: Assume that Q_e requires attention only finitely often and is satisfied only finitely often. If there is an infinite computable set S of stages such that, for any $s \in S$,

$$\forall\, y, s'\ ([y \leq s \leq s'\ \&\ y \in A_{s'+1} \setminus A_{s'}] \ \Rightarrow\ y \text{ is } e\text{-eligible at stage } s'+1) \tag{3.26}$$

holds, then requirement Q_e is met.

Proof: Fix an infinite computable set S such that (3.26) holds for all $s \in S$. In order to show that Q_e is met, w.l.o.g. we may assume that hypothesis (i) in Q_e holds, and it suffices to show that

$$A \leq_{\mathrm{T}} W_{i_0} \tag{3.27}$$

holds (hence hypothesis (iii) of Q_e fails).

For a proof of (3.27), it suffices to show that there is a function $s(z)$ $(z \geq 0)$ such that

$$A \upharpoonright z+1 = A_{s(z)} \upharpoonright z+1 \tag{3.28}$$

and such that $s(z)$ can be uniformly computed from $W_{i_0} \oplus (A \upharpoonright z)$. Obviously, this gives an inductive procedure for uniformly computing $A(z)$ from W_{i_0} for any $z \geq 0$.

The definition of $s(z)$ is as follows. Fix $s_0 \geq e$ such that Q_e neither requires attention nor is satisfied after stage s_0. Note that, by (i), for any z there are numbers v_z and r_z such that, for all sufficiently large stages s and for $v(z,s) = v_z$ and $r(z,s) = r_z$, the following hold.

$$s > \max(z+1, s_0) \tag{3.29}$$

$$A \upharpoonright z = A_s \upharpoonright z \tag{3.30}$$

$$A_s \upharpoonright z+1 = \{j_2\}_s^{W_{i_0,s} \oplus W_{i_1,s}} \upharpoonright z+1 \tag{3.31}$$

$$v(z,s) = \max\{u(W_{i_0,s} \oplus W_{i_1,s}; j_2, x, s) : x \leq z\} \tag{3.32}$$

$$W_{i_0} \upharpoonright v(z,s) = W_{i_0,s} \upharpoonright v(z,s) \tag{3.33}$$

$$\forall\, p \leq 1\ (W_{i_p,s} \upharpoonright v(z,s) = \{j_p\}_s^{A_s} \upharpoonright v(z,s)) \tag{3.34}$$

$$r(z,s) = \max(\{u(A_s; j_p, x, s) : p \leq 1 \;\&\; x < v(z,s)\} \cup \{z+1\}). \tag{3.35}$$

(Note that, for given z and s, if there are numbers $v(z,s)$ and $r(z,s)$ such that (3.29)–(3.35) hold then these numbers are uniquely determined by z and s via (3.32) and (3.35).) So there are infinitely many stages s as above with $s \in S$. Moreover, using $W_{i_0} \oplus (A \upharpoonright z)$, as an oracle, it is decidable whether, for a stage s, there are numbers $v(z,s)$ and $r(z,s)$ such that (3.29)–(3.35) hold. So, if we let $s(z)$ be the least stage s such that $s \in S$ (hence (3.26) holds) and, for some $v(z,s)$ and $r(z,s)$, (3.29)–(3.35) hold then $s(z)$ can be uniformly computed from $W_{i_0} \oplus (A \upharpoonright z)$.

It remains to show that (3.28) holds. Fix z, let $s = s(z)$ and, for a contradiction, assume that $A \upharpoonright z+1 \neq A_s \upharpoonright z+1$. Then, by (3.30),

$$z \in A \setminus A_s. \tag{3.36}$$

Since, by (i), $A(z) = \{j_2\}^{W_{i_0} \oplus W_{i_1}}(z)$, it follows by (3.31), (3.32) and (3.33) that

$$W_{i_1} \upharpoonright v(z,s) \neq W_{i_1,s} \upharpoonright v(z,s).$$

So we may fix the unique $t \geq s$ such that

$$W_{i_1,t+1} \upharpoonright v(z,s) \neq W_{i_1,t} \upharpoonright v(z,s) = W_{i_1,s} \upharpoonright v(z,s). \tag{3.37}$$

On the other hand, since (by (3.35)) $z < r(z,s)$, it follows from (3.36) that there is a stage $s' \geq s$ such that

$$A_{s'+1} \upharpoonright r(z,s) \neq A_{s'} \upharpoonright r(z,s). \tag{3.38}$$

Note that $s' \leq t$ for the least such stage s' since otherwise, by (3.37), by (3.34) and by (3.35),

$$\begin{aligned} W_{i_1,t+1} \upharpoonright v(z,s) \neq W_{i_1,s} &\upharpoonright v(z,s) \\ = \{j_1\}_s^{A_s} &\upharpoonright v(z,s) \\ = \{j_1\}_{t+1}^{A_{t+1}} &\upharpoonright v(z,s). \end{aligned}$$

But this implies that Q_e requires attention via clause (3.13) at stage $t+2$ which (by (3.29)) contradicts the choice of s_0.

So for the remainder of the argument, we may fix s' maximal such that $s \leq s' \leq t$ and (3.38) holds. We will show that Q_e requires attention via clause (3.14) at stage $t+1$ contrary to choice of s_0.

Fix the (unique) number y which enters A at stage $s'+1$. Then

$$z \leq y < r(z,s) < s \tag{3.39}$$

(where the first inequality holds by (3.30) and the last inequality holds by choice of $r(z,s)$ and s and our convention (3.2)) and, by maximality of s',

$$(A_{t+1} \upharpoonright r(z,s)) \setminus (A_{s'} \upharpoonright r(z,s)) = \{y\}. \tag{3.40}$$

Moreover, by $s \in S$ and (3.39), y is e-eligible at stage $s'+1$, say via x, u, v and r. So $x \in \omega^{[e]}$, $\{m\}_{s'}^{A_{s'}}(x) = 0$ and $x < u = u(A_{s'}; m, x, s') \leq y$ which, by (3.40), implies $\{m\}_t^{A_t}(x) = 0$. So, in order to argue that Q_e requires attention via clause (3.14) at stage $t+1$, it suffices to show

$$W_{i_1,t+1} \upharpoonright \min\{\gamma_{k_0,t}(x), \gamma_{k_1,t}(x)\} \neq W_{i_1,t} \upharpoonright \min\{\gamma_{k_0,t}(x), \gamma_{k_1,t}(x)\}. \tag{3.41}$$

For a proof of (3.41), first note that, by e-eligibility of y via x, u, v and r at stage $s'+1$,

$$v = \max\{u(W_{i_0,s'} \oplus W_{i_1,s'}; j_2, x', s') : x' \leq y\} < \gamma_{k_0,s'}(x), \gamma_{k_1,s'}(x).$$

So, since the standard markers are nondecreasing in the stages and since $s' \leq t$, by (the inequality in) (3.37), it suffices to show that $v(z,s) \leq v$. But this follows from $z \leq y$ and $s \leq s' \leq t$, since, by (the equality in) (3.37), by (3.33), and by definition of $v(z,s)$,

$$\begin{aligned} v(z,s) &= \max\{u(W_{i_0,s} \oplus W_{i_1,s}; j_2, x', s) : x' \leq z\} \\ &= \max\{u(W_{i_0,s'} \oplus W_{i_1,s'}; j_2, x', s') : x' \leq z\}. \end{aligned}$$

This completes the proof of (3.41) and the proof of Claim 3.3. □

The following two claims provide some more facts on e-eligibility to be used in the construction: The first claim shows when we can find e-eligible numbers while the second claim tells us how e-eligibility of a number can be preserved.

Claim 3.4: Assume that clauses (i) and (ii) in Q_e hold, and that $\{m\}^A$ is total and, for almost all numbers $x \in \omega^{[e]}$, $\{m\}^A(x) = 0$. Then, for any infinite A-computable subset E of $\overline{A}$ there is an infinite subset F of E such that, for any $y \in F$, there are numbers x_y, u_y, v_y and r_y such that y is e-eligible via x_y, u_y, v_y and r_y at almost all stages.

Proof: Fix x_0 minimal such that $\{m\}^A(x) = 0$ for all $x \geq x_0$ in $\omega^{[e]}$. By (i), by totality of $\{m\}^A$, and by choice of E, for any number $x \geq 0$ there are unique numbers $u(x)$, $y(x)$, $v(x)$, $r(x)$ and $s(x)$ such that

$$\begin{aligned} u(x) &= u(A; m, x), \\ y(x) &= \mu y \geq u(x)\ (y \in E), \\ v(x) &= \max\{u(W_{i_0} \oplus W_{i_1}; j_2, z) : z \leq y(x)\}, \\ r(x) &= \max\{u(A; j_p, z) : p \leq 1\ \&\ z \leq v(x)\}, \end{aligned}$$

and $s(x)$ is the least stage $s > r(x)$ such that $\{m\}_s^{A_s}(x)\downarrow$, $u(A_s; m, x, s) = u(x)$, $A_s(z) = \{j_2\}^{W_{i_0,s} \oplus W_{i_1,s}}(z)$ for all $z \leq y(x)$, $v(x) = \max\{u(W_{i_0,s} \oplus W_{i_1,s}; j_2, z, s) : z \leq y(x)\}$, $W_{i_p,s}(z) = \{j_p\}_s^{A_s}(z)$ for all $p \leq 1$ and $z \leq v(x)$, $r(x) = \max\{u(A_s; j_p, z, s) : p \leq 1\ \&\ z \leq v(x)\}$, and $A \upharpoonright r(x) = A_s \upharpoonright r(x)$.

Obviously, $u(x)$, $y(x)$, $v(x)$, $r(x)$ can be computed by A, $x < u(x) \leq y(x) < v(x) < r(x) < s(x)$, $y(x) \in E$, and, for any $x \geq x_0$ in $\omega^{[e]}$, $y(x)$ is e-preeligible via x, $u(x)$, $v(x)$ and $r(x)$ at all stages $s + 1 > s(x)$. So it suffices to show that there are infinitely many numbers $x \in \omega^{[e]}$ such that $v(x) < \gamma_{k_0}(x)$ and $\gamma_{k_1}(x)$. But, by A-computability of $v(x)$, this is immediate by the Standard Marker Lemma. (Namely, a first application yields an infinite A-computable subset $\hat{D}$ of $\{x : x \in \omega^{[e]}\ \&\ x \geq x_0\}$

such that $v(x) < \gamma_{k_0}(x)$, and a second application yields an infinite A-computable subset $\hat{\hat{D}}$ of $\hat{D}$ such that $v(x) < \gamma_{k_1}(x)$.) □

Claim 3.5: Let e, y, x, u, v, r, s and s' be given such that $s \leq s'$, $e \leq s'$, y is e-(pre)eligible via x, u, v and r at stage $s+1$, $A_{s'} \upharpoonright r = A_s \upharpoonright r$, and Q_e is neither satisfied at stage s' nor requires attention at stage $s'+1$. Then y is e-(pre)eligible via x, u, v and r at stage $s'+1$.

Proof: Note that if (3.24) holds at stage s then it holds at all stages $s' \geq s$ since the standard markers are nondecreasing in the stages. So it suffices to consider the case of preeligibility.

By assumption, (3.15)–(3.23) and (3.25) hold, and it suffices to show that (3.16), (3.17), (3.19), (3.20), (3.21), (3.22) and (3.23) remain true if we replace s by s'. Since, by (3.25), $x < u \leq y < v < r < s$ and since, by assumption, $A_{s'} \upharpoonright r = A_s \upharpoonright r$, this is immediate for (3.16), (3.17), (3.19), and (3.23) where the last equation holds since, for $z \leq v$, the use of the computations $\{j_p\}_s^{A_s}(z) \downarrow$ is bounded by r hence these computations are preserved through the end of stage s'. The latter implies that (by $A_{s'} \upharpoonright y+1 = A_s \upharpoonright y+1$) the remaining conditions (3.20), (3.21) and (3.22) are preserved too, unless, for some $p \leq 1$, $W_{i_p,s'} \upharpoonright v+1 \neq W_{i_p,s} \upharpoonright v+1$. If the latter happens, however, then there is a number x' such that

$$1 = W_{i_p,s'}(x') \neq W_{i_p,s}(x') = \{j_p\}_s^{A_s}(x') \downarrow = \{j_p\}_{s'}^{A_{s'}}(x') \downarrow .$$

So, since, by assumption, Q_e is not satisfied at stage s' and since $e \leq s'$, it follows that Q_e requires attention at stage $s'+1$ via clause (3.13) contrary to assumption. So the clauses (3.20)–(3.22) hold at stage s' too, which completes the proof of the claim. □

Claim 3.3 leads to the following strategy for meeting Q_e: if the follower $y(e'', s)$ of a noncomputability requirement $\mathrm{P}_{e''}$ of lower priority than Q_e has not yet been enumerated into A, $y(e'', s)$ is not e-eligible, and there is a number $y > y(e'', s)$ such that $y \notin A_s$ and y is e-eligible at stage $s+1$ then we replace $y(e'', s)$ by (the least such) y. Then, by Claim 3.4 and Claim 3.5 we will argue that if Q_e is not satisfied for some trivial reasons then (almost all) permanent followers which do not enter A will become e-eligible and from this we conclude that the premise of Claim 3.3 will be satisfied.

To synchronize the attempts to make followers e-eligible for the higher priority inaccessibility requirements Q_e, we introduce e-states.

The *e-state* of a number y at the end of stage s is the (binary) string $\sigma(e, y, s)$ of length e defined by

$$\forall\, e' < e \left(\sigma(e, y, s)(e') = 0 \Leftrightarrow \right.$$
$$\left. y \text{ is } e'\text{-eligible at stage } s+1 \text{ or } \mathrm{Q}_{e'} \text{ is satisfied at the end of stage } s\right).$$

e-states are ordered by the standard lexicographical ordering, i.e., $\sigma < \sigma'$ if there is a number $e' < e$ such that $\sigma(e') \neq \sigma'(e')$ and, for the least such e', $\sigma(e') < \sigma'(e')$; and $\sigma \leq \sigma'$ if $\sigma < \sigma'$ or $\sigma = \sigma'$. In addition, we use the partial ordering $\preceq$ on $\{0, 1\}^e$, where $\sigma \preceq \sigma'$ if $\sigma(e') \leq \sigma'(e')$ for all $e' < e$; and we write $\sigma \prec \sigma'$ if $\sigma \preceq \sigma'$ and $\sigma \neq \sigma'$. Note that $\sigma \prec \sigma'$ implies $\sigma < \sigma'$, but the converse in general fails. In order to distinguish between $<$ ($\leq$) and $\prec$ ($\preceq$), we say that σ is *(weakly) less* than σ' if $\sigma < \sigma'$ ($\sigma \leq \sigma'$) holds, and we say that σ *(weakly) precedes* σ' if $\sigma \prec \sigma'$ ($\sigma \preceq \sigma'$) holds. If we talk about the *least* e-state with a certain property, we always mean the least e-state with this property with respect to the lexicographical ordering $<$.

Strategy for meeting P_e revisited

In the spirit of the above comments, in order to make the action of the noncomputability requirements compatible with the strategy for meeting the inaccessibility requirements, we ensure that the e-state of the followers of P_e becomes as small as possible. Roughly speaking, this is ensured by replacing a follower y of P_e which is not yet in A and not yet realized by a larger number y' which is not yet in A and which has smaller e-state than y, whenever this is possible. To be more precise, if $y = y(e, s)$ is the follower of P_e at the end of stage s then a *declared* e-state denoted by $\hat{\sigma}(e, s)$ is attached to P_e. We ensure that the true e-state of y, $\sigma(e, y, s)$ weakly precedes the declared e-state $\hat{\sigma}(e, s)$ as long as y is not enumerated into A and not cancelled (if y is cancelled then the declared e-state is cancelled too), and if there is a number $y' > y$ which is not yet in A and which has an e-state less than the declared e-state of P_e then we replace the follower y by such a number y' and let the declared e-state be the true e-state of this number at the current stage.

The above is implemented by adding to (3.8) and (3.9) the following third clause causing P_e to require attention at stage $s + 1$.

P_e has follower y at the end of stage s, y is not realized at stage s, and there is a number y' such that $y < y' \leq s$, $y' \notin A_s$, and $\sigma(e, y', s) < \hat{\sigma}(e, s)$. (3.42)

If this clause applies and P_e receives attention then the follower y is replaced by the least such y' and the declared e-state of P_e at stage $s+1$ is defined to be $\sigma(e, y', s)$. Moreover, if P_e receives attention via clause (3.8) at stage $s+1$ and a new follower is appointed then the maximum e-state 1^e becomes the declared e-state of P_e at stage $s+1$, i.e., $\hat{\sigma}(e, s+1) = 1^e$.

(Note that, by clause (3.42), a P_e-follower y may be replaced at stage $s+1$ by a number $y' \leq s$. So, for given e, y and s, it will not be decidable anymore whether y will be a P_e follower at a stage $\geq s$.)

The construction

The construction is immediate by the above. For clarity of presentation we explicitly state it here using the previously given definition of requiring attention. Parameters depending on the stage which are attached to the requirements persist unless explicitly stated otherwise. If any requirement R_e becomes initialized at stage $s+1$ and it was satisfied at the end of stage s then it becomes unsatisfied. Moreover, if a noncomputability requirement P_e becomes initialized at stage $s+1$ and it has a follower $y = y(e, s)$ at the end of stage s then this follower is cancelled as well as the declared e-state (i.e., $y(e, s+1) \uparrow$ and $\hat{\sigma}(e, s+1) \uparrow$).

At stage 0, no requirement is satisfied and no noncomputability requirement has a follower (i.e., $y(e, 0) \uparrow$ and $\hat{\sigma}(e, 0) \uparrow$ for $e \geq 0$), and $A_0 = C_{n,0} = \emptyset$ for all $n \geq 0$. Stage $s+1 > 0$ is as follows.

Stage $s+1$. Fix k minimal such that R_k requires attention at stage $s+1$, fix e and $i \leq 2$ such that $k = 3e+i$, and distinguish the following three cases according to the type of requirement R_k.

Case 1: $k = 3e$, *i.e.,* $\mathrm{R}_k = \mathrm{P}_e$. If (3.8) holds then appoint $y = s+1$ as follower of P_e (i.e., let $y(e, s+1) = s+1$) and let $\hat{\sigma}(e, s+1) = 1^e$; if (3.9) holds then put the follower $y = y(e, s)$ of P_e into A and declare P_e to be *satisfied*; and if (3.42) holds then replace the follower $y = y(e, s)$ of P_e by the least number y' as there, i.e., let $y(e, s+1) = y'$ and let $\hat{\sigma}(e, s+1) = \sigma(e, y', s)$.

Case 2: $k = 3e+1$, *i.e.,* $\mathrm{R}_k = \mathrm{N}_e$. Declare N_e to be *satisfied.*

Case 3: $k = 3e+2$, *i.e.,* $\mathrm{R}_k = \mathrm{Q}_e$. Then declare Q_e to be *satisfied.* Moreover if (3.14) holds then, for the least x as there, put x into C_n (where $e = \langle n, m\rangle$ for some m).

In any case, declare that requirement R_k *receives attention* and is *active* at stage $s+1$, and initialize all requirements $\mathrm{R}_{k'}$ with $k' > k$.

Verification

Obviously the construction satisfies (3.7). Hence (3.6) holds. So it suffices to show that all requirements are met.

We start with some observations on the followers and the declared states of the noncomputability requirements.

Obviously, for any e, P_e has no follower at the end of stage 0 and, for any $s \geq 0$, there is at most one follower of P_e at the end of stage $s+1$. So we may let $y(e,s)$ denote the follower of P_e at the end of stage s and let $y(e,s)\uparrow$ indicate that there is no such follower. Since at any stage at most one requirement acts and only the noncomputability requirements enumerate numbers into A, (3.10) is immediate. Moreover, by a straightforward induction on stage s, the indices e for which $y(e,s)$ is defined is an initial segment of ω of size $\leq s$, the followers (when defined) are nondecreasing in the stage and strictly increasing in the index, and a follower is bounded by its index from below and by the stage from above, i.e.,

$$\exists\, e \leq s\ (y(e',s)\downarrow \Leftrightarrow\ e' < e) \tag{3.43}$$

$$s < s' \ \&\ y(e,s)\downarrow\ \&\ y(e,s')\downarrow \Rightarrow\ y(e,s) \leq y(e,s') \tag{3.44}$$

$$e < e' \ \&\ y(e',s)\downarrow \Rightarrow\ y(e,s) < y(e',s) \tag{3.45}$$

$$y(e,s)\downarrow \Rightarrow\ e < y(e,s) \leq s \tag{3.46}$$

hold. Also note that if P_e has no follower at some stage then any of its later followers is greater than this stage, i.e.,

$$s < s' \ \&\ y(e,s)\uparrow\ \&\ y(e,s')\downarrow \Rightarrow\ y(e,s') > s, \tag{3.47}$$

and if P_e becomes active then the lower priority followers are cancelled hence

$$e < e' \ \&\ y(e,s) \neq y(e,s+1) \Rightarrow\ y(e',s+1)\uparrow. \tag{3.48}$$

Moreover, if the follower y is assigned to P_e at stage $s+1$ and P_e neither is initialized nor becomes active by stage $s' > s$ then, by cancellation of the lower priority followers at stage $s+1$, no number $\leq s+1$ enters A after stage s and prior to stage $s'+1$ whence

$$s < s' \ \&\ y(e,s) \neq y(e,s+1) \downarrow = y(e,s') \notin A_{s'} \ \Rightarrow\ A_{s'} \upharpoonright s+2 = A_s \upharpoonright s+2. \tag{3.49}$$

The declared state of a noncomputability requirement is defined at the end of a stage if and only if the requirement has a follower at the end of this stage, and the declared state changes if and only if the follower changes. Moreover, if a follower is appointed then the declared state assumes its maximum possible value (hence is weakly preceded by the e-state of the new follower at the previous stage), and if a follower is replaced with another number then the declared state is decreased and coincides with the true e-state of the new follower at the previous stage.

$$\hat{\sigma}(e,s) \downarrow \Leftrightarrow\ y(e,s) \downarrow \tag{3.50}$$

$$\hat{\sigma}(e,s) \neq \hat{\sigma}(e,s+1) \ \Leftrightarrow\ y(e,s) \neq y(e,s+1) \tag{3.51}$$

$$\hat{\sigma}(e,s) \neq \hat{\sigma}(e,s+1) \downarrow \Rightarrow\ \sigma(e, y(e,s+1), s) \preceq \hat{\sigma}(e,s+1) \tag{3.52}$$

$$\hat{\sigma}(e,s) \downarrow \neq \hat{\sigma}(e,s+1) \downarrow \Rightarrow\ \hat{\sigma}(e,s) > \hat{\sigma}(e,s+1) = \sigma(e, y(e,s+1), s). \tag{3.53}$$

Finally, observe that the declared states satisfy the following weak monotonicity property.

$$e < e' \ \&\ y(e,s) \notin A_s \ \&\ y(e',s) \downarrow \Rightarrow\ \hat{\sigma}(e,s) \leq \hat{\sigma}(e',s) \upharpoonright e. \tag{3.54}$$

Namely, for a contradiction, assume that $e < e'$, $y(e,s) \notin A_s$ and $y(e',s) \downarrow$ hold but $\hat{\sigma}(e',s) \upharpoonright e < \hat{\sigma}(e,s)$. Fix $s' < s$ minimal such that $y(e',t) = y(e',s)$ for all t with $s'+1 \leq t \leq s$. Then $y(e',s') \neq y(e',s'+1) \downarrow$ and (by (3.51) and by assumption) $\hat{\sigma}(e',s'+1) = \hat{\sigma}(e',s) < 1^{e'}$. So $P_{e'}$ receives attention at stage $s'+1$ according to clause (3.42) whence, for $y = y(e',s'+1)$, $y(e',s') < y \leq s'$, $y \notin A_{s'}$ and $\sigma(e',y,s') = \hat{\sigma}(e',s'+1)$ hence $\sigma(e',y,s') \upharpoonright e < \hat{\sigma}(e,s)$. On the other hand (by (3.48) and (3.51)) $y(e,s') = y(e,s)$ and $\hat{\sigma}(e,s') = \hat{\sigma}(e,s)$. Since (by (3.45)) $y(e,s') < y(e',s')$ and since $y(e,s') = y(e,s) \notin A_{s'}$, it follows that P_e requires attention at stage $s'+1$ according to clause (3.42) (or clause (3.9)) via y, which gives the desired contradiction.

Call a follower y of P_e *permanent* if $y = y(e,s)$ for all sufficiently large s and call an e-state σ the *permanent declared e-state* of P_e if $\sigma = \hat{\sigma}(e,s)$ for all sufficiently large s. Call requirement R_e *permanently satisfied at stage s* if R_e is satisfied at stage s and not initialized later (hence satisfied at all stages $s' \geq s$), and call R_e *permanently satisfied* if it is permanently satisfied at some stage.

Claim 3.6: Any requirement R_n requires attention only finitely often. Moreover, any requirement P_e has a permanent follower.

Proof: The proof is by induction on n. Fix n and, by inductive hypothesis, fix a stage $s_0 > n$ such that no requirement $\mathrm{R}_{n'}$ with $n' < n$ requires attention after stage s_0. Then R_n receives attention at any stage $s+1 > s_0$ at which it requires attention and R_n is not initialized after stage s_0. So if R_n is a lowness or inaccessibility requirement then R_n will require attention at most once after stage s_0, since if it requires attention then it receives attention and is satisfied from this stage on (hence does not require attention anymore).

So, for the remainder of the proof, we may assume that R_n is a noncomputability requirement, say $\mathrm{R}_n = \mathrm{P}_e$. If P_e has no follower at the end of stage s_0 then (by $s_0 > n \geq e$) it requires and receives attention via clause (3.8) at stage $s_0 + 1$. So, since P_e is not initialized after stage s_0, $y(e, s) \downarrow$ for all $s > s_0$. So P_e does not require attention via (3.8) after stage $s_0 + 1$, and P_e requires attention at most once via (3.9) at a stage $s \geq s_0 + 1$ since if this happens then P_e becomes permanently satisfied at this stage hence does not require attention later. Finally, since $y(e, s)$ is defined for all $s \geq s_0+1$ it follows, by (3.50), that $\hat{\sigma}(e, s) \downarrow$ for such s. It follows, by (3.53), that $\hat{\sigma}(e, s)$ is weakly decreasing after stage s_0 and $\hat{\sigma}(e, s+1) < \hat{\sigma}(e, s)$ if and only if P_e becomes active via clause (3.42) at stage $s+1$. So P_e can act (hence require attention) via (3.42) at most $2^e - 1$ times after stage $s_0 + 1$. So $\mathcal{P}_e$ requires attention only finitely often, and, for the follower $y(e, s+1)$ existing at the last stage $s+1$ at which P_e acts, $y(e, s') = y(e, s+1)$ for all $s' > s$ hence $y(e, s+1)$ is permanent. □

Note that, by Claim 3.6 and by (3.50) and (3.51), requirement P_e does not only have a permanent follower but also a permanent declared e-state. Let $y(e)$ be the permanent follower of requirement P_e and let $\hat{\sigma}(e)$ be the permanent declared e-state of P_e. Moreover, let $t_e + 1$ be the stage at which the permanent follower $y(e)$ is appointed. Note that, by (3.46) and (3.45), $e < y(e) < y(e')$ for $e < e'$. Moreover, it is immediate by Claim 3.6 and construction, that

$$A(y(e)) \neq \{e\}(y(e)) \tag{3.55}$$

(for any $e \geq 0$). Indeed, suppose that $A(y(e)) = 0$. Then, P_e is not permanently satisfied and if $\{e\}(y(e)) = 0$, then P_e would require attention infinitely often via (3.9), contradicting Claim 3.6. Now suppose that $A(y(e)) = 1$. If P_e requires attention at stage $t_e + 1$ via (3.8), then $y(e) = t_e + 1$, and (3.46) and the construction imply that $y(e) \notin A_{t_e+1}$, while if P_e requires attention at stage $t_e + 1$ via (3.42), we again have $y(e) \notin A_{t_e+1}$. It follows from (3.45) that after stage $t_e + 1$, $y(e)$ cannot be the follower of any requirement other than P_e. Thus, $A(y(e)) = 1$ implies that P_e received attention at some stage after $t_e + 1$ via (3.9) and $y(e)$ is realized, so again $A(y(e)) \neq \{e\}(y(e))$.

This immediately implies that the noncomputability requirements are met, and Claim 3.6 easily implies that the lowness requirements are met too.

Claim 3.7: Any requirement P_e and N_e is met.

Proof: The first part of the claim is immediate by (3.55). The proof of the second part is as follows. If $\{e\}_s^{A_s}(e)\downarrow$ for infinitely many stages s then, for the least stage $s \geq e$ such that no requirement of higher priority than N_e requires attention after stage s and $\{e\}_s^{A_s}(e)\downarrow$, N_e becomes active at stage $s+1$ thereby cancelling all lower priority followers. So (by choice of s and by (3.47)) $A \restriction s+1 = A_s \restriction s+1$ which, by our convention (3.2), ensures that $\{e\}^A(e) = \{e\}_s^{A_s}(e)\downarrow$. □

It remains to show that the inaccessibility requirements Q_e are met. For the remainder of the proof, fix $e \geq 0$, and fix $n, m, i_0, i_1, j_0, j_1, j_2, k_0, k_1$ such that $e = \langle n, m\rangle$ and $n = \langle i_0, i_1, j_0, j_1, j_2, k_0, k_1\rangle$. Moreover, by Claim 3.6, fix the greatest stage s_0 such that Q_e or a higher priority requirement requires attention at stage s_0. Note that $s_0 \geq e+1$ (since P_e has higher priority than Q_e, P_e eventually requires attention, and P_e requires attention only at stages $s+1 > e$). Moreover, by initialization,

$$\forall\, e'' > e\ (y(e'', s_0)\uparrow\ \&\ \hat{\sigma}(e'', s_0)\uparrow). \tag{3.56}$$

The following observation will be useful for showing that Q_e is met.

Claim 3.8: (a) Let s, s' and y be given such that $s_0 \leq s < s'$ and $A_{s'} \restriction s+2 = A_s \restriction s+2$. Then $\sigma(e+1, y, s') \preceq \sigma(e+1, y, s)$.

(b) Let e'', s' and y be given such that $e < e''$, $s_0 \leq s'$, and $y = y(e'', s') \notin A_{s'}$. Then $\sigma(e+1, y, s') \preceq \hat{\sigma}(e'', s') \restriction e+1$.

Proof: (a) Given $e' \leq e$ such that $\sigma(e+1,y,s)(e')=0$, it suffices to show that

$$\sigma(e+1,y,s')(e')=0. \tag{3.57}$$

If $Q_{e'}$ is satisfied at stage s then, by $s_0 \leq s$, $Q_{e'}$ is permanently satisfied at stage s_0 hence satisfied at stage s' whence (3.57) holds. Otherwise, there are numbers x, u, v and r such that y is e'-eligible via x, u, v and r at stage $s+1$, and it suffices to show that y is e'-eligible via x, u, v and r at stage $s'+1$ too. But, since $s_0 \leq s$ and $r < s$, this is immediate by Claim 3.5.

(b) Fix $s < s'$ minimal such that $y(e'', s+1) = y(e'', s') = y$. Then, by (3.49), $A_{s'} \upharpoonright s+2 = A_s \upharpoonright s+2$ and, for all t with $s+1 \leq t \leq s'$, $y(e'',t) = y(e'',s') = y$ whence $s_0 \leq s$ by (3.56). So, by part (a) of the claim, $\sigma(e+1,y,s') \preceq \sigma(e+1,y,s)$. On the other hand, by choice of s, it follows, by (3.52), that $\sigma(e+1,y,s) \preceq \hat{\sigma}(e'',s+1) \upharpoonright e+1$ and, by (3.50) and (3.51), that $\hat{\sigma}(e'',s') = \hat{\sigma}(e'',s+1)$. So $\sigma(e+1,y,s') \preceq \hat{\sigma}(e'',s') \upharpoonright e+1$. □

Claim 3.9: Q_e is met.

Proof: If Q_e is permanently satisfied then, for the greatest stage $s+1$ at which Q_e receives attention, Q_e is not initialized later whence $A \upharpoonright s+2 = A_s \upharpoonright s+2$. So Q_e is met by Claim 3.2. Hence, for the remainder of the proof we may assume that Q_e is not satisfied at any stage $s \geq s_0$. Moreover, w.l.o.g. we may assume that the clauses (i) and (ii) in Q_e hold while clause $(*)$ fails (since, otherwise, Q_e is obviously met). So, in particular, the assumptions made in Claim 3.3 and Claim 3.4 hold.

By the above and by Claim 3.3, in order to show that Q_e is met, it suffices to show that there is an infinite computable set S of stages s satisfying (3.26). For giving such a set S, we need some more notation and observations.

Let $e' \leq e$ and $s \geq s_0$. A number y is *truly* e'*-eligible via* x, u, v, r *at stage* $s+1$ if y is e'-eligible via x, u, v, r at stage $s+1$ and $A \upharpoonright r = A_s \upharpoonright r$; y is *truly* e'*-eligible at stage* $s+1$ if y is truly e'-eligible via some numbers x, u, v and r at stage $s+1$; and the $(e+1)$-state of y *truly precedes* the $(e+1)$-state σ at stage s ($\sigma(e+1,y,s) \preceq_t \sigma$ for short) if, for any $e' \leq e$ such that $\sigma(e')=0$ and $Q_{e'}$ is not satisfied at the end of stage s_0, y is truly e'-eligible at stage $s+1$.

Note that if $e' \leq e$, $s \geq s_0$, $Q_{e'}$ is not satisfied at stage s_0 and y is truly e'-eligible at stage $s+1$ then (by (3.25)) $y \notin A$ and, by choice of s_0 and

Claim 3.5, y is (truly) e'-eligible at stage $s'+1$ for all stages $s' \geq s$. It follows that, for any number y, any stages s and s' such that $s' \geq s \geq s_0$ and any $(e+1)$-states σ and σ' such that $\sigma \preceq \sigma'$ and such that the $(e+1)$-state of y truly precedes σ at stage s, it holds that the $(e+1)$-state of y at stage s' truly precedes σ', i.e.,

$$\forall\, s, s', y \geq 0\; \forall\, \sigma, \sigma' \in \{0,1\}^{e+1}$$
$$([s_0 \leq s \leq s' \;\&\; \sigma(e+1,y,s) \preceq_t \sigma \;\&\; \sigma \preceq \sigma'] \Rightarrow \sigma(e+1,y,s') \preceq_t \sigma') \tag{3.58}$$

holds. Moreover, obviously, for $s \geq s_0$, $\sigma(e+1,y,s) \preceq_t \sigma$ implies $\sigma(e+1,y,s) \preceq \sigma$. Conversely, (by (3.25))

$$\forall\, s \geq s_0\; \forall\, y \geq 0\; \forall\, \sigma \in \{0,1\}^{e+1}$$
$$([\sigma(e+1,y,s) \preceq \sigma \;\&\; A \restriction s+2 = A_s \restriction s+2] \Rightarrow \sigma(e+1,y,s) \preceq_t \sigma). \tag{3.59}$$

Now let σ^* be the least $(e+1)$-state σ (with respect to the ordering $\leq$) such that, for infinitely many numbers $y \notin A$, there is a stage $s \geq s_0$ at which the $(e+1)$-state of y truly precedes σ, i.e.,

$$\sigma^* = \mu\, \sigma \in \{0,1\}^{e+1}[\overset{\infty}{\exists}\, y \notin A\; \exists\, s \geq s_0\; (\sigma(e+1,y,s) \preceq_t \sigma)]$$

(note that σ^* exists since A is co-infinite and, for any y and $s \geq s_0$, the $(e+1)$-state of y truly precedes 1^{e+1} at stage s), and fix $e_0 > e$ minimal such that

$$y(e_0) \notin A \tag{3.60}$$

and, for any $y \geq e_0$ with $y \notin A$, there is no stage $s \geq s_0$ and no $\sigma < \sigma^*$ such that the $(e+1)$-state of y truly precedes σ at stage s, i.e.,

$$\forall\, y \geq e_0\; \forall\, s \geq s_0\; \forall\, \sigma < \sigma^*\; (y \notin A \;\Rightarrow\; \sigma(e+1,y,s) \not\preceq_t \sigma) \tag{3.61}$$

holds (note that e_0 exists since, by choice of σ^*, (3.61) holds for almost all numbers e_0 while, by (3.55), (3.60) holds for infinitely many e_0).

The following properties of σ^* will be crucial.

$$\forall\, e'' \geq e_0\; \forall\, s > t_{e_0}\; (\hat{\sigma}(e'',s)\downarrow \Rightarrow\; \sigma^* \leq \hat{\sigma}(e'',s) \restriction e+1) \tag{3.62}$$

$$\forall\, e'' \geq e_0\; (y(e'') \notin A \;\Rightarrow\; \hat{\sigma}(e'') \restriction e+1 = \sigma^*) \tag{3.63}$$

$$\sigma^*(e) = 0. \tag{3.64}$$

The proof of (3.62) is as follows. Since $y(e_0) \notin A$ and $\hat{\sigma}(e_0,s) = \hat{\sigma}(e_0, t_{e_0}+1)$ for all $s > t_{e_0}$, by (3.54) (and (3.50)), it suffices to show that $\sigma^* \leq \hat{\sigma}(e_0, t_{e_0}+1) \upharpoonright e+1$. This is done as follows. By choice of t_{e_0} and by (3.60), P_{e_0} becomes active via clause (3.8) or (3.42) at stage $t_{e_0}+1$ — hence $\sigma(e_0, y(e_0), t_{e_0}) \preceq \hat{\sigma}(e_0, t_{e_0}+1)$ — and $A \upharpoonright t_{e_0}+2 = A_{t_{e_0}} \upharpoonright t_{e_0}+2$. So, by (3.59), $\sigma(e+1, y(e_0), t_{e_0}) \preceq_t \hat{\sigma}(e_0, t_{e_0}+1) \upharpoonright e+1$. Since $e_0 \leq y(e_0)$ it follows by choice of e_0 that $\sigma^* \leq \hat{\sigma}(e_0, t_{e_0}+1) \upharpoonright e+1$, which completes the proof of (3.62).

For a proof of (3.63), fix $e'' \geq e_0$ such that $y(e'') \notin A$, and, for a contradiction, assume that $\hat{\sigma}(e'') \upharpoonright e+1 \neq \sigma^*$. Then, by (3.62), $\sigma^* < \hat{\sigma}(e'') \upharpoonright e+1$. But, by choice of $\sigma^*$and (3.58) this implies that there is a number $y > y(e'')$ such that $y \notin A$ and the $(e+1)$-state of y strictly precedes $\hat{\sigma}(e'') \upharpoonright e+1$ at all sufficiently large stages. So $\mathrm{P}_{e''}$ requires attention infinitely often via clause (3.42) contrary to Claim 3.6. (Note that, by $y(e'') \notin A$ and (3.55), the permanent follower $y(e'')$ is never realized.)

Finally, for a proof of (3.64), for a contradiction assume that $\sigma^*(e) = 1$. Then, by (3.61), there is no number y and no stage s such that

$$\begin{gathered} y \notin A \;\&\; e_0 \leq y \;\&\; s \geq s_0 \;\&\; \sigma(e+1, y, s) \preceq_t \sigma^* \;\& \\ y \text{ is truly } e\text{-eligible at stage } s+1. \end{gathered} \tag{3.65}$$

(Namely, for such y and s, $\sigma(e+1, y, s) \preceq_t \sigma$ for the string $\sigma < \sigma^*$ defined by $\sigma(e) = 0$ and $\sigma(e') = \sigma^*(e')$ for $e' < e$.) So, in order to get the desired contradiction, it suffices to show that there is a number y and a stage s satisfying (3.65). Let E be the A-computably enumerable set

$$E = \{y \notin A : \exists\, e'' \geq e_0 \;\exists\, s \geq s_0 \;(y(e'', s) \neq y(e'', s+1) = y \;\&$$

$$A \upharpoonright s+2 = A_s \upharpoonright s+2 \;\&\; \hat{\sigma}(e'', s+1) \upharpoonright e+1 = \sigma^*)\}.$$

Then, for any $y \in E$, $y \notin A$, $e_0 \leq y$ (by (3.46)), and, by (3.52), (3.59) and (3.58), $\sigma(e+1, y, s) \preceq_t \sigma^*$ for almost all stages s. So it only remains to show that there is a number $y \in E$ which is truly e-eligible at some stage $s \geq s_0$ hence at almost all stages s. But the existence of such a number y follows by Claim 3.4 since E is infinite — namely, for any number $e'' \geq e_0$ such that $y(e'') \notin A$, $y(e'') \in E$ since $y(e'', t_{e''}) \neq y(e'', t_{e''}+1) = y(e'')$, $A \upharpoonright t_{e''}+2 = A_{t_{e''}} \upharpoonright t_{e''}+2$ (by (3.49)), and $\hat{\sigma}(e'', t_{e''}+1) \upharpoonright e+1 = \sigma^*$ (by (3.63)) — and since any infinite A-c.e. set contains an infinite A-computable subset. This completes the proof of (3.64).

Now S is defined by

$$\begin{gathered} S = \{s > t_{e_0} : \forall\, e'' > e_0 \;(y(e'', s)\downarrow \;\& \\ y(e'', s) \notin A_s \;\Rightarrow\; \hat{\sigma}(e'', s+1) \upharpoonright e+1 = \sigma^*)\}. \end{gathered} \tag{3.66}$$

Obviously, S is computable. Moreover, S is infinite. Namely, for any $e'' > e_0$ such that $y(e'') \notin A$, $y(e''', t_{e''} + 1) = y(e''')$ and $y(e''') \in A$ if and only if $y(e''') \in A_{t_{e''}+1}$ for all $e''' \leq e''$. So, for any such e'', $t_{e''} + 1 \in S$ by (3.63). Infinity of S follows since there are infinitely many e'' with $y(e'') \notin A$.

It remains to show that the elements s of S satisfy (3.26). So fix $s \in S$, a number $y \leq s$ and a stage $s' \geq s$ such that $y \in A_{s'+1} \setminus A_{s'}$. We have to show that y is e-eligible at stage $s'+1$. Fix e'' such that $y = y(e'', s')$. Since y enters A at stage $s' + 1 > t_{e_0} + 1$ it follows that $e_0 < e''$. Moreover, since $y \leq s$, it follows by (3.47) that $y(e'', t)$ is defined for all t with $s \leq t \leq s'$. So, by (3.53) and (3.51) and by choice of s,

$$\hat{\sigma}(e'', s') \restriction e + 1 \leq \hat{\sigma}(e'', s) \restriction e + 1 = \sigma^*$$

hence, by (3.62), $\hat{\sigma}(e'', s') \restriction e + 1 = \sigma^*$. It follows by Claim 3.8(b) that $\sigma(e+1, y, s')$ precedes σ^*. But, by (3.64), this implies that y is e-eligible at stage $s' + 1$.

This completes the proof of Claim 3.9. □

This completes the proof of Theorem 2.7.

4. Proof of Theorem 2.8

Given a high c.e. degree $\mathbf{a}$, it suffices to give c.e. degrees $\mathbf{a}_0$ and $\mathbf{a}_1$ such that

$$\mathbf{a} = \mathbf{a}_0 \vee \mathbf{a}_1 \tag{4.1}$$

and, for $i = 0, 1$,

$$\begin{gathered} \forall\, \mathbf{b_0}, \mathbf{b_1}\ [\mathbf{b_0} \vee \mathbf{b_1} = \mathbf{a_i} \Rightarrow \mathbf{a_i} \leq \mathbf{b_0} \quad \text{or} \\ \forall\, \mathbf{c_0}, \mathbf{c_1} \not\leq \mathbf{a_i}\ \exists\, \mathbf{d}\ (\mathbf{d} \leq \mathbf{b_1} \vee \mathbf{c_0}, \mathbf{b_1} \vee \mathbf{c_1}\ \&\ \mathbf{d} \not\leq \mathbf{a_i})] \end{gathered} \tag{4.2}$$

hold. (Note that (4.2) ensures that $\mathbf{a}_i$ is s.m.i. unless $\mathbf{a}_i = \mathbf{0}$. So, by $\mathbf{a} > \mathbf{0}$ and by (4.1), either $\mathbf{a}_0$ and $\mathbf{a}_1$ are s.m.i. or one of these degrees is $\mathbf{0}$ and $\mathbf{a}$ is s.m.i. In either case, this implies Theorem 2.8.)

The strategy for satisfying (4.2) is based on the corresponding strategy introduced in the proof of Theorem 2.7 above. For $i = 0, 1$ we enumerate c.e. sets A_i and auxiliary c.e. sets $C_{i,n}$ ($n \geq 0$) such that, for any numbers $n, m \geq 0$ and for the corresponding unique numbers $i_0, i_1, j_0, j_1, j_2, k_0, k_1$ and e such that $n = \langle i_0, i_1, j_0, j_1, j_2, k_0, k_1 \rangle$ and $e = \langle n, m \rangle$,

$$C_{i,n} \leq_{\mathrm{T}} W_{i_1} \oplus W_{k_0}, W_{i_1} \oplus W_{k_1} \tag{4.3}$$

holds and the *Inaccessibility Requirements*

$$\begin{aligned}\mathrm{Q}^i_e : \text{If} \quad & (i)\ W_{i_0} = \{j_0\}^A \ \&\ W_{i_1} = \{j_1\}^{A_i} \ \&\ A_i = \{j_2\}^{W_{i_0}\oplus W_{i_1}},\\ & (ii)\ W_{k_0}, W_{k_1} \not\leq_{\mathrm{T}} A_i, \text{ and}\\ & (iii)\ A_i \not\leq_{\mathrm{T}} W_{i_0}\\ \text{then } & (*)\ C_{i,n} \neq \{m\}^{A_i}\end{aligned}$$

are met. (In the following, when we consider one of the sets $C_{i,n}$ or one of the requirements Q^i_e, we tacitly assume that $n = \langle i_0, i_1, j_0, j_1, j_2, k_0, k_1\rangle$ and $e = \langle n, m\rangle$.)

Just as in the proof of Theorem 2.7, condition (4.3) is satisfied by guaranteeing

$$x \in C_{i,n,s+1} \setminus C_{i,n,s} \Rightarrow \exists\, z < \min\{\gamma_{k_0,s}(x), \gamma_{k_1,s}(x)\}\ (z \in W_{i_1,s+1} \setminus W_{i_1,s}), \tag{4.4}$$

and we use the finitary strategy for meeting requirement Q^i_e by diagonalization introduced there. So requirement Q^i_e *requires attention* at stage $s+1$ if $e \leq s$, Q^i_e is not satisfied at (the end of) stage s, and one of the following holds.

$$\exists\, x \left(\begin{array}{l} A_{i,s}(x) \neq \{j_2\}_s^{W_{i_0,s}\oplus W_{i_1,s}}(x)\downarrow, \\ \text{and, for } u = u(W_{i_0,s} \oplus W_{i_1,s}; j_2, x, s) \text{ and for } p \leq 1, \\ W_{i_p,s} \restriction u = \{j_p\}_s^{A_{i,s}} \restriction u \end{array}\right) \tag{4.5}$$

$$\exists\, x\, \exists\, p \leq 1\ (W_{i_p,s}(x) = 1 \neq \{j_p\}_s^{A_{i,s}}(x)\downarrow) \tag{4.6}$$

$$\exists\, x \in \omega^{[e]} \left(\{m\}_s^{A_{i,s}}(x) = 0 \ \&\ \exists\, z < \min\{\gamma_{k_0,s}(x), \gamma_{k_1,s}(x)\}\ (z \in W_{i_1,s+1} \setminus W_{i_1,s})\right). \tag{4.7}$$

The corresponding action (if Q^i_e receives attention at stage $s+1$) is essentially as before: Declare Q^i_e to be *satisfied*. Moreover if (4.7) holds then, for the least x as there, put x into $C_{i,n}$. Note that the latter action is consistent with (4.4). Moreover, as in the proof of Theorem 2.7 (compare with Claim 3.2), the following holds (for $i = 0, 1$) implying that Q^i_e is met if it is permanently satisfied.

Claim 4.1: Assume that Q_e^i requires and receives attention at stage $s+1$ and that $A_i \restriction s+1 = A_{i,s} \restriction s+1$. Then either clause (i) in Q_e^i fails or clause $(*)$ in Q_e^i holds. So, in particular, Q_e^i is met.

While in the basic construction, however, we attempted to guarantee $A \restriction s+1 = A_s \restriction s+1$ by cancelling the lower priority followers of the noncomputability requirements when Q_e received attention, in the current setting (where the noncomputability requirements are replaced by some coding requirements, see below) cancellation is not possible. So, instead, when Q_e^i receives attention at stage $s+1$, then it imposes a restraint $r_Q^i(e, s+1) = s+1$ on A_i. Correspondingly, we initialize Q_e^i later if (and only if) this restraint becomes injured. More formally, Q_e^i is *initialized* at stage $s'+1$ if $A_{i,s'+1} \restriction r_Q^i(e,s') \neq A_{i,s'} \restriction r_Q^i(e,s')$ (as in the previous constructions we assume that any parameters stay unchanged during the construction unless explicitly stated otherwise). If initialized at stage $s'+1$ then Q_e^i becomes *unsatisfied* at stage $s'+1$ and the restraint is lifted by setting $r_Q^i(e,s'+1) = 0$. If Q_e^i is satisfied at stage $s+1$ and not initialized later then we say that Q_e^i is *permanently satisfied* (at stage $s+1$). (Q_e^i can become initialized only by the action of a coding requirement (to be defined below) where the coding requirement may have higher or lower priority than Q_e^i. But, since a coding requirement has a choice whether it puts a code into A_0 or into A_1, just as in the proof of the Sacks splitting theorem, by enforcing the highest priority restraint, eventually Q_e^i will not be initialized anymore.)

As in the proof of Theorem 2.7, the direct strategy for meeting the inaccessibility requirements does not suffice to guarantee that these requirements are met. Unless Q_e^i is permanently satisfied (hence met by Claim 4.1) or met for some trivial reason, in addition it has to be ensured that numbers which enter A_i "late" are "eligible". To make this more precise, define *i-e-(pre)eligibility* of a number y (via numbers x, u, v and r) at stage $s+1$ just as e-(pre)eligibility in the proof of Theorem 2.7 but with the set A_i in place of A (i.e., by replacing A_s by $A_{i,s}$ in (3.15)–(3.24)); and, similarly, the *i-e-state* $\sigma_i(e,y,s)$ of a number y at stage s is defined as the e-state $\sigma(e,y,s)$ in the basic construction with A_i, $Q_{e'}^i$ and i-e'-eligibility in place of A, $Q_{e'}$ and e'-eligibility, respectively. Then, as there (compare with Claim 3.3), one can observe that the following holds (for $i = 0, 1$).

Claim 4.2: Assume that Q_e^i requires attention only finitely often and is satisfied only finitely often. If there is an infinite computable set S of stages such that, for any $s \in S$,

$$\forall\, y, s' \; ([y \leq s \leq s' \;\&\; y \in A_{i,s'+1} \setminus A_{i,s'}] \;\Rightarrow\; y \text{ is } i\text{-}e\text{-eligible at stage } s'+1) \tag{4.8}$$

holds, then requirement Q^i_e is met.

Moreover, as in the proof of Theorem 2.7 we can prove the following two claims (corresponding to Claims 3.4 and 3.5) giving sufficient conditions for the existence of i-e-eligible numbers and for the preservation of i-e-eligibility ($i = 0, 1$).

Claim 4.3: Assume that clauses (i) and (ii) in Q^i_e hold, and that $\{m\}^{A_i}$ is total and, for almost all numbers $x \in \omega^{[e]}$, $\{m\}^{A_i}(x) = 0$. Then, for any infinite A_i-computable subset E of $\overline{A_i}$ there is an infinite subset F of E such that, for any $y \in F$, there are numbers x_y, u_y, v_y and r_y such that y is i-e-eligible via x_y, u_y, v_y and r_y at almost all stages.

Claim 4.4: Let e, y, x, u, v, r, s and s' be given such that $s \leq s'$, $e \leq s'$, y is i-e-(pre)eligible via x, u, v and r at stage $s+1$, $A_{i,s'} \upharpoonright r = A_{i,s} \upharpoonright r$, and Q^i_e is neither satisfied at stage s' nor requires attention at stage $s'+1$. Then y is i-e-(pre)eligible via x, u, v and r at stage $s'+1$.

Having reviewed the basic features of the inaccessibility strategy, we now turn to the strategy for guaranteeing that, for $\mathbf{a}_i = deg(A_i)$, (4.1) holds. This strategy is based on *Martin permitting*, a sort of "almost - always" permitting argument pertaining to high c.e. degrees (see e.g. Soare [18], Chapter XI.2 for more details).

We start with some notation. Given an infinite c.e. set A and a computable 1-1 function a enumerating A, let $A_s = \{a(0), \ldots, a(s-1)\}$, let $\overline{a}(x, s)$ be the $(x+1)$th element of $\overline{A_s}$ in order, let $\overline{a}(x)$ be the $(x+1)$th element of $\overline{A}$ in order, and let

$$c_{\overline{A}}(x) = \mu s\, (\overline{a}(x, s) = \overline{a}(x)).$$

Now, since the given degree $\mathbf{a}$ is high, the following holds (see e.g. Theorem XI.2.1 in [18]).

Fact 4.5: There is a c.e. set $A \in \mathbf{a}$ and a computable 1-1 function a enumerating A such that $c_{\overline{A}}$ dominates all computable functions.

For the remainder of the proof fix such a set A and the corresponding enumeration function a and enumeration $(A_s)_{s \geq 0}$. Say that a number y is

M(artin)-permitted (by A) at stage $s+1$ if $\overline{a}(y, s+1) \neq \overline{a}(y, s)$. Then, by Fact 4.5, for any given computable function $f(y)$, y is M-permitted at a stage $> f(y)$ for almost all y.

Now in order to guarantee $A =_{\mathrm{T}} A_0 \oplus A_1$ (hence (4.1)), we use a combined Martin and marker permitting. To be more precise, we define a partial computable marker function $\beta(e, s)$ (i.e., a partial computable function $\beta : \omega \times \omega \to \omega$ with computable domain) with the following properties (for any $e, e', s, s', y \geq 0$).

Initially, the marker is nowhere defined, i.e.,

$$(\beta_0)\ \forall\, e\ (\beta(e, 0)\uparrow).$$

At any stage $s+1 > 0$ either, for the least e such that $\beta(e, s)\uparrow$, $\beta(e, s+1)$ becomes defined while all other marker positions remain unchanged, or a marker $\beta(e)$ defined at stage s may be moved to a higher position or lifted in which case the markers below remain unchanged while the markers above are lifted. Moreover, if the marker $\beta(e)$ is put down on a new position at stage $s+1$ then we let $\beta(e, s+1) = \langle s+1, 2^{s+1}\rangle > s+1$. So, more formally, the following holds (for all $s \geq 0$).

$$\begin{aligned}(\beta_{00})\ \exists\, e \leq s\ &\Big[\forall\, e' < e\ (\beta(e', s) = \beta(e', s+1)\downarrow)\ \&\\ &\beta(e, s) \neq \beta(e, s+1)\ \&\\ &(\beta(e, s+1) = \langle s+1, 2^{s+1}\rangle \vee (\beta(e, s)\downarrow\ \&\ \beta(e, s+1)\uparrow))\ \&\\ &\forall\, e'' > e\ (\beta(e'', s+1)\uparrow)\Big].\end{aligned}$$

Here we assume that the pairing function $\langle\ ,\ \rangle$ is not only strictly increasing in both arguments but also satisfies $x \leq \langle x, 0\rangle$ and $\langle x, 2^x\rangle < \langle x+1, 0\rangle$ for all $x \geq 0$. (The latter property will be used below in order to keep the traces defined there in order.)

As one can easily check (by induction on s), (β_0) and (β_{00}) guarantee that, at any stage s, β is defined on an initial segment of ω of length $\leq s$,

$$(\beta_1)\ \exists\, e \leq s\ (\beta(e', s)\downarrow \Leftrightarrow\ e' < e),$$

that β is strictly increasing on this initial segment,

$$(\beta_2)\ [e < e'\ \&\ \beta(e', s)\downarrow]\ \Rightarrow\ \beta(e, s) < \beta(e', s),$$

that, whenever defined, $\beta(e,s)$ is nondecreasing in s, exceeds e, and is bounded by $\langle s+1, 2^{s+1}\rangle$,

$$(\beta_3) \quad [s \leq s' \ \&\ \beta(e,s)\downarrow\ \&\ \beta(e,s')\downarrow]$$

$$\Rightarrow\ e < \beta(e,s) \leq \beta(e,s') \leq \langle s', 2^{s'}\rangle,$$

that the value of $\beta(e)$ at a stage $s+1$ where it is put down or moved to a new value is greater than this stage,

$$(\beta_4)\ \beta(e,s) \neq \beta(e,s+1)\downarrow \Rightarrow\ \beta(e,s+1) = \langle s+1, 2^{s+1}\rangle > s+1,$$

and that the values of the marker on different arguments differ (when defined) independent of the stages, i.e.,

$$(\beta_5)\ [e \neq e' \ \&\ \beta(e,s)\downarrow\ \&\ \beta(e',s')\downarrow]\ \Rightarrow\ \beta(e,s) \neq \beta(e',s').$$

Now, in order to guarantee that A is Turing equivalent to $A_0 \oplus A_1$, we guarantee that, eventually, the marker $\beta(e)$ comes to a final position $\beta^*(e) \in \omega$, i.e.,

$$(\beta_6)\ \beta^*(e) = \lim_{s\to\omega} \beta(e,s) < \omega \text{ exists},$$

and that the moves of the marker are related to the given enumeration of the given set A and the constructed enumerations of the sets A_i $(i = 0,1)$ under construction as follows.

If defined on e at stage s, the marker $\beta(e)$ is allowed to move or to be lifted at stage $s+1$ only if the old marker position is M-permitted, i.e.,

$$(\beta_7)\ \beta(e,s)\downarrow \neq \beta(e,s+1)\ \Rightarrow\ \overline{a}(\beta(e,s),s) \neq \overline{a}(\beta(e,s),s+1),$$

and any such move has to be made recognizable for $A_0 \oplus A_1$ by enumerating a new number y with $y < \beta(e,s)$ into A_0 or A_1 at stage $s+1$, i.e.,

$$(\beta_8)\ \beta(e,s)\downarrow \neq \beta(e,s+1)\ \Rightarrow\ \exists\, i \leq 1\ \exists\, y < \beta(e,s)(y \in A_{i,s+1} \setminus A_{i,s}).$$

Conversely, if a number y is enumerated into A_0 or A_1 at stage $s+1$ then this is witnessed by a corresponding move of β, i.e.,

$$(\beta_9) y \in A_{i,s+1} \setminus A_{i,s}\ \Rightarrow\ \exists\, e\, [\beta(e,s)\downarrow\ \&\ y \in [e, \beta(e,s)) \\ \&\ \beta(e,s) \neq \beta(e,s+1)].$$

Finally, for any element $a(s)$ of A,

$$(\beta_{10})\ \beta(a(s),s)\uparrow \text{ or } \beta(a(s),s+1)\uparrow.$$

Note that, by (β_4), (β_3), and (β_6), the last condition implies that, for $e \in A$, the final position $\beta^*(e)$ of the marker on e exceeds the stage at which the number e is enumerated into A, namely

$$(\beta_{10^*})\ \beta^*(a(s)) \geq s+1.$$

That the above rules guarantee $A =_{\mathrm{T}} A_0 \oplus A_1$ is shown as follows.

Claim 4.6: Assume that the partial computable marker β satisfies the conditions (β_1)–(β_{10}) for all e, e', s, s', y. Then $A \leq_{\mathrm{T}} \beta^*$, $\beta^* \leq_{\mathrm{T}} A$, $\beta^* \leq_{\mathrm{T}} A_0 \oplus A_1$, and $A_0 \oplus A_1 \leq_{\mathrm{T}} \beta^*$. So, in particular, $A =_{\mathrm{T}} A_0 \oplus A_1$.

Proof: Note that, by (β_6), $\beta^*(e) = \lim_{s\to\omega} \beta(e,s) \in \omega$ is well-defined and, by (β_3), $\beta^*(e) = \sup\{\beta(e,s) : s \geq 0 \ \&\ \beta(e,s)\downarrow\}$. So $A \leq_{\mathrm{T}} \beta^*$ holds since, by (the above and) (β_{10^*}), a number e is in A iff $e \in A_{\beta^*(e)}$; $\beta^* \leq_{\mathrm{T}} A$ holds since, by (β_7), $\beta^*(e) = \beta(e,s)$ for the least s such that $\beta(e,s)\downarrow$ and $\overline{a}(\beta(e,s),s) = \overline{a}(\beta(e,s))$; $\beta^* \leq_{\mathrm{T}} A_0 \oplus A_1$ holds since, by (β_8), $\beta^*(e) = \beta(e,s)$ for the least s such that $\beta(e,s)$ is defined and $A_i \upharpoonright \beta(e,s) = A_{i,s} \upharpoonright \beta(e,s)$ for $i = 0,1$; and $A_0 \oplus A_1 \leq_{\mathrm{T}} \beta^*$ holds since, by (β_9), it holds that, for any $y \geq 0$ and $i \leq 1$, $A_i(y) = A_{i,s}(y)$ for the least stage s such that $\beta^*(e) = \beta(e,s)$ for all $e \leq y$. □

The task to guarantee the above rules for the eth marker $\beta(e)$, in particular to guarantee that $\beta(e)$ comes to a limit and that, for $e = a(s)$, (β_{10}) holds, is assigned to the eth *coding requirement* (for $e \geq 0$)

$$\mathrm{P}_e : \ \beta^*(e) \in \omega \text{ exists and if } e = a(s) \text{ and } \beta(e,s)\downarrow \text{ then } \beta(e,s+1)\uparrow.$$

The priority ordering of the requirements is given by $\mathrm{P}_e = \mathrm{R}_{3e}$ and $\mathrm{Q}^i_e = \mathrm{R}_{3e+i+1}$ $(i = 0,1,\ e \geq 0)$.

The basic strategy for meeting P_e is twofold. First, if e is minimal such that $\beta(e,s)$ is undefined (and there is no higher priority coding requirement which becomes active at stage $s+1$) then P_e puts down marker $\beta(e)$ (according to (β_4)) by letting $\beta(e,s+1) = \langle s+1, 2^{s+1}\rangle$. Second, if e enters A at stage $s+1$, i.e., if $e = a(s)$, and $\beta(e,s)\downarrow$ then P_e lifts the marker $\beta(e)$ and all markers $\beta(e'')$ with $e < e''$.

Note that this action is consistent with the rules (β_1)–(β_5) and guarantees (β_6) and (β_{10}) where, for the former, one should note that $\beta(e)$ changes only if P_e or a higher priority coding requirement becomes active and that this will happen only finitely often. Moreover (by (β_3)), the above action is consistent with (β_7) since e entering A at stage $s+1$ M-permits the lifting of $\beta(e)$ at stage $s+1$. In order to ensure (β_8), however, when $\beta(e)$ is lifted at stage $s+1$ (in order to guarantee (β_{10})) then, for some $i \leq 1$, a *trace* $y < \beta(e,s)$ has to be put into A_i at stage $s+1$. This action in turn has to be made compatible with (β_9) for which it suffices to choose the trace y so that $y \in [e, \beta(e,s)) \setminus A_{i,s}$ holds. So, when $\beta(e)$ is put down at stage $s+1$ then at the same stage *i-e-traces* $\beta^i(e, s+1) \in [e, \beta(e, s+1)) \setminus A_{i,s}$ ($i = 0, 1$) are put down too, and one of these traces is enumerated into its corresponding set A_i if $\beta(e)$ is lifted later in order to ensure (β_{10}). The decision, which of the traces is chosen, is based on the current restraints. Namely, in order to guarantee that the restraint imposed by a requirement is eventually obeyed, we have to guarantee that, if some higher priority restraints might be injured, then the highest priority restraint which may be injured by the enumeration of a trace is protected by choosing the trace going into the other side (we refer to this as the *splitting condition* in the following).

Just as in the basic construction, the positive requirements P_e have to cooperate with the higher priority inaccessibility requirements in order to make the strategy for meeting the latter work. This requires some amendments of the just described basic P_e-strategy. In particular, if the current i-e-trace $\beta^i(e,s)$ of the coding requirement P_e can be replaced by a larger unused number y having i-e-state σ less than the (declared) i-e-state of $\beta^i(e,s)$ then we have to do so. Since such a number y may be bigger than the current position of the marker $\beta(e)$, this requires moving $\beta(e)$ beyond y in order to ensure that $y \in [e, \beta(e))$. Now moving $\beta(e)$ requires M-permission. By the domination property of A, in almost all cases, eventually we get such a permission, but we have to ensure that, while waiting for this permission, the i-e-state σ of y is preserved. In order to achieve this, we have to impose a restraint on A_i once we see the candidate y (compare with Claim 4.4). Moreover, once permission is given to move the marker $\beta(e)$, in addition we have to put one of the current traces of P_e into A_0 or A_1. Of course, we cannot use the i-e-trace since enumeration of $\beta^i(e,s) < y$ into A_i may destroy the i-e-state σ of y. So we have to enumerate the $(1-i)$-e-trace $\beta^{1-i}(e,s)$ into A_{1-i}. The latter, however, may cause the following two problems. First, this might violate the splitting condition. In order to prevent this from happening, whenever one of the traces is less than the restraint imposed by the higher priority requirements on the corresponding side then, once permission is given to move $\beta(e)$, we lift the marker (simultaneously enumerating the trace satisfying the splitting condition in its corresponding set) and start all over again with the P_e-strategy. Second, by enumerating the $(1-i)$-e-trace into A_{1-i}, this trace is used up and we have to assign a new $(1-i)$-e-trace which may have a greater $(1-i)$-e-state than the previous trace. So improving the i-e-state of the i-e-trace may worsen the $(1-i)$-e-state of the $(1-i)$-e-trace. In order to overcome the latter problem, we do not assign a single i-e-trace $\beta^i(e)$ to P_e but a sequence $\vec{\beta}^i(e) = (\beta^i_0(e), \ldots, \beta^i_{2^e}(e))$ of $2^e + 1$ i-e-traces and assign a declared i-e-state to the sequence (as a whole) where $\beta^i_0(e) < \beta^i_1(e) < \cdots < \beta^i_{2^e}(e)$, the i-e-state of any of these traces precedes the declared i-e-state, and, for $k < 2^e$, the trace y_{k+1} (for ease of notation, we write y_k for $\beta^i_k(e)$) is chosen big enough such that enumerating y_{k+1} into A_i will not affect the i-e-state of the smaller members $y_0, \ldots, y_k$ in the sequence. Since the declared i-e-state can be improved at most $2^e - 1$ times, this guarantees that there are enough of the current $(1-i)$-e-traces left for the necessary updates of i-e-traces (and vice versa) and we can preserve the i-e-states of the unused traces by enumerating the traces of the given sequence in decreasing order. Correspondingly to the basic construction, the first sequence of i-e-traces for P_e is assigned the declared i-e-state 1^e, and the current sequence of i-e-traces with state σ is only replaced by a new sequence if the states of all members of this new sequence precede a state $\sigma' < \sigma$ (so that the declared state can be improved to σ' by replacing the old sequence with the new sequence).

For the implementation of the just described P_e-strategy, we use the following definition. Given a binary string σ of length e, call a sequence $\vec{y} = (y_0, \ldots, y_{2^e})$ (of numbers) of length $2^e + 1$ *i-σ-z-eligible* at stage $s + 1$ if the following hold.

(1) $z < y_0 < y_1 < \cdots < y_{2^e} < s$.
(2) For $k \leq 2^e$, $y_k \notin A_{i,s}$.
(3) For $k \leq 2^e$, $\sigma_i(e, y_k, s) \preceq \sigma$. Moreover, for any $e' < e$ and $k \leq 2^e$ such that $\sigma(e') = 0$ and $\mathrm{Q}^i_{e'}$ is not satisfied at the end of stage s, there are numbers x_k, u_k, v_k and r_k such that y_k is i-e'-eligible via x_k, u_k, v_k and r_k at stage $s + 1$ and $r_k < y_{k+1}$ (where we let $y_{2^e+1} = s$).

As explained above, if the replacement of the current sequence $\vec{\beta}^i(e, s)$ of i-e-traces by a sequence of larger unrestrained traces allows to improve the (declared) i-e-state then the P_e-strategy attempts to do this in the following two steps. First, by imposing an appropriate restraint on A_i, the state of the potential replacement is preserved; then, once M-permission is given, the actual replacement takes place. The first part of this action is formalized by the following definition of requiring attention (while the second part will be described in Case 1 of Step 1 of stage $s+1$ of the construction below). Whether P_e requires attention at stage $s+1$ depends on the restraint $R^i(e,s)$ on A_i to be respected by P_e at stage $s+1$ while the restraint $R^i(e,s)$ depends on which higher priority coding requirements $\mathrm{P}_{e'}$, $e' < e$, require attention at stage $s+1$. So these concepts are defined simultaneously by induction on e.

The restraint imposed on A_i by the requirements of higher priority than P_e $(i = 0,1)$ (including the restraints which higher priority requirements which require attention at stage $s+1$ may want to impose) is denoted by $R^i(e,s)$ and is defined as follows. Let

$$R^i(e,s) = s+1$$

if there is a number $e' < e$ such that $\mathrm{Q}^i_{e'}$ requires attention at stage $s+1$ or $\mathrm{P}_{e'}$ requires attention via i at stage $s+1$, and let

$$R^i(e,s) = \max\{r^i(e',s) : e' < e\}$$

where

$$r^i(e',s) = \max\{r^i_P(e',s), r^i_Q(e',s)\}$$

otherwise.

If $\beta(e,s)\downarrow$, $r^0_P(e,s) = r^1_P(e,s) = 0$, and $\hat{\sigma}_0(e,s)\downarrow$ and $\hat{\sigma}_1(e,s)\downarrow$ are the declared 0-e-state and declared 1-e-state at the end of stage s, then requirement P_e *requires attention* via $(i,\sigma,\vec{y})$ at stage $s+1$ if $i \leq 1$, σ is a binary string of length e such that $\sigma < \hat{\sigma}_i(e,s)$, and $\vec{y}$ is an i-σ-$\max\{R^i(e,s),\beta(e,s)\}$-eligible sequence at stage $s+1$; P_e *requires attention via* i at stage $s+1$ if i is minimal such that P_e requires attention via $(i,\sigma,\vec{y})$ for some σ and $\vec{y}$; and P_e *requires attention* at stage $s+1$ if there is an $i \leq 1$ such that P_e requires attention via i at stage $s+1$.

If requirement P_e is *initialized* at stage s then $\beta(e,s)$ is lifted ($\beta(e,s)\uparrow$), and, for $i = 0,1$, the sequence of the i-e-traces is cancelled ($\vec{\beta}^i(e,s)\uparrow$), the declared i-e-state is cancelled ($\hat{\sigma}_i(e,s+1)\uparrow$), and the restraint imposed on A_i by P_e is cancelled ($r^i_P(e,s) = 0$).

Using the notation introduced above, the construction is as follows where at stage 0 all requirements are initialized.

Stage $s+1$. The stage consists of three steps. Either the first or the second step, however, is vacuous.

Step 1. Fix e minimal such that $\beta(e,s)\downarrow$, $\beta(e,s)$ is M-permitted at stage $s+1$, and one of the following holds.

(a) $e = a(s)$.
(b) For some $i \le 1$, $\beta_0^i(e,s) < R^i(e,s)$.
(c) For some $i \le 1$, $r_P^i(e,s) > 0$.

If there is no such e then proceed to Step 2. (Note that this happens only if $\beta(a(s),s)\uparrow$ since otherwise $\beta(a(s),s)$ is M-permitted at stage $s+1$.) Otherwise, distinguish the following two cases, perform the corresponding action, and afterwards move to Step 3 (skipping Step 2).

Case 1: Conditions (a) and (b) fail and, for some $i \le 1$ such that $r_P^i(e,s) > 0$, there is a binary string σ of length e such that $\sigma < \hat{\sigma}_i(e,s)$ and such that there is an i-σ-$\max\{R^i(e,s),\beta(e,s)\}$*-eligible sequence $\vec{y}$ at stage $s+1$.*

Then, for the least such i, the least corresponding σ, and the least corresponding $\vec{y}$, let

$\beta(e,s+1) = \langle s+1, 2^{s+1}\rangle$,
$\vec{\beta}^i(e,s+1) = \vec{y}$,
$\hat{\sigma}_i(e,s+1) = \sigma$,
$r_P^i(e,s+1) = 0$, and
declare that P_e becomes *i-active* at stage $s+1$.

Moreover, if there is a number $k \le 2^e$ such that $\beta_k^{1-i}(e,s) \notin A_{1-i,s}$ (below we will show that such a number exists) then, for the greatest such k, put $\beta_k^{1-i}(e,s)$ into $A_{1-i,s+1}$.

Case 2: Otherwise. For $i = 0,1$, let n_i be the least number $n < 3e$ such that, for some $e' < e$,

$$[n = 3e' \;\&\; \beta_0^i(e,s) < r_P^i(e',s)]$$
or
$$[n = 3e'+1+i \;\&\; \beta_0^i(e,s) < r_Q^i(e',s)]$$
or
$$[n = 3e' \;\&\; \mathrm{P}_{e'} \text{ requires attention via } i \text{ at stage } s+1]$$
or
$$[n = 3e'+1+i \;\&\; \mathrm{Q}_{e'}^i \text{ requires attention at stage } s+1]$$

holds, and let $n_i = 3e$ if no such number n exists; fix i_0 such that either $n_{i_0} < n_{1-i_0}$ or $i_0 = 0$ and $n_0 = n_1$; and do the following.

(1) If $\beta_0^{1-i_0}(e,s) \notin A_{1-i_0,s}$ then put $\beta_0^{1-i_0}(e,s)$ into $A_{1-i_0,s+1}$ (below we will show that $\beta_0^{1-i_0}(e,s) \notin A_{1-i_0,s}$ holds), and
(2) initialize P_e.

In either case, declare that P_e *becomes active* at stage $s+1$, and initialize all requirements $\mathrm{P}_{e'}$ with $e' > e$. Moreover, initialize all requirements $\mathrm{Q}^i_{e'}$ such that $A_{i,s+1} \upharpoonright r^i_Q(e',s) \neq A_{i,s} \upharpoonright r^i_Q(e',s)$. (Note that numbers can go into A_i in the first step of a stage only. So $A_{i,s+1}$ is given by the end of this step.)

Step 2. Fix e minimal such that $\beta(e,s)\uparrow$ and do the following (for $i = 0,1$).

Let $\beta(e,s+1) = \langle s+1, 2^{s+1}\rangle$,
let $\vec{\beta}^i(e,s+1) = (\langle s+1,0\rangle,\ldots,\langle s+1,2^e\rangle)$,
let $\hat{\sigma}_i(e,s+1) = 1^e$,
let $r^i_P(e,s+1) = 0$, and
declare that P_e *becomes active* at stage $s+1$.
Moreover, initialize all requirements $\mathrm{P}_{e'}$ with $e < e'$ (actually this action is vacuous since $\beta(e',s)\uparrow$ for $e' > e$).

Step 3. Fix e minimal as in Step 1 or — if Step 1 is vacuous — as in Step 2.
For any $e' < e$ such that $\mathrm{P}_{e'}$ requires attention at stage $s+1$, fix the unique $i \leq 1$ such that $\mathrm{P}_{e'}$ requires attention via i at stage $s+1$, let $r^i_P(e',s+1) = s+1$, and declare that $\mathrm{P}_{e'}$ *receives attention (via i)* at stage $s+1$.

Finally, for any $e' < e$ and $i \leq 1$ such that $A_{i,s+1} \upharpoonright s+1 = A_{i,s} \upharpoonright s+1$ and $\mathrm{Q}^i_{e'}$ requires attention at stage $s+1$, let $r^i_Q(e',s+1) = s+1$ and declare that $\mathrm{Q}^i_{e'}$ *receives attention* at stage $s+1$. Moreover, if (4.7) holds (for e' in place of e) then, for the least x as there, put x into $C_{i,n}$ (where $e' = \langle n,m\rangle$ for some m).

This completes the construction.

Note that in this construction the terms *receiving attention* and *becoming active* have different meanings. A coding requirement P_e can become active only in Step 1 or 2 of a stage whereas it can receive attention only in Step 3 (and the latter, but not the former, only occurs if P_e requires attention). An inaccessibility requirement Q^i_e does not become active at all but it may receive attention in Step 3 of a stage at which it requires attention. The coding requirement P_e may enumerate a number into A_0 or A_1 only if it becomes active (and this happens (if and) only if it becomes active in Step 1 of the stage). If a (coding or inaccessibility) requirement receives attention then it imposes a restraint (on one of the sets A_0 or A_1) but does not initialize any (coding or inaccessibility) requirements. If P_e becomes active then it initializes the lower priority coding requirements as well as the inaccessibility requirements which are injured by the action of P_e. (The action of P_e may also injure the restraint imposed by some higher priority coding requirement $\mathrm{P}_{e'}$, so that this restraint becomes useless. Still, if this happens, for technical convenience, we do not cancel the restraint. Similarly, we let such a higher priority requirement $\mathrm{P}_{e'}$ receive attention if it requires attention no matter whether or not the restraint it wants to impose became injured in Step 1 of stage $s+1$.)

Proof of Correctness.

Obviously the construction satisfies (4.4). Hence (4.3) holds. So, by Claim 4.6, it suffices to show that the marker β satisfies the conditions (β_1)–(β_{10}) (hence, in particular, the coding requirements P_e are met) and that the inaccessibility requirements Q^i_e are met.

We first show that the conditions (β_1)–(β_{10}) are satisfied where we proceed in three steps: First, following some basic observations on the moves of the marker β, we show that (β_1)–(β_5) are satisfied (Claim 4.7). Then, using some observations on the relations among the positions of the marker β on e and the corresponding traces and declared states together with the fact that the least i-e-trace does not enter A_i (Claim 4.8), we prove conditions (β_7)–(β_{10}) (Claim 4.9). Finally, for a proof of (β_6), we show that the coding requirements act and require attention only finitely often and the inaccessibility requirements require attention only finitely often, from which we deduce that not only the marker β on e but also the restraints, traces and declared states come to a limit (Claim 4.10).

Note that at any stage $s+1>0$ there is a unique number e, in the following denoted by e_{s+1}, such that P_e becomes active at stage $s+1$. Since all coding requirements are initialized at stage 0, since all coding requirements $\mathrm{P}_{e'}$ with $e'>e_{s+1}$ are initialized at stage $s+1$, and since — unless $\beta(e_{s+1},s)\downarrow$ — e_{s+1} is the least e such that $\beta(e,s)\uparrow$, it follows by a straightforward induction on s that (β_0) and (for $e=e_{s+1}$) (β_{00}) hold. As observed above, this implies

Claim 4.7: Conditions (β_1)–(β_5) are satisfied.

To verify conditions (β_7)–(β_{10}) next (and for the remainder of the proof in general), we start with some observations on the relations between the marker $\beta(e,s)$ and the corresponding traces $\beta^i_k(e,s)$ and their declared *i*-*e*-states $\hat{\sigma}_i(e,s)$ and introduce some notation.

It is immediate by construction that (for $i \leq 1$ and $e, s \geq 0$)

$$\beta(e,s)\downarrow \Leftrightarrow \ \vec{\beta}^i(e,s)\downarrow \Leftrightarrow \ \hat{\sigma}_i(e,s)\downarrow \tag{4.9}$$

holds. (In the following we will tacitly use this fact.) So, if we call a stage s an *e-stage* if $\beta(e,s)$ is defined, then the parameters $\beta(e,s)$, $\vec{\beta}^i(e,s)$ and $\hat{\sigma}_i(e,s)$ attached to P_e are defined at stage s if and only if s is an e-stage. Also note that, by (β_1), any e-stage s is greater than e and any e-stage is an e'-stage for all $e' \leq e$. Moreover, for any e-stage s, $\vec{\beta}^i(e,s) = (\beta^i_0(e,s),\ldots,\beta^i_{2^e}(e,s))$ where $\beta^i_0(e,s) < \beta^i_1(e,s) < \cdots < \beta^i_{2^e}(e,s)$. When dealing with these traces, we sometimes identify the vector $\vec{\beta}^i(e,s)$ (if defined) with the corresponding finite set, i.e., let $\vec{\beta}^i(e,s) = \{\beta^i_0(e,s),\ldots,\beta^i_{2^e}(e,s)\}$, and we write $\vec{\beta}^i(e,s) < \vec{\beta}^i(e',s')$ if $\vec{\beta}^i(e,s)$ and $\vec{\beta}^i(e',s')$ are defined and $\beta^i_{2^e}(e,s) < \beta^i_0(e',s')$ (i.e., $\max \vec{\beta}^i(e,s) < \min \vec{\beta}^i(e',s')$) and, similarly, $\vec{\beta}^i(e,s) < y$ ($y < \vec{\beta}^i(e,s)$) if $\vec{\beta}^i(e,s)$ is defined and $\beta^i_{2^e}(e,s) < y$ ($y < \beta^i_0(e,s)$).

Moreover, the following notions will be useful. We say that $\vec{\beta}^i(e)$ is *newly appointed at stage* $s+1$ if P_e is active in Step 2 of stage $s+1$ (i.e., iff $\vec{\beta}^i(e,s)\uparrow \neq \vec{\beta}^i(e,s+1)\downarrow$), and we say that $\vec{\beta}^i(e)$ *becomes upgraded at stage* $s+1$ if P_e is i-active at stage $s+1$ (i.e., iff $\vec{\beta}^i(e,s)\downarrow \neq \vec{\beta}^i(e,s+1)\downarrow$). For any e-stage s, we let $s(e,s)$ be the greatest stage $\leq s$ at which $\vec{\beta}^i(e)$ is newly appointed (note that s does not depend on i since $\vec{\beta}^0(e)$ and $\vec{\beta}^1(e)$ are newly appointed at the same stages), and we let $s^i(e,s)$ be the greatest stage $\leq s$ at which $\vec{\beta}^i(e)$ is newly appointed or becomes upgraded. Moreover, we call e-stages s' and s'' *e-equivalent* ($s' \sim_e s''$) if any stage s with $\min(s',s'') \leq s \leq \max(s',s'')$ is an e-stage, and we say that $\vec{\beta}^i(e,s)$ is *upgraded* if s is an e-stage and $\vec{\beta}^i(e)$ becomes upgraded at stage $s^i(e,s)$. Note that, for an e-stage s, $s(e,s)$ is the least stage $s' \leq s$ with $s' \sim_e s$ and that $s(e,s) \leq s^i(e,s) \leq s$. Also note that

$$\beta(e,s)\downarrow \Rightarrow \ \vec{\beta}^i(e,s) = \vec{\beta}^i(e,s^i(e,s))\downarrow \tag{4.10}$$

since P_e neither becomes i-active nor is initialized at any stage t with $s^i(e,s) < t \leq s$. Moreover, if $\vec{\beta}^i(e)$ is newly appointed at stage $s+1$ then, by (β_1) and by construction, $e \leq s$, $\vec{\beta}^i(e, s+1) = (\langle s+1, 0\rangle, \ldots, \langle s+1, 2^e\rangle)$ and $\beta(e, s+1) = \langle s+1, 2^{s+1}\rangle$ whence

$$\begin{aligned}\beta(e,s)\downarrow \& \ s(e,s) = s^i(e,s) \Rightarrow e < s^i(e,s) &\leq \vec{\beta}^i(e, s^i(e,s)) \\ &= (\langle s^i(e,s), 0\rangle, \ldots, \langle s^i(e,s), 2^e\rangle) \\ &< \langle s^i(e,s), 2^{s^i(e,s)}\rangle = \beta(e, s^i(e,s))\end{aligned} \tag{4.11}$$

holds, while if $\vec{\beta}^i(e)$ is upgraded at stage $s+1$ then $\beta(e,s) < \vec{\beta}^i(e, s+1) < s$, and none of the new i-e-traces has been previously enumerated into A_i (and none is enumerated into A_i at stage $s+1$) whence

$$\begin{aligned}\beta(e,s)\downarrow \ \& \ s(e,s) < s^i(e,s) \Rightarrow\ &\beta(e, s^i(e,s)-1) < \vec{\beta}^i(e, s^i(e,s)) < s^i(e,s) \\ &< \langle s^i(e,s), 2^{s^i(e,s)}\rangle = \beta(e, s^i(e,s)) \\ &\text{and} \\ &\vec{\beta}^i(e, s^i(e,s)) \cap A_{i, s^i(e,s)} = \emptyset\end{aligned} \tag{4.12}$$

holds. Together with (β_3) the preceding three observations imply that the traces $\vec{\beta}^i(e,s)$ (if defined) are contained in the open interval $(e, \beta(e,s))$,

$$\beta(e,s)\downarrow \Rightarrow\ e < \vec{\beta}^i(e,s) < \beta(e,s) \leq \langle s, 2^s\rangle, \tag{4.13}$$

that the traces $\vec{\beta}^i(e,s')$ (if defined) exceed $\langle s, 2^s\rangle$ for any previous stage s at which the traces were undefined,

$$s < s' \ \&\ \beta(e,s)\uparrow \ \&\ \beta(e,s')\downarrow \Rightarrow\ \langle s, 2^s\rangle < \vec{\beta}^i(e,s') \tag{4.14}$$

and that the traces $\vec{\beta}^i(e,s)$ are nondecreasing in s (where defined),

$$s < s' \ \&\ \beta(e,s)\downarrow \ \&\ \beta(e,s')\downarrow \Rightarrow\ \vec{\beta}^i(e,s) = \vec{\beta}^i(e,s') \ \vee\ \vec{\beta}^i(e,s) < \vec{\beta}^i(e,s'). \tag{4.15}$$

(For the latter two facts, recall that the pairing function is strictly increasing in either argument and satisfies $\langle s, 2^s\rangle < \langle s+1, 0\rangle$ for all s, and observe that, for e-equivalent stages $s < s'$, either $\vec{\beta}^i(e,s) = \vec{\beta}^i(e,s') = \vec{\beta}^i(e, s^i(e,s))$ (namely if $s^i(e,s) = s^i(e,s')$) or $\vec{\beta}^i(e,s) < \beta(e, s^i(e,s)) < \vec{\beta}^i(e,s')$ (namely if $s^i(e,s) < s^i(e,s')$).) It follows that, at any stage s, the sets of i-traces of the coding requirement are pairwise different, in fact

$$e < e' \ \&\ \beta(e',s)\downarrow \Rightarrow\ \vec{\beta}^i(e,s) < \beta(e, s^i(e,s)) < \vec{\beta}^i(e',s) \tag{4.16}$$

holds. For a proof of (4.16), fix e, e', s such that $e < e'$ and s is an e'-stage. By (4.10) and (4.13), it suffices to show that $\beta(e, s^i(e,s)) < \vec{\beta}^i(e',s)$. But, since $\beta(e, s^i(e,s)) = \langle s^i(e,s), 2^{s^i(e,s)}\rangle$ and since $\mathrm{P}_{e'}$ is initialized at stage $s^i(e,s)$, this is immediate by (4.14).

Since a number y is enumerated into A_i at stage $s+1$ only if y is an i-e-trace (for some $e \geq 0$) at stage s, (4.13) implies

$$A_{i,s+1} \subseteq \omega \upharpoonright \langle s, 2^s \rangle. \tag{4.17}$$

So if $\vec{\beta}^i(e)$ is newly appointed at stage $s+1$ then, by (4.11), the newly appointed i-e-traces are not in $A_{i,s+1}$. Since the corresponding observation for traces becoming upgraded at stage $s+1$ trivially holds (compare (4.12)),

$$\beta(e,s)\downarrow \Rightarrow \ \vec{\beta}^i(e, s^i(e,s)) \cap A_{i,s^i(e,s)} = \emptyset \tag{4.18}$$

follows. Moreover,

$$\begin{gathered} s \sim_e s+1 \ \& \ A_{i,s+1} \upharpoonright \langle s^i(e,s), 2^{s^i(e,s)} \rangle \neq A_{i,s} \upharpoonright \langle s^i(e,s), 2^{s^i(e,s)} \rangle \ \Rightarrow \\ [\ \mathrm{P}_e \text{ is } (1-i)\text{-active at stage } s+1 \ \& \\ \exists\, k \leq 2^e \ (A_{i,s+1} \setminus A_{i,s} = \{\beta^i_k(e, s^i(e,s))\})] \,. \end{gathered} \tag{4.19}$$

Namely, since P_e becomes active at stage $s^i(e,s)$, all lower priority requirements $\mathrm{P}_{e''}$, $e < e''$, are initialized at stage $s^i(e,s)$ hence (by (4.14)) enumerate only numbers $> \langle s^i(e,s), 2^{s^i(e,s)} \rangle$ into A_i after stage $s^i(e,s)$, and, since $s+1$ is an e-stage, no higher priority requirement $\mathrm{P}_{e'}$, $e' < e$, may act at stage $s+1$. So P_e is the only requirement which may enumerate a number into A_i at stage $s+1$ and — since $s+1$ is an e-state — this happens only if P_e is $(1-i)$-active at stage $s+1$ whence the claim follows by (4.10).

Since the i-e-traces are enumerated into A_i in decreasing order, (4.18) and (4.19), together with (4.10), (4.13) and the fact that no number enters A_i at stage $s^i(e,s)$, imply (for $k \leq 2^e$)

$$\begin{gathered} \beta(e,s)\downarrow \ \& \ \beta^i_k(e, s^i(e,s)) \notin A_{i,s} \ \Rightarrow \\ A_{i,s} \upharpoonright \beta^i_{k+1}(e, s^i(e,s)) = A_{i,s^i(e,s)-1} \upharpoonright \beta^i_{k+1}(e, s^i(e,s)) \\ (\text{where we let } \beta^i_{2^e+1}(e, s^i(e,s)) = \beta(e, s^i(e,s))). \end{gathered} \tag{4.20}$$

Next we turn to the declared states. Note that, correspondingly to (4.10),

$$\beta(e,s)\downarrow \Rightarrow \ \hat{\sigma}_i(e,s) = \hat{\sigma}_i(e, s^i(e,s))\downarrow \tag{4.21}$$

holds (hence $\vec{\beta}^i(e,s)$ is upgraded iff $\hat{\sigma}_i(e,s) < 1^e$). Moreover, if defined at stages s and $s+1$, then $\hat{\sigma}_i(e,s) = \hat{\sigma}_i(e,s+1)$ unless P_e becomes i-active at stage $s+1$, in which case $\hat{\sigma}_i(e,s+1) < \hat{\sigma}_i(e,s)$. Hence

$$s' \leq s'' \ \& \ s' \sim_e s'' \ \Rightarrow \ \hat{\sigma}_i(e,s'') \leq \hat{\sigma}_i(e,s')\downarrow \tag{4.22}$$

and

$$s \sim_e s+1 \;\Rightarrow (\hat{\sigma}_i(e,s+1) < \hat{\sigma}_i(e,s) \Leftrightarrow \text{ P}_e \text{ is } i\text{-active at stage } s+1) \tag{4.23}$$

hold. Since the declared i-e-state can assume only 2^e different values hence can be decreased at most $2^e - 1$ times, (4.22) and (4.23) imply

$$\beta(e,s)\downarrow \Rightarrow \; |\{s' : s \sim_e s' \;\&\; \text{P}_e \text{ is } i\text{-active at stage } s'\}| < 2^e. \tag{4.24}$$

Since there are $2^e + 1$ i-e-traces (if defined) and since i-e-traces are enumerated into A_i in decreasing order (compare (4.20)) it follows (from (4.24) with $1-i$ in place of i) by (4.18), (4.19) and (4.20) that

$$\beta(e,s)\downarrow \Rightarrow \begin{array}{c} \beta^i_0(e,s^i(e,s)) \notin A_{i,s} \text{ and} \\ A_{i,s} \restriction \beta^i_1(e,s^i(e,s)) = A_{i,s^i(e,s)-1} \restriction \beta^i_1(e,s^i(e,s)). \end{array} \tag{4.25}$$

By (4.10), the latter implies

Claim 4.8: Let s be an e-stage. Then, for $i \leq 1$, $\vec{\beta}^i(e,s) = \vec{\beta}^i(e,s^i(e,s))$, $\beta^i_0(e,s) \notin A_{i,s}$, and $A_{i,s} \restriction \beta^i_1(e,s) = A_{i,s^i(e,s)-1} \restriction \beta^i_1(e,s)$.

Note that Claim 4.8 implies that if P_e is i-active at stage $s+1$ then there is a number $k \leq 2^e$ such that $\beta^{1-i}_k(e,s) \notin A_{1-i,s}$, and, for the greatest such k, the $(1-i)$-e-trace $\beta^{1-i}_k(e,s)$ is enumerated into A_{1-i} at stage $s+1$; and, similarly, if Case 2 of Step 1 applies to stage $s+1$ then $\beta^{1-i_0}_0(e,s)$ is not in $A_{1-i_0,s}$ and $\beta^{1-i_0}_0(e,s)$ is enumerated into A_{1-i_0} at stage $s+1$. So if P_e becomes active via Step 1 at stage $s+1$ then, for some $i \leq 1$, there is an i-e-trace $\beta^i_k(e,s)$ which is newly enumerated into A_i at stage $s+1$. Since, for the least e such that $\beta(e,s)\downarrow \neq \beta(e,s+1)$ (if any), P_e becomes active in Step 1 of stage $s+1$ it follows that

$$\begin{array}{c} \beta(e,s)\downarrow \neq \beta(e,s+1) \;\&\; \forall\, e' < e\; (\beta(e',s) = \beta(e',s+1)) \;\Rightarrow \\ \exists\, i \leq 1\; \exists\, k \leq 2^e\; (\beta^i_k(e,s) \in A_{i,s+1} \setminus A_{i,s}) \end{array} \tag{4.26}$$

holds. This observation is crucial for establishing the following claim.

Claim 4.9: Conditions (β_7)–(β_{10}) are satisfied.

Proof: For a proof of (β_7) note that, for the least e such that $\beta(e,s)\downarrow\neq \beta(e,s+1)$, P_e becomes active in Step 1 of stage $s+1$ whence $\beta(e,s)$ is M-permitted. By (β_2) this implies (β_7). (β_8) is immediate by (4.26), (4.13), and (β_2). For a proof of (β_9) assume $y \in A_{i,s+1} \setminus A_{i,s}$. Then $y = \beta^i_k(e,s)$ for some e and $k \leq 2^e$ where P_e becomes active in Step 1 of stage $s+1$ whence $\beta(e,s+1) \neq \beta(e,s)\downarrow$. So (β_9) follows from (4.13). Finally, for a proof of (β_{10}), w.l.o.g. assume that $\beta(a(s), s) \downarrow$. Then, since $a(s) < \beta(a(s), s)$, $\beta(a(s), s)$ is M-permitted at stage $s + 1$. So either $\mathrm{P}_{a(s)}$ becomes active according to Case 2 in Step 1 of stage $s+1$ or a higher priority requirement P_e, $e < a(s)$, becomes active. In either case, $\mathrm{P}_{a(s)}$ is initialized at stage $s + 1$, hence, $\beta(a(s), s + 1) \uparrow$. □

Next we show that all requirements require attention only finitely often, that the coding requirements become active only finitely often, and that the marker β reaches a final position $\beta^*(e) \in \omega$ for all $e \geq 0$ whence the marker condition (β_6) is satisfied too.

Claim 4.10: (a) For $e \geq 0$, the coding requirement P_e becomes active only finitely often, requires attention only finitely often and is initialized only finitely often. Moreover, $\beta^*(e) = \lim_s \beta(e, s) \in \omega$ whence, in particular, (β_6) is satisfied.

(b) For $e \geq 0$ and $i \leq 1$, the inaccessibility requirement Q^i_e requires attention only finitely often and is initialized only finitely often.

Proof: The two parts of the claim are proven simultaneously by induction on the index n of the corresponding requirements $\mathrm{R}_n = \mathrm{P}_e$ and $\mathrm{R}_n = \mathrm{Q}^i_e$.

Fix n and, by inductive hypothesis, fix a stage $s_0 > 0$ such that no requirement $\mathrm{R}_{n'}$ with $n' < n$ requires attention, becomes active, or is initialized after stage $s_0 - 1$. So any restraint imposed by such a higher priority requirement at stage s_0 is permanent and is bounded by s_0.

Next fix $s_1 \geq s_0$ such that, for any coding requirement $\mathrm{P}_{e''}$ which becomes active after stage s_1 and for any $i \leq 1$ and $s \geq s_1$ such that $\beta^i_0(e'', s)$ is defined, $s_0 < \beta^i_0(e'', s)$. The existence of such a stage s_1 is established as follows. First note that, by (4.13), $\beta^i_0(e'', s) \leq s_0$ implies that $e'' < s_0$. So, by (4.14), it suffices to show that any requirement $\mathrm{P}_{e''}$ which becomes active infinitely often is initialized infinitely often. But this is immediate by (4.24).

Note that, by choice of s_1, for any requirement $\mathrm{P}_{e''}$ which becomes active according to Case 2 in Step 1 of a stage $s+1 > s_1$, $n \leq n_i$ for the parameter n_i ($i = 0, 1$) defined there. Namely, by $s_1 \geq s_0$, $3e'' \geq n$, no requirement $\mathrm{R}_{n'}$ with $n' < n$ requires attention at stage $s+1$, and the restraint imposed by such a requirement $\mathrm{R}_{n'}$ is bounded by s_0 hence is less than $\beta^i_0(e'', s)$ by choice of s_1.

Now distinguish the following two cases according to the type of requirement R_n.

Case 1: $\mathrm{R}_n = \mathrm{P}_e$ *(i.e., $n = 3e$).* It suffices to show that

$$\exists\, s\, \forall\, s' \geq s\, (\beta(e, s') \downarrow) \tag{4.27}$$

holds. Namely, fix s as in (4.27). Then, by (4.24), P_e becomes active only finitely often after stage s. Since P_e can be initialized only at a stage where some $\mathrm{P}_{e'}$ with $e' \leq e$ becomes active, it follows that P_e is initialized only finitely often. Since $\beta(e, s') \downarrow \neq \beta(e, s'+1) \downarrow$ implies that P_e becomes active at stage $s'+1$, it follows that $\beta^*(e) = \lim_{s' \to \omega} \beta(e, s')$ exists and, by (4.27), $\beta^*(e) \in \omega$. Finally, in order to show that P_e requires attention only finitely often, let s' be the last stage at which some $P_{e'}$ with $e' \leq e$ becomes active. It suffices to observe that P_e receives attention at any stage $> s'$ at which it requires attention and that P_e receives attention at most once after stage s' since the restraint which is imposed then will never be cancelled.

Now, for a proof of (4.27), for a contradiction assume that $\beta(e, s) \uparrow$ for infinitely many s. Fix $s_2 > s_1$ such that $a(s) > e$ for all $s \geq s_2$ and such that $\beta(e, s_2) \uparrow$. By the latter, P_e becomes active in Step 2 of stage $s_2 + 1$ hence $\beta(e, s_2+1) \downarrow$ and $r^0_P(e, s_2+1) = r^1_P(e, s_2+1) = 0$. So, by assumption, we may fix $s_3 > s_2$ minimal such that $\beta(e, s_3+1) \uparrow$. Since no higher priority requirement may act at stage s_3+1 (by $s_3 \geq s_0$), it follows that P_e becomes active according to Case 2 in Step 1 of stage s_3+1. Moreover, since $e < a(s_3)$ (by $s_3 \geq s_2$) and since, for $i \leq 1$, $R^i(e, s_3) = R^i(e, s_0 - 1) \leq s_0 < \beta^i_0(e, s_3)$ (by choice of s_0 and by $s_3 \geq s_1$), the conditions (a) and (b) given there fail. So (c) must hold, i.e., there is a (unique) number $i \leq 1$ such that $r^i_P(e, s_3) > 0$. On the other hand, since P_e does not become active according to Case 1, there is no string $\sigma \in \{0,1\}^e$ and no vector $\vec{y}$ such that $\sigma < \hat{\sigma}_i(e, s_3)$ and $\vec{y}$ is i-σ-max$\{R^i(e, s_3), \beta(e, s_3)\}$-eligible at stage $s_3 + 1$.

Now, by $r^i_P(e, s_2) = 0 < r^i_P(e, s_3)$, fix s maximal such that $s_2 \leq s < s_3$ and $r^i_P(e, s) = 0$. Then P_e requires and receives attention via i at stage $s+1$ whence there is a string $\sigma \in \{0,1\}^e$ and a vector $\vec{y}$ such that $\sigma < \hat{\sigma}_i(e, s)$ and $\vec{y}$ is i-σ-max$\{R^i(e, s), \beta(e, s)\}$-eligible at stage $s+1$, and, by maximality of s, $r^i_P(e, s') = s+1$ for all s' with $s < s' \leq s_3$. By the latter, P_e does not become active at any such stage s' (since $r^i_P(e, s') = 0$ for any stage s' at which P_e is active) hence $\hat{\sigma}_i(e, s) = \hat{\sigma}_i(e, s_3)$ and $\beta(e, s) = \beta(e, s_3)$. So, since, by choice of s_0, $R^i(e, s) = R^i(e, s_3)$ too, in order to get the desired contradiction it suffices to show that $\vec{y}$ is i-σ-max$\{R^i(e, s), \beta(e, s)\}$-eligible at stage $s_3 + 1$. For this sake, by definition of i-σ-max$\{R^i(e, s), \beta(e, s)\}$-eligibility and by Claim 4.4, it suffices to show that $A_{i,s_3} \upharpoonright s + 1 = A_{i,s} \upharpoonright s + 1$ since, by choice of s_0, a requirement $Q^i_{e'}$ with $e' < e$ is satisfied at stage s iff it is satisfied at stage s_3, and does not require attention after stage s_0.

Now since neither P_e nor a higher priority requirement becomes active at a stage $s'+1$ where $s \leq s' < s_3$, for a proof of $A_{i,s_3} \upharpoonright s+1 = A_{i,s} \upharpoonright s+1$, it suffices to show that no lower priority requirement enumerates a trace $< s+1$ into A_i at such a stage $s'+1$. So fix $\mathrm{P}_{e''}$, $e < e''$, and, for a contradiction, assume that $\beta^i_k(e'',s') < s+1$ is in $A_{i,s'+1} \setminus A_{i,s'}$. Since, by choice of s', either P_e requires attention via i at stage $s'+1$ or $r^i_P(e,s') = s+1$, it follows that $\beta^i_0(e'',s') < s+1 \leq R^i(e'',s'+1)$. So $\mathrm{P}_{e''}$ becomes active according to Case 2 in Step 1 of stage $s'+1$. Moreover, for the parameters n_0 and n_1 defined there, $n_i \leq 3e = n$ (since P_e requires attention via i at stage $s'+1$ or $r^i_P(e,s') = s+1$). On the other hand, by $s' > s_1$, $n \leq n_{1-i}$ as observed above. But, since, for a stage s' such that P_e requires attention via i at stage $s'+1$ or $r^i_P(e,s') = s+1$, P_e does not require attention via $1-i$ at stage $s'+1$ and $r^{1-i}_P(e,s') = 0$, the latter implies that $n_{1-i} > n$ hence $n_i < n_{1-i}$. But, by construction, this implies that $\mathrm{P}_{e''}$ does not enumerate a trace in A_i at stage $s'+1$ which gives the desired contradiction and completes the proof of (4.27).

Case 2: $\mathrm{R}_n = \mathrm{Q}^i_e$ *(i.e.,* $n = 3e+1+i$*).* If Q^i_e does not require attention after stage s_1, then it can only be initialized once after stage s_1 and the result holds for the requirement, so we can assume that Q^i_e requires attention at stage $s+1 > s_1$ and it suffices to show that Q^i_e receives attention and is permanently satisfied at stage $s+1$.

In order to show that Q^i_e receives attention at stage $s+1$, it suffices to show that $A_{i,s+1} \upharpoonright s+1 = A_{i,s} \upharpoonright s+1$. For a contradiction, assume that this is not the case. Then, for some $e'' > e$, $\mathrm{P}_{e''}$ becomes active at stage $s+1$ and enumerates a trace $\beta^i_k(e'',s) \leq s$ into A_i. Since, by Q^i_e requiring attention, $R^i(e'',s) = s+1$ it follows that $\beta^i_0(e'',s) < R^i(e'',s)$. So $\mathrm{P}_{e''}$ acts according to Case 2 of Step 1 of stage s+1 and, for the parameters n_0 and n_1 defined there, $n_{1-i} \leq n_i$ since otherwise $\mathrm{P}_{e''}$ would not enumerate a trace into A_i. But, by Q^i_e requiring attention, $n_i \leq 3e+1+i = n$ while on the other hand, as observed above, $n \leq n_{1-i}$. Moreover, by definition, n_{1-i} cannot attain the value $3e+1+i$. So $n_i < n_{1-i}$ giving the desired contradiction.

The proof that Q^i_e is permanently satisfied at stage $s+1$ is similar. For a contradiction, fix $s' > s$ minimal such that Q^i_e is initialized at stage $s'+1$. Then $A_{i,s'} \upharpoonright r^i_Q(e,s') \neq A_{i,s'+1} \upharpoonright r^i_Q(e,s')$ where, by minimality of s', $r^i_Q(e,s') = r^i_Q(e,s+1) = s+1$. So (by the priority ordering of the requirements and by choice of s_0) a coding requirement $\mathrm{P}_{e''}$ with $e'' > e$ becomes active at stage $s'+1$ and enumerates a trace $\beta^i_k(e'',s') < r^i_Q(e,s')$

into A_i. Since $r^i_Q(e,s') \leq R^i(e'',s')$ and $\beta^i_0(e'',s') \leq \beta^i_k(e'',s')$, it follows that $\mathrm{P}_{e''}$ acts according to Case 2 in Step 1 of stage $s'+1$ and that $n_i \leq 3e+1+i = n$. Moreover, since $\mathrm{P}_{e''}$ enumerates $\beta^i_k(e'',s')$ into A_i, $n_{1-i} \leq n_i$. But this is impossible, since (as observed above) $n \leq n_{1-i}$ by choice of s_1 and $n_{1-i} \neq 3e+1+i$ by definition. □

By Claim 4.10, the parameters attached to the coding and inaccessibility requirements come to a limit. So we may let

$$\vec{\beta}^i(e) = (\beta^i_0(e), \ldots, \beta^i_{2^e}(e)) = \lim_{s\to\omega} \vec{\beta}^i(e,s) \in \omega^{2^e+1},$$

$$\hat{\sigma}_i(e) = \lim_{s\to\omega} \hat{\sigma}_i(e,s),$$

$$r^i_P(e) = \lim_{s\to\omega} r^i_P(e,s) < \omega,$$

and

$$r^i_Q(e) = \lim_{s\to\omega} r^i_Q(e,s) < \omega.$$

Moreover, we let

$$r_P(e) = \max\{r^0_P(e), r^1_P(e)\} \text{ and } r_P(e,s) = \max\{r^0_P(e,s), r^1_P(e,s)\},$$

and we let t_e+1 and t^i_e+1 be the last stages at which P_e becomes active, and at which the i-e-traces become newly defined or upgraded, respectively. Note that $t^i_e \leq t_e$ and, for $e < e'$, $t_e < t^i_{e'}$. Moreover, for all $s \geq t_e$, $\beta(e,s+1) = \beta(e,t_e+1) = \beta^*(e)$, while, for all stage $s \geq t^i_e$, $\vec{\beta}^i(e,s+1) = \vec{\beta}^i(e,t^i_e+1) = \vec{\beta}^i(e)$, $\hat{\sigma}^i(e,s+1) = \hat{\sigma}^i(e,t^i_e+1) = \hat{\sigma}^i(e)$, and $s^i(e,s+1) = t^i_e+1$ (hence, in particular, $s+1 \sim_e t^i_e+1$). Also note that, by (4.25), the latter imply

$$\beta^i_0(e) \notin A_i \text{ and } A_i \upharpoonright \beta^i_1(e) = A_{i,t^i_e} \upharpoonright \beta^i_1(e). \tag{4.28}$$

It remains to show that the inaccessibility requirements Q^i_e are met.

Claim 4.11: Q^i_e is met.

Proof: For the proof, fix $e \geq 0$ and $i \leq 1$ and fix the least stage s_0 such that no requirement of higher priority than P_{e+1} becomes active or requires attention or is initialized after stage s_0 (note that Q^0_e and Q^1_e, hence Q^i_e, have higher priority than P_{e+1}).

Note that, by choice of s_0,

$$e < e'' \ \& \ \beta(e'', s')\downarrow \ \& \ s_0 < s^i(e'', s') - 1 \ \& \ y = \beta^i_k(e'', s') \notin A_{i,s'} \ \Rightarrow \\ \sigma_i(e+1, y, s') \preceq \hat{\sigma}_i(e'', s') \restriction e+1 \tag{4.29}$$

holds. (Compare with Claim 3.8 in the proof of Theorem 2.7.) The proof is as follows. By (4.10), $\vec{\beta}^i(e'', s') = \vec{\beta}^i(e'', s^i(e'', s'))$ hence $y = \beta^i_k(e'', s^i(e'', s'))$. It follows, by construction, that $\sigma_i(e'', y, s^i(e'', s') - 1) \preceq \hat{\sigma}_i(e'', s^i(e'', s'))$ and, for any $e' \leq e$ such that $\hat{\sigma}_i(e'', s^i(e'', s'))(e') = 0$ and $\mathrm{Q}^i_{e'}$ is not satisfied at stage $s^i(e'', s') - 1$, y is i-e'-eligible at stage $s^i(e'', s')$ via numbers x, u, v and r where $r < \beta^i_{k+1}(e'', s^i(e'', s'))$ (and where $\beta^i_{2^{e''}+1}(e'', s^i(e'', s')) = \langle s^i(e'', s'), 2^{s^i(e'', s')} \rangle$ by convention). Since $s_0 < s^i(e'', s') - 1$, it follows by (4.20) and by Claim 4.4, that, for any such e', y is i-e'-eligible at stage $s'+1$ too. Since, by $s_0 < s^i(e'', s') - 1$, for $e' \leq e$, $\mathrm{Q}^i_{e'}$ is satisfied at stage s' iff $\mathrm{Q}^i_{e'}$ is satisfied at stage $s^i(e'', s') - 1$ it follows that $\sigma_i(e+1, y, s') \preceq \hat{\sigma}_i(e'', s^i(e'', s'))$. So the claim follows by (4.21).

The remainder of the proof is organized as the proof of the corresponding Claim 3.9 in the proof of Theorem 2.7. As there w.l.o.g. we may assume that Q^i_e is not satisfied at any stage $s \geq s_0$ and that the clauses (i) and (ii) in Q^i_e hold while clause $(*)$ fails. So, in particular, the assumptions made in Claim 4.2 and Claim 4.3 hold, and it suffices to show that there is an infinite computable set S of stages s satisfying (4.8).

For the definition of such a set S, first define *true* i-e'-*eligibility* and the i-$(e+1)$-state of y *truly preceding* the $(e+1)$-state σ at stage s ($\sigma_i(e+1, y, s) \preceq_t \sigma$) just as the corresponding notions in the proof of Theorem 2.7 with A_i, $\mathrm{Q}^i_{e'}$ and i-e'-eligibility in place of A, $\mathrm{Q}_{e'}$ and e'-eligibility, respectively. In addition, say that $\sigma_i(e+1, y, s)$ precedes σ at stage s *via* r if, for any $e' \leq e$ such that $\sigma(e') = 0$ and $\mathrm{Q}^i_{e'}$ is not satisfied at stage s_0, y is i-e'-eligible via numbers x, u, v and r' at stage $s+1$ such that $r' \leq r$, and say that $\sigma_i(e+1, y, s)$ truly precedes σ at stage s *via* r if $\sigma_i(e+1, y, s)$ precedes σ at stage s via r and $A_i \restriction r = A_{i,s} \restriction r$. Then, obviously, the analogs of (3.58) and (3.59) hold and, for the newly introduced notion, we obtain the following variant of (3.59):

$$\forall\, s \geq s_0 \ \forall\, y \geq 0 \ \forall\, \sigma \in \{0,1\}^{e+1} \\ ([\sigma_i(e+1, y, s) \preceq \sigma \text{ via } r \ \& \ A_i \restriction r = A_{i,s} \restriction r] \Rightarrow \sigma_i(e+1, y, s) \preceq_t \sigma \text{ (via } r)). \tag{4.30}$$

Then, based on these modified notions, define σ^* correspondingly too:

$$\sigma^* = \mu\, \sigma \in \{0,1\}^{e+1} [\overset{\infty}{\exists}\, y \notin A_i \ \exists\, s \geq s_0 \ (\sigma_i(e+1, y, s) \preceq_t \sigma)].$$

Finally, fix $e_0 > e$ minimal such that

$$s_0 < t^i_{e_0}, \tag{4.31}$$

$$r_P(e_0) = 0 \tag{4.32}$$

and

$$\forall\, y \geq e_0\ \forall\, s \geq s_0\ \forall\, \sigma < \sigma^*\ (y \notin A_i\ \Rightarrow\ \sigma_i(e+1, y, s) \not\leq_t \sigma) \tag{4.33}$$

hold. Since (4.31) and (4.33) hold for almost all numbers e_0, in order to show that such a number e_0 exists, it suffices to show that there are infinitely many $e'' > e$ such that $r_P(e'') = 0$. For a contradiction, assume that this is not the case and fix $e_1 > e$ such that $r_P(e'') > 0$ for all $e'' \geq e_1$. Then, for any coding requirement $\mathrm{P}_{e''}$ with $e'' \geq e_1$ and for any stage $s + 1$ at which $\mathrm{P}_{e''}$ becomes active (hence $r_P(e'', s + 1) = 0$) and $\beta(e'', s + 1) \downarrow$, there is a stage $s' > s$ such that $\mathrm{P}_{e''}$ receives attention at stage $s' + 1$. So the function f which assigns to any stage s with $\beta(e'', s) \neq \beta(e'', s + 1) \downarrow$ for some $e'' \geq e_1$ the least stage $s' > s + 1$ at which $r_P(e'', s') > 0$ (while $f(s) = s$ otherwise) is total and computable. Since, for such e'' and s, $s < \beta(e'', s+1)$ it follows that, for almost all such pairs (e'', s), $\beta(e'', s+1)$ is M-permitted at some stage $s'' > f(s)$. So, unless $\mathrm{P}_{e''}$ became active or initialized at a stage t with $s + 1 < t \leq s''$, $\beta(e'', s'') = \beta(e'', s + 1)$ and $r_P(e'', s'') > 0$ whence $\mathrm{P}_{e''}$ becomes active at stage $s'' + 1$ or becomes initialized by some higher priority requirement becoming active. It follows that, for almost all e'' either the marker $\beta(e'')$ is moved infinitely often or is eventually permanently undefined. But this contradicts (β_6).

The properties (3.62), (3.63) and (3.64) of σ^*, which were crucial in the proof of Theorem 2.7, here become as follows.

$$\forall\, e'' \geq e_0\ \forall\, s > t_{e_0}\ (\hat{\sigma}_i(e'', s) \downarrow \Rightarrow\ \sigma^* \leq \hat{\sigma}_i(e'', s) \restriction e + 1) \tag{4.34}$$

$$\exists\, e_1 \geq e_0\ \forall\, e'' \geq e_1\ (\hat{\sigma}_i(e'') \restriction e + 1 = \sigma^*) \tag{4.35}$$

$$\sigma^*(e) = 0. \tag{4.36}$$

For a proof of (4.34), for a contradiction, fix $e'' \geq e_0$ minimal such that $\hat{\sigma}_i(e'', s) \restriction e + 1 < \sigma^*$ for some $s > t_{e_0}$ and let s be the least such stage. Distinguish the following two cases depending on whether $e'' = e_0$ or $e_0 < e''$.

Case 1: $e'' = e_0$. Then, by $s > t_{e_0}$, $\hat{\sigma}_i(e_0, s) = \hat{\sigma}_i(e_0, t^i_{e_0} + 1) = \hat{\sigma}_i(e_0)$ (and $\vec{\beta}^i(e_0) = \vec{\beta}^i(e_0, t^i_{e_0} + 1)$) hence $\hat{\sigma}_i(e_0) \restriction e + 1 < \sigma^*$. So P_{e_0} becomes i-active at stage $t^i_{e_0} + 1$. It follows, by construction, that $\vec{\beta}^i(e_0)$ is i-$\hat{\sigma}_i(e_0)$-$\max\{R^i(e_0, t^i_{e_0}), \beta(e_0, t^i_{e_0})\}$-eligible at stage $t^i_{e_0} + 1$ hence $\sigma_i(e + 1, \beta^i_0(e_0), t^i_{e_0}) \preceq \hat{\sigma}_i(e_0) \restriction e + 1$ via $\beta^i_1(e_0)$. Since, by (4.28), $\beta^i_0(e_0) \notin A_i$ and $A_i \restriction \beta^i_1(e_0) = A_{i,t^i_{e_0}} \restriction \beta^i_1(e_0)$, it follows by (4.30) that $\sigma_i(e{+}1, \beta^i_0(e_0), t^i_{e_0}) \preceq_t \hat{\sigma}_i(e_0) \restriction e + 1$. So, since $e_0 < \beta^i_0(e_0)$, $\hat{\sigma}_i(e_0) \restriction e + 1 \not< \sigma^*$ by (4.33) contrary to assumption.

Case 2: $e_0 < e''$. Let $s' = s^i(e'', s) - 1$. Then $t_{e_0} < s'$ since, by P_{e_0} becoming active at stage $t_{e_0} + 1$, $\mathrm{P}_{e''}$ is initialized at stage $t_{e_0} + 1$, and (since $\hat{\sigma}_i(e'', s) < 1^{e''}$ by assumption) $\mathrm{P}_{e''}$ becomes i-active at stage $s' + 1$. So, by construction, there is an i-$\hat{\sigma}_i(e'', s)$-$\max\{R^i(e'', s'), \beta(e'', s')\}$-eligible sequence $\vec{y}$ at stage $s'+1$. By $e_0 < e''$ it follows that the sequence $\vec{y} \restriction 2^{e_0}+1$ is i-$\hat{\sigma}_i(e'', s) \restriction e_0$-$\max\{R^i(e_0, s'), \beta(e_0, s')\}$-eligible at stage $s' + 1$. Since, by minimality of e'', $\hat{\sigma}_i(e'', s) \restriction e + 1 < \hat{\sigma}_i(e_0, s') \restriction e + 1$ (hence $\hat{\sigma}_i(e'', s) \restriction e_0 < \hat{\sigma}_i(e_0, s')$) and since, by choice of e_0, $r_P(e_0, s') = 0$, it follows that P_{e_0} requires attention at stage $s' + 1$ hence $r_P(e_0, s' + 1) = s' + 1$. Since P_{e_0} neither is initialized nor becomes active after stage $t_0 + 1$, it follows that $r_P(e_0) > 0$ contrary to choice of e_0.

The proof of (4.35) is indirect too. For a contradiction, assume that (4.35) fails. Then, by (4.34), $\sigma^* < \hat{\sigma}_i(e'') \restriction e+1$ — hence $\sigma^* < \hat{\sigma}_i(e'', t_{e''} + 1) \restriction e + 1$ — for infinitely many numbers $e'' \geq e_0$. On the other hand, by choice of σ^*, for any $e'' \geq e_0$ and z there is a sequence $\vec{y}$ and a stage s' such that $\vec{y}$ is i-$\sigma^* 1^{e''-e-1}$-z-eligible at all stages $s \geq s'$. It follows that, for any $e'' \geq e_0$ and any stage $s + 1 > s_0$ such that $\mathrm{P}_{e''}$ becomes active at stage $s + 1$ and $\sigma^* < \hat{\sigma}_i(e'', s + 1) \restriction e + 1$, there will be a least stage $s' > s$ such that $\mathrm{P}_{e''}$ requires attention at stage $s' + 1$ or is initialized at this stage. So, by M-permitting, we may argue that for almost all (e'', s) as above either $\mathrm{P}_{e''}$ becomes active after stage $s + 1$ or is initialized later whence $t_{e''} \neq s$. But this contradicts the assumption.

Finally, for a proof of (4.36), for a contradiction, assume that $\sigma^*(e) = 1$. Define $\sigma \in 2^{e+1}$ by $\sigma \restriction e = \sigma^* \restriction e$ and $\sigma(e) = 0$. Then, $\sigma < \sigma^*$ and we will show

$$\overset{\infty}{\exists}\, y \notin A_i \; \exists s \geq s_0 \; (\sigma_i(e + 1, y, s) \preceq_t \sigma) \tag{4.37}$$

which contradicts the definition of σ^*.

To establish (4.37), define

$$E = \{y \notin A_i : \exists\, s \geq s_0\ (\sigma_i(e+1, y, s) \preceq_t \sigma^*)\}.$$

Then, E is clearly computably enumerable in A_i, and we claim that E is infinite. To see this, fix $e_1 \geq e_0$ as in (4.35) and for any $e'' \geq e_1$, consider $y = \beta_0^i(e'')$ and $s = t^i_{e''}$. Then, $s \geq s_0$ and, by (4.28), $y \notin A_i$. By construction and (4.35), $\sigma_i(e{+}1, y, s) = \sigma_i(e'', y, s) \upharpoonright e{+}1 \preceq \hat{\sigma}_i(e'', s{+}1) \upharpoonright e{+}1 = \sigma^*$ and $\sigma_i(e{+}1, y, s)$ precedes σ^* via $\beta_1^i(e'')$, so by (4.28) and (4.30), $\sigma_i(e{+}1, y, s) \preceq_t \sigma^*$ and thus $y \in E$. By (4.16), for $e'' \neq e'''$, $\beta_0^i(e'') \neq \beta_0^i(e''')$, so E is infinite.

It follows that E has an infinite A_i-computable subset E', and, by Claim 4.3, there is an infinite subset F of E' such that for every $y \in F$, y is truly i-e-eligible at all sufficiently large stages. For any $y \in F$, we also have $\sigma_i(e+1, y, s) \preceq_t \sigma^*$ for all sufficiently large stages s, so for any $y \in F$, $\exists s \geq s_0(\sigma_i(e+1, y, s) \preceq_t \sigma)$, establishing (4.37), and hence (4.36) by contradiction.

Now S is defined by

$$S = \{s > t_{e_1} : \forall\, e'' > e_1\ (\hat{\sigma}_i(e'', s)\downarrow \Rightarrow\ \hat{\sigma}_i(e'', s) \upharpoonright e+1 = \sigma^*)\} \qquad (4.38)$$

where e_1 is as in (4.35).

Obviously, S is computable. Moreover, S is infinite since, by (4.35), $t^i_{e''} + 1 \in S$ for all $e'' > e_1$. It remains to show that the elements s of S satisfy (4.8). So fix $s \in S$, a number $y \leq s$ and a stage $s' \geq s$ such that $y \in A_{i,s'+1} \setminus A_{i,s'}$. We have to show that y is i-e-eligible at stage $s'+1$. Fix e'' and k such that $y = \beta_k^i(e'', s')$. Since y enters A_i at stage $s'+1 > t_{e_1}+1$ it follows that $e_1 < e''$. Moreover, since $y \leq s$, it follows (by (4.14)) that s and s' are e''-equivalent whence by (4.22), by $s \in S$ and by (4.34),

$$\hat{\sigma}_i(e'', s') \upharpoonright e+1 \leq \hat{\sigma}_i(e'', s) \upharpoonright e+1 = \sigma^* \leq \hat{\sigma}_i(e'', s') \upharpoonright e+1,$$

i.e., $\hat{\sigma}_i(e'', s') \upharpoonright e{+}1 = \sigma^*$. Since, by (4.29), $\sigma_i(e{+}1, y, s') \preceq \hat{\sigma}_i(e'', s') \upharpoonright e{+}1$, it follows that $\sigma_i(e+1, y, s')$ precedes σ^*. So y is i-e-eligible at stage $s'+1$ by (4.36).

This completes the proof of Claim 4.11. □

5. Proof of Theorem 2.12

The proof combines the basic construction of an s.m.i. degree with the permitting technique first used by Dekker in [7] and then formalized by

Yates in [20]. Given noncomputable c.e. sets $B_0, \dots, B_{n-1}$ $(n \geq 1)$, it suffices to construct pairwise disjoint c.e. sets $A_0, \dots, A_{n-1}$ such that, for $A = A_0 \cup \cdots \cup A_{n-1}$ and $\mathbf{a} = deg(A)$,

$$A_i \text{ is noncomputable } (i \leq n-1), \tag{5.1}$$

$$A_i \leq_{\mathrm{T}} B_i \ (i \leq n-1) \tag{5.2}$$

and (3.5) hold. Since the construction is very similar to the basic construction of a low s.m.i. degree given in the proof of Theorem 2.7, we only point out the necessary changes.

Let $(B_{i,s})_{s \geq 0}$ be a computable enumeration of the given set B_i $(i \leq n-1)$. The finite part of A_i enumerated by the end of stage s is denoted by $A_{i,s}$, and we let $A_s = A_{0,s} \cup \cdots \cup A_{n-1,s}$.

In order to ensure (3.5), just as in the proof of Theorem 2.7, we enumerate auxiliary c.e. sets C_n $(n \geq 0)$ and ensure that the requirements Q_e are met and condition (3.7) is satisfied. (The index n of C_n will (in the rest of this section only) be used implicitly in the index e of Q_e and should not be confused with the n used in the statement of the theorem. All future uses of n in this section will refer to this number n introduced above.) In order to ensure that A_i is noncomputable, i.e., in order to ensure (5.1), we meet the requirements

$$\mathrm{P}_{ne+i} : A_i \neq \{e\}$$

for $e \geq 0$ and $i \leq n-1$. Condition (5.2) is satisfied by permitting, i.e., for $i \leq n-1$ and all numbers x and stages s we ensure

$$x \in A_{i,s+1} \setminus A_{i,s} \ \Rightarrow \ B_{i,s+1} \upharpoonright x+1 \neq B_{i,s} \upharpoonright x+1. \tag{5.3}$$

The requirements are ordered by $\mathrm{R}_{2e} = \mathrm{P}_e$ and $\mathrm{R}_{2e+1} = \mathrm{Q}_e$. The strategy for meeting the inaccessibility requirements Q_e is exactly as in the proof of Theorem 2.7, and e-(pre)eligibility and e-states are defined as there. The strategy for meeting the noncomputability requirements P_e has to be refined, since now the enumeration of a (realized) follower into A_i requires permitting by B_i (according to condition (5.3)). So a single follower will not be sufficient, and — as usual for constructions by permitting — we appoint a new follower if all existing followers are realized and the requirement is not satisfied. Since, for the sake of the inaccessibility requirements, the e-states of the P_e-followers have to be optimized, this requires that a declared state is assigned to each follower (not just to the requirement as in the basic construction). We let $y_p(e,s)$ be the pth follower of P_e in order of magnitude at the end of stage s (and write $y_p(e,s)\uparrow$ if there is no such follower). Then, if $y_p(e,s)$ is defined, a declared e-state $\hat{\sigma}_p(e,s)$ is assigned to this follower at the end of stage s. This leads to the following revision of the definition for a noncomputability requirement requiring attention and the corresponding action.

For $e = ne' + i$, P_e *requires attention* at stage $s+1$ if $e \leq s$, P_e is not satisfied at (the end of) stage s, and one of the following holds.

$$\text{There is a realized follower } y \text{ of } \mathrm{P}_e \text{ at the end of stage } s \text{ (i.e., a follower } y \text{ such that } \{e'\}_s(y) = 0) \text{ and } B_{i,s+1} \upharpoonright y+1 \neq B_{i,s} \upharpoonright y+1. \tag{5.4}$$

$$\text{There is a follower } y \text{ of } \mathrm{P}_e \text{ at the end of stage } s, \text{ say } y = y_p(e,s), \text{ and there is a number } y' \text{ such that } y < y' \leq s,\ y' \notin A_s, \text{ and } \sigma(e,y',s) < \hat{\sigma}_p(e,s). \tag{5.5}$$

$$\text{All followers of } \mathrm{P}_e \text{ at the end of stage } s \text{ (if any) are realized.} \tag{5.6}$$

If P_e receives attention then P_e becomes active via the first of the above clauses which holds, and the action is as follows. If (5.4) holds then the least y as there is enumerated into A_i and the requirement P_e is declared to be satisfied. If (5.5) holds (and (5.4) does not hold), then for the least y as there, for the corresponding p and for the least corresponding y', y is replaced by y', the declared state of $y_p(e,s+1)$ is set to $\hat{\sigma}_p(e,s+1) = \sigma(e,y',s)$ and all P_e-followers $y'' > y$ which exist at the end of stage s are cancelled (note that, by $y < y'$, the latter ensures that $y' = y_p(e,s+1)$). Finally, if (5.6) holds (and (5.4) and (5.5) do not hold), then $y = s+1$ is appointed as follower and, for p such that $y = y_p(e,s+1)$, the declared state of y is set to $\hat{\sigma}_p(e,s+1) = 1^e$.

If P_e becomes active at stage $s+1$ then, just as in the basic construction, all requirements of lower priority are initialized, where, for a noncomputability requirement P_n, becoming initialized now means that all followers and their declared states are cancelled and the requirement is declared to be unsatisfied. Moreover, we say that P_e becomes *active via* p at stage $s+1$ if either P_e becomes active according to clause (5.4) or (5.5) and $y = y_p(e,s)$ for the least y as there or P_e becomes active according to clause (5.6) and $y_p(e,s+1) = s+1$ is appointed at stage $s+1$.

Up to this modification of P_e requiring attention and the corresponding changes in the activity of P_e the construction is the same as in the proof of

Theorem 2.7. The proof of correctness follows the lines of the corresponding part of the proof of Theorem 2.7 whence we only give a rough sketch in the following.

The observations on the relations among the followers for different requirements made there directly carry over. For the followers $y_0(e,s),\dots,y_p(e,s)$ of a requirement P_e at the end of stage s at which the requirement is not satisfied, one should observe that for any follower $y_l(e,s)$ ($l \leq p$), $y_l(e,s) \notin A_s$ and $y_l(e,s)$ is not a follower of any other requirement at stage s. Moreover, for $l < p$, the stage at which $y_l(e,s)$ is appointed (according to clause (5.5) or (5.6)) precedes the corresponding stage for $y_{l+1}(e,s)$ and $\hat{\sigma}_l(e,s) \leq \hat{\sigma}_{l+1}(e,s)$.

Since at any stage $s+1$ at most one follower is enumerated into at most one of the sets A_i, the above in particular implies that the sets A_i are pairwise disjoint.

The argument that any requirement requires attention only finitely often (see Claim 3.6) now becomes a bit more sophisticated for the noncomputability requirements: Given P_e ($e = ne'+i$), by inductive hypothesis, fix a stage $s_0 > e$ such that no higher priority requirement requires attention after stage s_0, and, for a contradiction, assume that P_e requires attention infinitely often. Then P_e is not initialized after stage s_0, P_e is not satisfied after stage s_0, and P_e receives attention at any stage $s+1 > s_0$ at which it requires attention. Note that once $y_p(e,s)$ is appointed and not cancelled afterwards, P_e can become active via p at most 2^e times (since, whenever this happens, the declared state $\hat{\sigma}_p(e,s')$ is decreased). Since, by assumption, P_e acts infinitely often, it follows, by induction on p, that, for any p there is a permanent follower $y_p(e)$ with a permanent declared state $\hat{\sigma}_p(e)$ and $A_i(y_p(e)) = \{e'\}(y_p(e)) = 0$. Moreover, $y_p(e,s') = y_p(e)$ for all $s' \geq s$ where $s > s_0$ is minimal such that, for all $p' \leq p$, $y_{p'}(e,s) = y_{p'}(e)$ and $\hat{\sigma}_{p'}(e,s) = \hat{\sigma}_{p'}(e)$. Since $\hat{\sigma}_p(e)$ is nondecreasing in p and bounded by 1^e, $\hat{\sigma}_p(e)$ comes to a limit for all sufficiently large p hence is computable in p. It follows that the least stage $s(p) > s_0$ such that $y_p(e) = y_p(e,s)$ for $s \geq s(p)$ and $y_p(e)$ is realized at stage $s(p)$ can be computed. Since, by choice of s_0, (5.4) fails for all stages $s \geq s_0$ and since $y_p(e) \geq p$, it follows that $B_i \upharpoonright p = B_{i,s(p)} \upharpoonright p$. So B_i is computable contrary to assumption, giving the desired contradiction.

As in the proof of Theorem 2.7, the fact that all requirements require attention only finitely often easily implies that the noncomputability requirements are met (see Claim 3.7). So it only remains to show that the inaccessibility requirements Q_e are met too. Here the proof given in Theorem 2.7 (see Claim 3.9) directly carries over. Note that Claim 3.2, Claim 3.3, Claim 3.4 and Claim 3.5 proven there hold in the given context too. So, as in the proof of Claim 3.9, given a stage s_0 such that neither Q_e nor a higher priority requirement requires attention after stage s_0, it suffices to show that — assuming that Q_e is not satisfied at any stage $s \geq s_0$ and that the clauses (i) and (ii) in Q_e hold while clause $(*)$ fails — there is an infinite computable set S of stages s satisfying (3.26). Such a set S is obtained as in the proof Theorem 2.7 by making some obvious changes: in the definition of stage e_0 there, condition (3.60) has to be replaced by the condition that P_{e_0} is not permanently satisfied (note that in the old context (3.60) is equivalent to this fact), and, correspondingly, the set S now consists of the stages $s > t_{e_0}$ satisfying

$$\forall\, e'' > e_0\ \forall\, p \geq 0\ (y_p(e'', s)\downarrow\ \&\ \mathrm{P}_{e''} \text{ is not satisfied at stage } s \\ \Rightarrow\ \hat{\sigma}_p(e'', s+1) \restriction e+1 = \sigma^*)$$

where now $t_{e_0} + 1$ is the last stage at which P_{e_0} becomes active.

6. Open Problems

We conclude the chapter with some open problems.

Open Problem 6.1: Can Theorem 2.8 be extended to show that any high degree can be split into two high s.m.i. degrees?

An affirmative answer to Open Problem 6.1 would show that any generator of the high degrees is a join generator of the high degrees.

The next open problem asks if Theorem 2.8 can be generalized in a different direction.

Open Problem 6.2: Can every high$_2$ or even every non-low$_2$ degree be split into two s.m.i. degrees?

An affirmative answer to Open Problem 6.2 would show that every generator of the c.e. degrees generates the high$_2$ (or even the non-low$_2$) degrees under join.

Open Problem 6.3: Is the class of s.m.i. degrees definable in **R**?

Neither the definition of strong meet inaccessibility nor any of the three equivalent conditions given in Lemma 2.2 are obviously first-order.

Open Problem 6.4: Does every strongly meet inaccessible degree satisfy Condition (2.3)?

Since Condition (2.3) is clearly first-order, an affirmative answer to Open Problem 6.4 together with Lemma 2.3 would imply an affirmative answer to Open Problem 6.3.

References

1. Ambos-Spies, Klaus. Contiguous r.e. degrees. Computation and proof theory (Aachen, 1983), 1–37, Lecture Notes in Math., 1104, Springer, Berlin, 1984.
2. Ambos-Spies, Klaus. On pairs of recursively enumerable degrees. Trans. Amer. Math. Soc. 283 (1984), 507–531.
3. Ambos-Spies, Klaus. Generators of the recursively enumerable degrees. Recursion theory week (Oberwolfach, 1984), 1–28, Lecture Notes in Math., 1141, Springer, Berlin, 1985.
4. Ambos-Spies, Klaus; Soare, Robert I. The recursively enumerable degrees have infinitely many one-types. Ann. Pure Appl. Logic 44 (1989), 1–23.
5. Ambos-Spies, Klaus; Ding, Decheng; Fejer, Peter A. Embedding distributive lattices preserving 1 below a nonzero recursively enumerable Turing degree. Logical methods (Ithaca, NY, 1992), 92–129, Progr. Comput. Sci. Appl. Logic, 12, Birkhäuser, Boston, MA, 1993.
6. Ambos-Spies, Klaus; Lempp, Steffen; Slaman, Theodore A. Generating sets for the recursively enumerable Turing degrees. Computational prospects of infinity. Part II. Presented talks, 1–22, Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap., 15, World Sci. Publ., Hackensack, NJ, 2008.
7. Dekker, J. C. E. A theorem on hypersimple sets. Proc. Amer. Math. Soc. 5 (1954), 791–796.
8. Ding, Decheng. Every p-generic r.e. degree is meet-inaccessible. Chinese J. Contemp. Math. 14 (1993), 331–337.
9. Downey, Rodney G.; Lempp, Steffen. Contiguity and distributivity in the enumerable Turing degrees. J. Symbolic Logic 62 (1997), 1215–1240.
10. Fejer, Peter A. The density of the nonbranching degrees. Ann. Pure Appl. Logic 24 (1983), 113–130.
11. Lachlan, A. H. Lower bounds for pairs of recursively enumerable degrees. Proc. London Math. Soc. (3) 16 (1966), 537–569.
12. Lachlan, A. H. Bounding minimal pairs. J. Symbolic Logic 44 (1979), 626–642.
13. Ladner, Richard E.; Sasso, Leonard P., Jr. The weak truth table degrees of recursively enumerable sets. Ann. Math. Logic 8 (1975), 429–448.
14. Nies, André. Parameter definability in the recursively enumerable degrees. J. Math. Log. 3 (2003), 37–65.
15. Sacks, Gerald E. Degrees of unsolvability. Princeton University Press, Princeton, N.J. 1963 ix+174 pp.
16. Shoenfield, Joseph. Degrees of unsolvability. North-Holland Publishing Co., Amsterdam, 1971.

17. Slaman, Theodore A. The density of infima in the recursively enumerable degrees. International Symposium on Mathematical Logic and its Applications (Nagoya, 1988). Ann. Pure Appl. Logic 52 (1991), 155–179.
18. Soare, Robert I. Recursively enumerable sets and degrees. A study of computable functions and computably generated sets. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1987. xviii+437 pp.
19. Stob, Michael. wtt-degrees and T-degrees of r.e. sets. J. Symbolic Logic 48 (1983), 921–930.
20. Yates C. E. M. Three theorems on the degrees of recursively enumerable sets. Duke Math. J. 32 (1965), 461–468.
21. Yates, C. E. M. A minimal pair of recursively enumerable degrees. J. Symbolic Logic 31 (1966), 159–168.
22. Zhang, Qing Long. The density of the meet-inaccessible r.e. degrees. J. Symbolic Logic 57 (1992), 585–596.